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CONCERNING TOPOLOGICAL CONVERGENCE OF SETS

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In this paper a convergence of nets of subsets of a topological space is defined. Fundamental properties of this convergence are derived and applied to the set of all points of a connected compact Hausdorff space K at which the space K is not locally connected. In this connection a generalisation of the theorem of R. L. Moore is given (theorem 4.7).

1. TERMINOLOGY AND NOTATION

With small modifications the terminology and notation of J. Kelley [2] is used throughout. For convenience we recall all definitions relating to Moore-Smith convergence.

1.1. A binary relation $\geqq$ *directs* a set A if the set A is non-void, the relation $\geqq$ is transitive and reflexive, and for each m and n in A, there exists an element $p \in A$ such that both $p \geqq m$ and $p \geqq n$. A *directed set* is a pair $(A, \geqq)$ such that the relation $\geqq$ directs the set A. If no confusion is possible then we do not indicate the relation $\geqq$ directing the set A. The subset B of a directed set A is said to be *cofinal* in A if and only if for each a in A there exists a $b \in B$ such that $b \geqq a$; B is said to be *residual* in A if the set A-B is not cofinal in A. It is clear that every residual subset is cofinal and every cofinal subset is directed.

1.2. A *net* is a pair $(S, \geqq)$ such that S is a function and $\geqq$ directs the domain of S. As in the case of directed sets we shall sometimes denote the net $(S, \geqq)$ merely by S. If S is a function whose domain contains A and A is directed by $\geqq$, then $\{S_a, a \in A, \geqq\}$ (or merely $\{S_a, A, \geqq\}$ eventually $\{S_a, A\}$) is the net $(S \mid A, \geqq)$ where $S \mid A$ is S restricted to A. A net $\{S_a, A, \geqq\}$ is *in* a set B if and only if $S[A] \subset B$, i. e., $S_a \in B$ for each $a \in A$; it is *eventually in* B if and only if there exists an element a of A such that, if $a' \in A$ and $a' \geqq a$, then $S_{a'} \in B$; the net S is *frequently in* B if and only if for each $a \in A$ there exists an element a_0 of A such that $a_0 \geqq a$ and $S_{a_0} \in B$.

1.3. Let $S = \{S_a, A, \geqq\}$ be a net and let $B = (B, >)$ be a directed set. The net S is said to be *cofinal in* B if and only if the following condition is satisfied:

(i) for each b in B there exists an element a_0 of A such that

$$a \in A\,, \quad a \geqq a_0 \Rightarrow S_a \in B\,,\ S_a > b\,.$$

The net S is said to be *residual in* B if and only if it is cofinal in B and

(ii) for each a_0 in A the set

$$B \cap S\,[\{a;\, a \in A,\, a \geqq a_0\}]$$

is residual in B.

1.4. If $S = \{S_a, A\}$ is a net and if a net $\pi = \{\pi(b), B\}$ is in A and cofinal (residual) in A, then the net $S \odot \pi = \{S_{\pi(b)}, B\}$ is said to be a *subnet* (*residual subnet*, respectively) of the net S.

1.5. Let P be a topological space. A net $x = \{x(a), A\}$ in P *converges* to $x_0 \in P$ (in symbols $\lim x = x_0$) if and only if for each neighborhood U of the point x_0 the set

$$(*) \qquad \{a;\, a \in A\,, \quad x(a) \in U\}$$

is residual in A. A point x_0 is said to be a *cluster point* of the net x, if for each neighborhood U of the point x_0 the set (*) is cofinal in A.[1])

1.6. It is easy to show that if $x_0 = \lim x$, then x_0 is a cluster point of x. If a net x converges to x_0, then every subnet of x converges to x_0. If x_0 is a cluster point of a net x, then there exists a subnet $x \bigcirc \pi$ of the net x such that $\lim x \bigcirc \pi = x_0$. Proofs are contained in [2].

2. THE TOPOLOGICAL CONVERGENCE OF SETS

If S is a set, then $\exp S$ denotes the family of all subsets of the set S. In the present section we assume that $P \neq \Phi$ is a topological space.

2.1. Definition. Let $M = \{M_a, A\}$ be a net in $\exp P$. *The topological upper limit* $\limsup M$ (*lower limit* $\liminf M$) of the net M is the set of all points $x \in P$ satisfying the following condition: The set

$$\{a;\, a \in A\,,\ M_a \cap U \neq \Phi\}$$

is cofinal (residual, respectively) in A for each neighborhood U of the point x. Evidently

$$\limsup M \supset \liminf M\,.$$

[1]) Equivalently, a point x_0 is a cluster point of x if and only if the net x is frequently in every neighborhood of the point x_0. Also $\lim x = x_0$ if and only if the net x is eventually in every neigborhood of the point x_0.

