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THE FRATTINI SUBGROUPS OF ABELIAN GROUPS

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In the present paper the form of the Frattini subgroup of an abelian group is described. Further, there is proved that every abelian group is the Frattini subgroup of suitable groups; in the class of all abelian groups with this property there exists a minimal one, unique up to isomorphism (the φ-closure of the given group). The paper concludes with some applications concerning the Φ- and φ-series of an abelian group and the study of generating systems of abelian groups.

1. INTRODUCTION

The *Frattini subgroup* $\Phi(G)$ of a (in general non-abelian) group G is defined as the intersection of all maximal proper subgroups of the group G, if G has maximal subgroups; otherwise one puts $\Phi(G) = G$. We can also characterize the Frattini subroup $\Phi(G)$ in terms of the "non-generators" of G, that is of those elements which can be omitted from any generating system of the group G without loss of the property of being a generating system of the whole group: $\Phi(G)$ is just the subgroup of all such non-generators.

Many papers have been dedicated to the investigation of the Frattini subgroups; the papers of G. A. MILLER [8] and W. GASCHÜTZ [5] also discuss finite abelian groups. The present article extends the results of the latter to arbitrary abelian groups. In § 2 there is described the form of the Frattini subgroup of an abelian group: The Frattini subgroup of an abelian group G is the subgroup of those elements $g \in G$ for which the equations

$$p \, . \, x = g \tag{1,1}$$

are solvable in the group G for any prime p (Theorem 1). In contradistinction to the non-commutative case, the Frattini subgroup of an abelian group is fully invariant (compare *e. g.* [9]). In § 3 there is proved that every abelian group is the Frattini subgroup of some abelian group; in the class of all abelian groups the Frattini subgroups of which are isomorphic to a given abelian group, there exists a minimal one, unique up to isomorphism (Theorem 4, Appendix to

Theorem 4): the so-called $\boldsymbol{\varphi}$-closure of the group G.[1]) Theorem 4,4 in B. H. NEUMANN's paper [9] implies that there exist non-abelian groups which cannot be Frattini subgroups of any group whatsoever; from the results of § 3 it follows that the assumption of non-commutativity cannot be weakened in this theorem (for finite groups this has already been shown by Gaschütz [5]). In the case of abelian groups we obtain from the latter a positive answer to a problem of B. H. Neumann from the end of his paper [9]: If a finite abelian group G is the Frattini subgroup of some group, then there even exists a finite group whose Frattini subgroup is isomorphic to G. The final section of this paper, § 4, is devoted to some applications of previous results.

By a group we shall mean throughout an abelian group written additively. The symbols $G_1 + G_2$ resp. $\sum_{\delta \in \Delta} G_\delta$ will denote the direct sum of groups G_1 and G_2 resp. G_δ $(\delta \in \Delta)$, G/H the quotient group G modulo H and pG the subgroup of a group G of all elements $p \cdot g$ with $g \in G$. For any non-void subset $\mathfrak{M}$ of G, $\{\mathfrak{M}\}$ is used to denote the subgroup of G generated by the elements of $\mathfrak{M}$; thus $\{\mathfrak{G}\} = G$ means that $\mathfrak{G}$ is a generating system of the group G. By the symbols $\mathfrak{A} \cup \mathfrak{B}$, $\mathfrak{A} \cap \mathfrak{B}$ and $\mathfrak{A} \setminus \mathfrak{B}$ we shall denote the set-theoretical union, intersection and difference of sets $\mathfrak{A}$ and $\mathfrak{B}$, respectively. $\mathfrak{A} \subset \mathfrak{B}$ means that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$. The set of all primes shall be denoted by Π, the greatest common divisor of integers m and n by (m, n) and the power of a set $\mathfrak{M}$ by $\mathrm{m}(\mathfrak{M})$.

The concept of the *rank* of a group G as the cardinal number of a maximal linearly independent set of the group G is well-known (see *e. g.* [7]); let us denote it by $\mathrm{r}(G)$. A set $\mathfrak{G} = (g_\delta)_{\delta \in \Delta}$ of non-zero elements of G is called *D-independent* if for any finite subset $(g_{\delta_i})_{i=1,2,\ldots,n}$ of $\mathfrak{G}$ a relation

$$k_1 \cdot g_{\delta_1} + k_2 \cdot g_{\delta_2} + \ldots + k_n \cdot g_{\delta_n} = 0 \text{ with integers } k_1, k_2, \ldots, k_n$$

implies $k_i \cdot g_{\delta_i} = 0$ $(i = 1, 2, \ldots, n)$. By the *D-rank* of a group G we understand the cardinal number of a maximal D-independent set of the group G containing only elements of infinite or prime power orders; we shall denote it by $\mathrm{r}_D(G)$ (for properties of the D-rank of an abelian group thus defined see [2]). Thus, especially, the equality $\mathrm{r}_D(G) = \mathrm{r}(G)$ follows for a torsion-free G.