If $\limsup M = \liminf M$, then the net M is said to be *convergent* and the set $\limsup M$ is denoted by $\lim M$ and called *topological limit* of the net M. In this case we say that the net M *converges* to the set $\lim M$.

2.2. Proposition. *Let $M = \{M_a, A\}$ be a net in $\exp P$. A point x_0 belongs to $\limsup M$ if and only if there exists a net $\{\pi(b), B\}$ cofinal in A, and points $x(b) \in M_{\pi(b)}$ such that $x_0 = \lim \{x(b), B\}$. A point x_0 belongs to $\liminf M$ if and only if there exists a net $\{\pi(b), B\}$ residual in A, and points $x(b) \in M_{\pi(b)}$ such that $\lim \{x(b), B\} = x_0$.*

Proof. Sufficiency. Suppose that there exists a net $\{\pi(b), B\}$ cofinal (residual, respectively) in A and points $x(b) \in M_{\pi(b)}$ such that $x_0 = \lim \{x(b), B\}$. If U is a neighborhood of the point x_0, then there exists a $b_0 \in B$ such that

$$b \in B\,,\ b \geqq b_0 \Rightarrow x(b) \in U\,.$$

According to 1.3 the set $\pi[\{b; b \in B, b \geqq b_0\}]$ is cofinal (residual, respectively) in A and consequently the set

$$\{a; a \in A\,,\ M_a \cap U \neq \Phi\}$$

containing the set $\pi[\{b; b \in B, b \geqq b_0\}]$ is cofinal (residual, respectively) in A. It follows that $x_0 \in \limsup M$ ($x_0 \in \liminf M$, respectively).

Necessity. Suppose that $x_0 \in \limsup M$ ($x_0 \in \liminf M$, respectively). Let $\mathfrak{B}$ be a local base at the point x_0. The set $\mathfrak{B}$ is directed by inclusion $\subset$. Let

$$B = \{(a, U);\ a \in A\,,\ U \in \mathfrak{B}\,,\ U \cap M_a \neq \Phi\}\,.$$

Define

$$(a, U) \succ (a_1, U_1) \Leftrightarrow a \geqq a_1, U \subset U_1\,.$$

Evidently the relation $\succ$ directs the set B. Putting

$$\pi((a, U)) = a$$

for $(a, U) \in B$ we can easily show that the net $\{\pi(b), B\}$ is cofinal (residual, respectively) in A. We prove cofinality only; residuality may be proved by similar arguments. The condition (i) of 1.3 is evident. To prove the condition (ii) of 1.3 we choose an arbitrary $b_0 = (a_0, U_0)$ in B and a_1 in A. We have to find $a \in \pi[\{b; b \in B, b \geqq b_0\}]$ such that $a \geqq a_1$. Since $x_0 \in \limsup M$, there is an element a of A such that

$$a \geqq a_0\,,\quad a \geqq a_1\,,\quad U_0 \cap M_a \neq \Phi\,.$$

It follows that $(a, U_0) \in B$, $(a, U_0) \succ b_0$ and $\pi((a, U_0)) = a$. Choose

$$x((a, U)) \in U \cap M_a$$

for $(a, U) \in B$. It is easy to show that the net $\{x(b), B\}$ converges to x_0. The proof is complete.

The following proposition is a consequence of our definitions:

2.3. Proposition. *Let M be a net in* exp P *and let N be a subnet of M. Then*

$$\limsup M \supset \limsup N \supset \liminf N \supset \liminf M .$$

2.4. Corollary. *If* $\lim M = M_0$ *and N is a subnet of the net M, then* $\lim N = M_0$.

2.5. Proposition. *If $M = \{M_a, A\}$ is a net in* exp P, *then the sets* $\limsup M$ *and* $\liminf M$ *are closed.*

Proof. Suppose that a point x belongs to the closure of the set $M_0 = \limsup M$. If U is a neighborhood of the point x, then $U \cap M_0 \neq \Phi$ and U is a neighborhood of some point $y \in U \cap M_0$. According to the definition 2.1 the set $\{a; a \in A, M_a \cap U \neq \Phi\}$ is cofinal in A. Since U was an arbitrary neighborhood of the point x we conclude that x belongs to $\limsup M$. By similar arguments we may prove that $\liminf M$ is closed. Similar arguments prove:

2.6. *Let $\{M_a, A\}$ be a net in* exp P. *Then* $\limsup \{M_a, A\} = \limsup \{\overline{M}_a, A\}$, $\liminf \{M_a, A\} = \liminf \{\overline{M}_a, A\}$.

2.7. Theorem. *If a net M in exp P does not converge to a set F, then there exists a subnet N of M such that no subnet of N converges to F.*

Proof. Suppose that a net $M = \{M_a, A\}$ does not converge to the set F. The net M is either convergent or $\limsup M - \liminf M \neq \Phi$. If the net M converges to some set M_0, then $M_0 \neq F$ and by 2.4 every subnet of M converges to M_0; therefore no subnet of M converges to F. There remains the case $\limsup M - \liminf M \neq \Phi$. Choose x in the set $\limsup M - \liminf M$. According to the definition 2.1 there exists an open neighborhood U of the point x such that the sets

$$A_1 = \{a;\ a \in A\ ,\ M_a \cap U \neq \Phi\}\ ,\ A_2 = \{a\ ;\ a \in A\ ;\ M_a \cap U = \Phi\}$$

are cofinal in A. If $x \in F$, then we put $N = \{M_a, A_2\}$. The point x does not belong to $\limsup N$. By 2.3 the point x belongs to the set $F - \limsup S$ for every subnet S of N. Finally, there remains the case x non $\in F$. Let $\mathfrak{V}$ be a local base at the point x. Let

$$B = \{(a, U);\ \ a \in A,\ \ U \in \mathfrak{V},\ \ U \cap M_a \neq \Phi\}\ .$$

The set B is directed by the relation $\succcurlyeq$ defined in the proof of 2.2. Let $\pi((a, U)) = a$ for $(a, U) \in B$. Put $N = M \circ \pi$. It is easy to show that N is a subnet of M and $x \in \liminf N$. By 2.3 the point x belongs to the set $\liminf S$ for every subnet S of N. It follows that $\liminf S - F \neq \Phi$ for every subnet S of N. The proof is complete.

2.8. Proposition. *Let $\{M_a, A\}$ be a decreasing net in* exp P (i. e., $M_{a_1} \subset M_{a_2}$ *for* $a_1 \geqq a_2$). *Then*

$$\lim M = \bigcap \{\overline{M}_a; a \in A\}\ .$$

Let $\{M_a, A\}$ be an increasing net in exp P. *Then* $\lim M = \overline{\bigcup \{M_a; a \in A\}}$.

Proof. Evidently $P \supset \liminf M \supset \bigcap \{\overline{M}_a; a \in A\} = F$. Suppose that there is a point x in $P - F$. There exist $a_0 \in A$ and a neighborhood U of the point x such that $M_{a_0} \cap U = \Phi$. The net M is decreasing and consequently

$$a \in A, \quad a \geqq a_0 \Rightarrow U \cap N_a = \Phi .$$

It follows that x non $\in \limsup M$. Hence we have $\limsup M \subset F$.

The second assertion may be proved by similar arguments.

2.9. *Let* $M = \{M_a, A\}$ *and* $N = \{N_a, A\}$ *be nets in* $\exp P$. *Let* $M_a \supset N_a$ *for each* a *in* A. *Then* $\limsup M \supset \limsup N$ *and* $\liminf M \supset \liminf N$.

This is an immediate consequence of definition 2.1.

2.10. Theorem. *Let* $\{M_a, A\}$ *be a net in* $\exp P$. *Then*

$$\limsup M = \bigcap_{a_0 \in A} \overline{\bigcup \{M_a; a \in A, a \geqq a_0\}} .$$

Proof. Put $N_{a_0} = \overline{\bigcup \{M_a; a \in A, a \geqq a_0\}}$ for each a_0 in A. According to 2.8 and 2.9 we have

$$\bigcap \{N_a; a \in A\} = \lim \{N_a, A\} \supset \limsup M .$$

Conversely, let $x \in \bigcap \{N_a; a \in A\}$. If U is an open neighborhood of the point x and $a_0 \in A$, then

$$U \cap \bigcup \{M_a; a \in A , a \geqq a_0\} \neq \Phi ,$$

and therefore the set $U \cap M_{a_1}$ is non-void for some $a_1 \geqq a_0$. Hence the set

$$\{a; a \in A, U \cap M_a \neq \Phi\}$$

is cofinal in A. It follows that $x \in \limsup M$.

2.11. *Let* $M = \{M_a, A\}$ *be a net in* $\exp P$. *Then*

$$\liminf M \supset \overline{\bigcup_{a_0 \in A} \bigcap \{\overline{M}_a; a \in A, a \geqq a_0\}}$$

The proof is evident.

2.12. Theorem. *Let* $\mathfrak{A}$ *be the set of all cofinal subsets of a directed set* A. *Let* $\{M_a, A\}$ *be a net in* $\exp P$. *Then*

$$\liminf \{M_a, A\} = \bigcap_{A' \in \mathfrak{A}} \limsup \{M_a, A'\} = \bigcap_{A' \in \mathfrak{A}} \bigcap_{a_0 \in A'} \overline{\bigcup \{M_a, a \in A', a \geqq a_0\}} .$$

Proof. The inclusion $\subset$ is an immediate consequence of 2.3. Suppose x non $\in$ . . $\liminf M$. There exists a neighborhood U of the point x such that the set

$$A_1 = \{a; a \in A, M_a \cap U \neq \Phi\}$$

is not residual in A. It follows that $A_2 = A - A_1$ is cofinal in A and x non $\in$ . . $\limsup \{M_a, A_2\}$.

2.13. *Let* $M = \{M_a, A\}$ *and* $N = \{N_a, A\}$ *be nets in* $\exp P$. *Then*

$$\limsup \{M_a \cup N_a, A\} = \limsup \{M_a, A\} \cup \limsup \{N_a, A\} ,$$
$$\liminf \{M_a \cup N_a, A\} \supset \liminf \{M_a, A\} \cup \liminf \{N_a, A\} .$$

Consequently $\lim \{M_a \cup N_a, A\} = \lim \{M_a, A\} \cup \lim \{N_a, A\}$ *provided that both limits on the right side exist.*

Proof. By 2.9 we have the inclusions $\supset$. Suppose that the point x does not belong to the set $\limsup M \cup \limsup N$. Hence there exist neighborhoods U_1 and U_2 of the point x such that the sets

$$A_1 = \{a;\ a \in A,\ U_1 \cap M_a \neq \Phi\},\ A_2 = \{a;\ a \in A,\ U_2 \cap N_a \neq \Phi\}$$

are not cofinal in A. The set $U_3 = U_1 \cap U_2$ is a neighborhood of the point x and the set

$$A_3 = \{a;\ a \in A,\ (M_a \cup N_a) \cap U_3 \neq \Phi\}$$

is contained in $A_1 \cup A_2$. It follows that A_3 is not cofinal in A, and consequently the point x does not belong to the set $\limsup \{M_a \cup N_a, A\}$. The proof is complete.

2.14. Theorem. *Let the space* P *be regular. Every net in* $\exp P$ *has a convergent subnet.*

Proof. Let $M = \{M_a, A\}$ be a net in $\exp P$. Let $\mathfrak{B}$ be the set of all those open subsets U of P for which the set $\{a;\ a \in A,\ M_a \subset U\}$ is cofinal in A. There exists a maximal multiplicative subfamily $\mathfrak{B}'$ of $\mathfrak{B}$. Put

$$F = \bigcap \{\overline{U};\ U \in \mathfrak{B}'\}\ .$$

We shall prove that some subnet of M converges to F. Put

$$L = \{(a, U);\ \ a \in A,\ \ U \in \mathfrak{B}',\ \ M_a \subset U\}\ .$$

We order the set L in the following manner:

$$(a, U) \succ (a_1, U_1) \Leftrightarrow a \geqq a_1\ ,\ \ U \subset U_1\ .$$

The relation $\succ$ directs the set L. Indeed, if $(a_i, U_i) \in L$, $i = 1, 2$, then there exists an $a' \in A$ such that both $a' \geqq a_1$ and $a' \geqq a_2$. The set $U = U_1 \cap U_2$ belongs to $\mathfrak{B}'$ and consequently, there exists an $a \in A$, $a \geqq a'$, such that $M_a \subset U$. Hence

$$(a, U) \in L,\ \ (a, U) \succ (a_i, U_i)\quad (i = 1, 2)\ .$$

For $(a, U) \in L$ let $\pi((a, U)) = a$. The net π is cofinal in A. Indeed, the set

$$\pi[\{(a, U);\ (a, U) \in L,\ (a, U) \succ (a_0, U_0)\}]$$

contains the cofinal set $\{a;\ a \in A,\ a \geqq a_0,\ M_a \subset U_0\}$. Now we shall prove that

$$(*)\qquad \lim M \circ \pi = F\ .$$

Choose $U_0 \in \mathfrak{B}'$. There exists an $a_0 \in A$ such that $M_{a_0} \subset U_0$. It follows that

$$(a, U) \in L,\ (a, U) \succ (a_0, U_0) \Rightarrow M_a \subset U_0\ .$$

According to 2.8 and 2,9 we have $\limsup M \circ \pi \subset \overline{U}_0$. Therefore

$$\limsup M \circ \pi \subset \bigcap \{\overline{U};\ U \in \mathfrak{B}'\} = F\ .$$

If $\liminf M \circ \pi = P$ then (*) is evident. Choose

$$x \in P - \liminf M \circ \pi .$$

Since the space P is regular, there exists a neighborhood V of the point x such that the set

$$L_1 = \{(a, U);\ (a, U) \in L,\ \overline{V} \cap M_a = \Phi\}$$

is cofinal in L (since $L - L_1$ is not residual for some V). Put $V' = P - \overline{V}$. We shall prove that $V' \in \mathfrak{B}'$. According to the definition of the family $\mathfrak{B}'$ it is sufficient to show that $V' \cap U \in \mathfrak{B}$ for each U in $\mathfrak{B}'$. Suppose the contrary, that for some $U_0 \in \mathfrak{B}'$ the set

$$A_1 = \{a; a \in A,\ M_a \subset U_0 \cap V'\}$$

is not cofinal in A. It follows that the set $A_2 = A - A_1$ is residual in A. Hence there is an element a_0 of A such that

(**) $$a \in A,\ a \geqq a_0 \Rightarrow a \text{ non} \in A_1 .$$

But this contradicts the cofinality of the set L_1 in L. In fact, there is $a_1 \geqq a_0$ such that $(a_1, U_0) \in L$. The set L_1 is cofinal in L and therefore there exists an $(a, U) \in L_1$ such that $(a, U) \succcurlyeq (a_1, U_0)$. In consequence $a \geqq a_0$ and

$$M_a \subset U \cap V' \subset U_0 \cap V' ,$$

i. e. $a \in A_1$. This contradicts (**).

We have proved that $V' \in \mathfrak{B}'$. The set V is a neighborhood of the point x and hence

$$x \text{ non} \in \overline{P - V} \supset \overline{V'} \supset F .$$

In consequence $\liminf M \circ \pi \supset F$. The proof is complete.

2.15. Let $\mathfrak{F}$ be the set of all closed subsets of P. For each $\Phi \subset \mathfrak{F}$ let $\mathbf{C}(\Phi)$ be the set of all sets $F \in \mathfrak{F}$ such that $F = \lim M$ for some net M in Φ. Then

(i) $$\Phi \subset \mathbf{C}(\Phi) ,$$

(ii) if Φ is finite, then $\mathbf{C}(\Phi) = \Phi$,

(iii) $\mathbf{C}(\Phi_1 \cup \Phi_2) = \mathbf{C}(\Phi_1) \cup \mathbf{C}(\Phi_2)$.

The property (i) is trivial. Let $M = \{M_a, A\}$ be a net in Φ and let $\lim M = F$. If Φ is finite, then there exists a set $F_1 \in \Phi$ such that the set

$$A_1 = \{a;\quad a \in A,\quad M_a = F_1\}$$

is cofinal in A. The net $\{M_a, A_1\}$ is a subnet of M and $\lim \{M_a, A_1\} = F_1$. It follows by 2.4 that $F = F_1$. The property (ii) is thus proved. The inclusion $\supset$ in (iii) is evident. Let $M = \{M_a, A\}$ be a net in $\Phi_1 \cup \Phi_2$ and let $F = \lim M$. Put

$$A_1 = \{a; a \in A, M_a \in \Phi_1\}, A_2 = A - A_1 .$$

Either A_1 or A_2 is cofinal in A. It follows that either $\{M_a, A_1\}$ or $\{M_a, A_2\}$ is a subnet of M. Hence $F \in \mathbf{C}(\Phi_1) \cup \mathbf{C}(\Phi_2)$. The proof of properties (i) — (iii) is complete.

The closure operator $\mathbf{C}$ defines a structure less restrictive than a topology in the sence of J. Kelley. In general the axiom

$$\text{(iV)} \qquad \mathbf{C}[\mathbf{C}(\Phi)] = \mathbf{C}(\Phi)$$

might not be fulfilled. The topologies satisfying axioms (i) — (iii) are investigated extensively in the Čech's book [1]. The basic concepts of general topology remain meaningful for topologies satisfying (i) — (iii) only.

3. THE TOPOLOGICAL CONVERGENCE IN COMPACT SPACES

In this section we assume that P is a compact Hausdorff topological space. $\mathfrak{F}$ denotes the set of all closed subsets of the space P.

3.1. Proposition. *Let $M = \{M_a, A\}$ be a net in $\mathfrak{F}$. If an open subset U of P contains the set* lim sup M, *then there exists an element a_0 of A such that*

$$a \in A, \; a \geqq a_0 \Rightarrow M_a \subset U .$$

Proof. Suppose the contrary, that there exists a cofinal subset A_1 of A such that

$$a \in A_1 \Rightarrow M_a - U \neq \Phi .$$

Choose $x(a) \in M - U$ for each a in A_1. P is compact and therefore (see [2], p. 136) there exists a convergent subnet $x \circ \pi$ of the net $x = \{x(a), A_1\}$. The set $P - U$ is closed and the net $x \circ \pi$ is in P U. It follows that

$$\lim x \circ \pi \in P - U .$$

According to 2.2 we have $\lim x \circ \pi \in \limsup M$. But this is impossible, since $\limsup M \subset U$.

2.3. Definition. Suppose that for each d in a set $D \neq \Phi$ there is given a directed set $(A_d, >_d)$. The cartesian product $X\{A_d; d \in D\}$ is the set of all functions f on D such that $f_d\ (= f(d))$ is a member of A_d for each d in D. The product directed set is

$$(X\{A_d; d \in D\}, \geqq)$$

where, if f and g are members of the product, then $f \geqq g$ if and only if $f(d) >_d$. $>_d g(d)$ for each d in D. The product order is $\geqq$. It is easy to verify that the relation $\geqq$ directs this cartesian product.

3.3. Theorem on iterated limits. *Let D be a directed set, and A_d a directed set for each d in D. Let F be the product directed set $X\{A_d; d \in D\}$. Let L be the* *product directed set* $D \times F$. *Let for each* $d \in D$ $\{M^d_a, a \in A_d\}$ *be a net in* $\mathfrak{F}$ *and* $\lim M^d = \lim \{M^d_a, a \in A_d\} = M'_d$. *Let* $\lim M' = \lim \{M'_d, D\} = M$. *Then*

$$\lim \{M^d_{f(d)}, (d, f) \in L\} = M .$$

Proof. I. First we shall prove that

$$(*) \qquad \liminf \{M^d_{f(d)}, (d, f) \in L\} \supset M .$$

Let x be a point in M and U an open neighborhood of the point x. Since $\lim . \{M'_d, D\} = M$, the set

$$D' = \{d; d \in D, \quad U \cap M'_d \neq \Phi\}$$

is residual in D. For each d in D' the set

$$A'_d = \{a; \quad a \in A_d, \quad M^d_a \cap U \neq \Phi\}$$

is residual in A_d, because $\lim \{M^d_a; a \in A_d\} = M'_d$ and $U \cap M'_d \neq \Phi$. Choose $d_0 \in D$ so that

$$d \in D, \quad d \geqq d_0 \Rightarrow d \in D' .$$

For each $d \geqq d_0$ choose $a(d) \in A'_d$ so that

$$a \in A_d, \quad a \geqq a(d) \Rightarrow a \in A'_d .$$

Choose a function $f_0 \in F$ so that $f_0(d) = a(d)$ for $d \in D$, $d \geqq d_0$.

To prove (*) it is sufficient to show that

$$(d, f) \in L, (d, f) \geqq (d_0, f_0) \Rightarrow U \cap M^d_{f(d)} \neq \Phi .$$

But this is evident. In fact, $f(d) \geqq f_0(d) = a(d)$ and by definition of $a(d)$ we have $U \cap M^d_{f(d)} \neq \Phi$.

II. It remains to prove that

$$\limsup \{M^d_{f(d)}; (d, f) \in L\} \subset M .$$

It is sufficient to show that

$$(**) \qquad \limsup \{M^d_{f(d)}, (d, f) \in L\} \subset U$$

for every open set U containing the set M since

$$M = \bigcap \{\overline{U}; U \text{ is open, } M \subset U\} .$$

By 3.1 there exists an element d_0 in D such that

$$d \in D, \quad d \geqq d_0 \Rightarrow M'_d \subset U .$$

According to 3.1 for each element d following d_0 in D there exists an element $a(d)$ in A_d such that

$$a \in A_d, \quad a \geqq a(d) \Rightarrow M^d_a \subset U .$$

Choose an element f_0 in F such that $f_0(d) = a(d)$ for each element d following d_0 in D. By similar arguments as in the first part of the proof if follows that

$$(d, f) \in L, \quad (d, f) \geqq (d_0, f_0) \Rightarrow M^d_{f(d)} \subset U .$$

By 2.9 we have (**). The proof is complete.

3.4. *The closure operator* **C** *defined in 2.15 satisfies the condition* (iV) *of 2.15. It follows that* **C** *defines a topology for the set* $\mathfrak{F}$. *This topology will be called the topology induced by the topological convergence. The space* $\mathfrak{F}$ *will be denoted by* 2^P. *This topology agrees with usual topology for* 2^P.

Proof. The condition (iV) follows from 3.3 (vide [2], Chapter II., Theorem 9).

4. APPLICATION

4.1. Definition. A *continuum* is a compact connected Hausdorff space containing at least two points.

4.2. Definition. Let P be a topological space. A continuum $L \subset P$ is said to be a *continuum of convergence in* P if there exists a net $K = \{K_a, A\}$ such that for each a in A the space K_a is a continuum, $\lim K = L$ and $K_a \cap L = \Phi$ for each a in A.

4.3. Proposition. Let P be a compact Hausdorff space. The set C of all connected closed subsets of the space P is closed in 2^P.

Proof. Let $K = \{K_a, A\}$ be a net in C, $\lim K = L$. Suppose the contrary, that set L is not connected. There exist disjoint open sets U_1 and U_2 such that $U_1 \cup U_2 \supset L$, $U_1 \cap L \neq \Phi \neq U_2 \cap L$. According to 3.1 there exists an element a_0 in A such that

$$a \in A, \quad a \geqq a_0 \Rightarrow K_a \subset U_1 \cup U_2 .$$

Since $L \neq \Phi$ we may assume that $K_a \neq \Phi$ for each a in A. The sets K_a are connected and therefore either $K_a \subset U_1$ or $K_a \subset U_2$ for $a \geqq a_0$. But this is impossible. Indeed, the set U_i $(i = 1, 2)$ is a neighborhood of some point x_i in L and consequently there exists an element a_i in A such that

$$a \in A, \ a \geqq a_i \Rightarrow U_i \cap K_a \neq \Phi .$$

Choose an element a in A following a_0, a_1 and a_2. Then $U_i \cap K_a \neq \Phi$ $(i = 1, 2)$ and either $K_a \subset U_1$ or $K_a \subset U_2$. This is a contradiction since the sets U_1 and U_2 are disjoint.

4.4. Theorem. *Let P be a continuum and let N be the set of all points at which the space P is not locally connected.*[2]) *The set N is a union of a family of continua of convergence in P, that is, for each x in N there is a continuum of convergence L in P such that $x \in L \subset N$.*

Proof. Let x by a point in N. There exists a closed neighborhood E of the point x such that the component C of the point x in E is not a neighborhood of the point x. Choose a closed neighborhood F of the point x so that $F \subset \operatorname{Int} E$.

[2]) A space P is said to be locally connected at the point $x \in P$ if and only if the family of all connected neighborhoods of the the point x is a local base at x.

Let $\mathfrak{B}$ be the set of all open neighborhoods B of the point x with $B \subset E$. The set $\mathfrak{B}$ is directed by inclusion. For each B in $\mathfrak{B}$ choose $y(B) \in (B - C)$. The function $y = \{y(B), \mathfrak{B}\}$ is a net in E and

$$(*) \qquad \lim y = x$$

Let $C(B)$ be the component of the point $y(B)$ in E. Since $y(B)$ not $\in C$, we have

$$(**) \qquad C(B) \cap C = \Phi \ (B \in \mathfrak{B}) .$$

Let $D(B)$ be the component of the point $y(B)$ in F (for $B \in \mathfrak{B}$). By theorem 2.14 there exists a convergent subnet $D \circ \pi$ of the net $D = \{D(B), \mathfrak{B}\}$. Put $L = \lim D \circ \pi$. The set L contains the point x, since $y(B) \in D(B)$ and $\lim y = x$. By 4.3 the set L is connected. The set F is closed and $D[\mathfrak{B}] \subset \exp F$ and hence $L \subset F$.[3]) The set L is connected and hence it is contained in some component of the set F. But $x \in L$ and consequently $L \subset C$, since C is the component of the point x in $E \supset F$. P is a continuum and hence the set $D(B) \cap \mathrm{Fr}(F)$ ($\mathrm{Fr}(F)$ denotes the boundary of the set F) is non-void for each B in $\mathfrak{B}$. It follows that $L \cap \mathrm{Fr}(F) \neq \Phi$. The set F is a neighborhood of the point x and hence x non $\in$ non $\in \mathrm{Fr}(F)$. It follows that the set L contains at least two points. In consequence the set L is a continuum. It remains to prove that $L \subset N$. Suppose to the contrary that there exists a point x_0 in $L - N$. The set E is a neighborhood of the point x_0 and consequently the component K of the point x_0 in E is a neighborhood of the point x_0 But $K = C$, since C is a component of the set E and $x_0 \in L \subset C$. This leads to a contradicition. Indeed, the point x_0 belongs to $\lim D \circ \pi$ and hence $C \cap D(B) \neq \Phi$ for each B in $\mathfrak{B}$ following some $B_0 \in \mathfrak{B}$. On the other hand $D(B) \subset C(B)$ and $C(B) \cap C = \Phi$ for each B in $\mathfrak{B}$. The proof is complete.

4.5. Corollary. *If a continuum P contains no continuum of convergence, then P is locally connected.*

4.6. Proposition. *Let K be a compact Hausdorff space. Let $\mathfrak{A}$ be the set of all components of the space K. $\mathfrak{A}$ is an upper continuous decomposition of the space K. The quotient space $K \mid \mathfrak{A}$ is a subset of a Cantor discontinuum. In consequence, the quotient space $K \mid \mathfrak{A}$ is totally disconnected.*

Proof. In compact spaces the concepts of component and quasi-component are identical. Let $\mathfrak{B}$ be a family of both open and closed sets in K such that every component of the space K is the intersection of a sub-family of $\mathfrak{B}$. Let F be the set of all functions from the set $\mathfrak{B}$ to the discrete space whose only elements are 0 and 1, with the product topology. (F is a Cantor space.) Define a continuous map φ from K into F as follows: for $x \in K$, $f = \varphi(x)$ set

$$f(x) = \begin{cases} 1 \text{ if } x \in B , \\ 0 \text{ if } x \in K - B . \end{cases}$$

[3]) Vide 2.8 and 2.9.

Evidently the map φ is closed and hence φ is a quotient map and $K' = \varphi[K]$ is the quotient space. It is easy to show that family of all sets of the form $\varphi^{-1}(y)$ where $y \in K'$ is the family $\mathfrak{A}$ of all components of the space K. Hence $\mathfrak{A}$ is an upper continuous decomposition. The space F is totally disconnected and hence the space $K' \subset F$ is totally disconnected. The proof is complete.

4.7. Theorem. *Let K be a continuum and let N be the set of all points at which the space K is not locally connected. Let $\mathfrak{A}$ be the decomposition of the space K consisting of the points $x \in (K - \overline{N})$ and of the components of the set $\overline{N}$. Then the quotient space $K \mid \mathfrak{A}$ is a connected and locally connected compact Hausdorff space.*

Proof. Let f be the quotient map from K onto $K \mid \mathfrak{A}$. It is clear that f is closed and the partial map $f \mid K - \overline{N}$ is a homeomorphism. Consequently the space $f[K]$ is locally connected at each point belonging to $f[K - \overline{N}]$. By 4.6 the space $f[\overline{N}]$ is totally disconnected. It follows that $f[\overline{N}]$ contains no continuum. According to 4.5, the space $f[K]$ is locally connected. The space $f[K]$ is compact and connected as a continuous image of a continuum. The proof is complete.

Note. Let K be a compact Hausdorff topological space. With some small modifications the theorems of § 39 of [3] hold for 2^K.

References

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Резюме

ТОПОЛОГИЧЕСКАЯ СХОДИМОСТЬ МНОЖЕСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

В настоящей статье определена сходимость ,,сетей" подмножеств любого топологического пространства. Сеть — это отображение, определенное на некотором направленном множестве, и направленное множество — это частично упорядоченное множество, которое со всякими двумя элементами a и b содержит элемент c такой, что $c \geqq a$ и $c \geqq b$. Для всякой сети $M = \{M_a; a \in A\}$ подмножеств топологического пространства P (A обозначает направленное множество индексов) определяются замкнутые

множества $\overline{\lim}\, M$ и $\underline{\lim}\, M$ следующим образом: $x \in P$ принадлежит множеству $\overline{\lim}\, M$ ($\underline{\lim}\, M$), если для всякой окрестности U точки x множество

$$A_1 = \{a;\ a \in A,\ U \cap M_a \neq \Phi\}$$

кофинально (резидуально) в A, т. е., если $a \in A$, то для некоторого $a_1 \in A$, $a_1 \geqq a$ (существует $a_0 \in A$ так, что $a \geqq a_0 \Rightarrow a \in A_1$). Говорим, что M сходится к множеству F, если $\overline{\lim}\, M = \underline{\lim}\, M = F$, и пишем $F = \lim M$.

Естественным образом определяется понятие подсети (1.4). Оказывается, что все теоремы, касающиеся обычной топологической сходимости последовательностей подмножеств топологического пространства имеют место для определенной нами сходимости.

Пусть P — топологическое пространство. 2^P обозначает совокупность всех замкнутых подмножеств пространства P. Для всякого $\Phi \subset 2^P$ определяется замыкание $\overline{\Phi}$ обычным образом, т. е., $M_0 \in \overline{\Phi}$ тогда и только тогда, если некоторая сеть элементов множества Φ сходится к M_0. Оказывается, что

$$\Phi \subset \overline{\Phi}\,, \quad \overline{\Phi_1 \cup \Phi_2} = \overline{\Phi}_1 \cup \overline{\Phi}_2\,, \quad (\overline{\Phi}) = (\emptyset)\,.$$

Если P компактно (т. е., бикомпактно), то также $\overline{\overline{\Phi}} = \overline{\Phi}$. Оказывается, что в этом случае обычная топология для 2^P и определенная нами топология совпадают.

Теорема (2.14). *Для всякой сети подмножеств регулярного пространства существует сходящаяся подсеть, т. е., если P регулярно, то 2^P компактно.*

В последней части применяются предыдущие результаты к некоторым вопросам теории континуумов.