By an *elementary* group we understand a group in which all non-zero elements are of a prime order. A group G^* is said to be *divisible* if $pG^* = G^*$ holds for any $p \in \Pi$, or alternately, if every equation of the form (1,1) admits a solution in G^* for any $g \in G$ and any $p \in \Pi$ (see [6]). A group having no non-zero divisible subgroup is called *reduced.* Every abelian group can be embedded in a so-called *divisible closure* $\overline{G}$, *i. e.* in a minimal divisible group $\overline{G}$ containing G, which is unique up to isomorphie extensions of the identical isomorphism of the group G (see *e. g.* A. G. KUROŠ [7]). For any non-zero element

[1]) However, the $\boldsymbol{\varphi}$-closure thus defined does not have the property $\boldsymbol{\varphi}(\boldsymbol{\varphi}(G)) = \boldsymbol{\varphi}(G)$ usually demanded in the abstract concept of a closure relation (compare *e. g.* BIRKHOFF [1]).

$g \in \overline{G}$ there exists a positive integer n such that $n \cdot \overline{g} \neq 0$, $n \cdot \overline{g} \in G$ (see *e. g.* Hilfssatz 2 in [3]); thus, especially, $\mathrm{r}_D(G) = \mathrm{r}_D(\overline{G})$. On the other hand, any divisible group G^* containing G each element of which has the mentioned property is obviously a divisible closure of the group G; especially, if $\mathrm{r}_D(G)$ is finite and G^* is a divisible group for which the relations $G \subseteq G^*$ and $\mathrm{r}_D(G) = \mathrm{r}_D(G^*)$ hold, then G^* is a divisible closure of the group G.

Let us, moreover, add that by the *characteristic* $\chi(g)$ of an element $g \in G$ in the group G we shall understand the sequence

$$\chi(g) = (k_1, k_2, \ldots, k_i, \ldots)\,, \tag{1,2}$$

where k_i $(i = 1, 2, \ldots)$ is the maximal non-negative integer n for which the equation

$$p_i^n \,.\, x = g \tag{1,3}$$

is solvable in G, if such an n exists and $k_i = \infty$ otherwise, *i. e.* if the equation (1,3) admits a solution in G for any positive integer n ($p_1 < p_2 < \ldots < p_i < \ldots$ are assumed to be all primes).

2, THE FRATTINI SUBGROUP OF AN ABELIAN GROUP

First of all let us prove several lemmas from which Theorem 1 will immediately follow.

Lemma 1. *Let G be an abelian group; then the relation*

$$\Phi(G) \subseteq pG \tag{2,1}$$

holds for every prime p.

Proof. The relation (2,1) is trivial if $pG = G$. In the contrary case the quotient group $\tilde{G} = G/pG$ is a non-zero elementary group; therefore evidently $\Phi(\tilde{G}) = 0$. Thus there exist maximal subgroups $\tilde{M}_\delta$ of the group $\tilde{G}$ $(\delta \in \Delta)$ such that

$$\bigcap_{\delta \in \Delta} \tilde{M}_\delta = 0\,. \tag{2,2}$$

The corresponding subgroups M_δ, $pG \subseteq M_\delta \subset G$, are maximal in G and by (2,2) we have $\bigcap_{\delta \in \Delta} M_\delta = pG$. This implies the relation (2,1).

From Lemma 1 one can easily deduce

Lemma 2. *For an abelian group G there holds the relation*

$$\Phi(G) \subseteq \bigcap_{p \in \Pi} pG\,.$$

We proceed to prove

Lemma 3. *For an abelian group G there holds the relation*

$$\Phi(G) \supseteq \bigcap_{p \in \Pi} pG\,.$$

Proof. Suppose

$$g_0 \in \bigcap_{p \in \Pi} pG \tag{2,3}$$

and let M be an maximal subgroup of G not containing the element g_0. Hence

$$\{M \cup (g_0)\} = G \tag{2,4}$$

and there exists a prime q such that

$$q \cdot g_0 \in M . \tag{2,5}$$

According to (2,3) we can find an element g^* such that

$$q \cdot g^* = g_0 , \tag{2,6}$$

which by (2,4) can be written in the form

$$g^* = m + k \cdot g_0 , \quad m \in M , \quad 0 < k < q .$$

Thus by (2,6)

$$g_0 = q \cdot m + kq \cdot g_0 ;$$

according to (2,5) we immediately deduce that $g_0 \in M$ which contradicts with our assumption; and the lemma follows.[2])

Lemma 2 and 3 imply

Theorem 1.[3]) *The Frattini subgroup of an abelian group G is of the form*

$$\Phi(G) = \bigcap_{p \in \Pi} pG .$$

An immediate consequence of Theorem 1 is the following corollary which characterises divisible groups:

Corollary 1.[3]) *An abelian group is divisible if and only if $\Phi(G) = G$.*

Remark 1. If we take into account the alternative definition of the Frattini subgroups by means of generating systems mentioned in § 1 we obtain easily the following result of [4]: *A necessary and sufficient condition for a non-zero group G to be divisible is that every generating system $\mathfrak{G}$ of G is strongly reducible* (*i. e.* $\mathfrak{G} \setminus (g)$ is again a generating system of G for any element $g \in \mathfrak{G}$).

The following implications are easisly verified:

$$G = \sum_{\delta \in \Delta} G_\delta \Rightarrow pG = \sum_{\delta \in \Delta} pG_\delta \Rightarrow \bigcap_{p \in \Pi} pG = \sum_{\delta \in \Delta} (\bigcap_{p \in \Pi} pG_\delta) ;$$

obtaining the following result (which is, however, easily proved directly):

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[2]) For the prime q there holds simply $qG \subseteq M$, since the quotient group G/M is of the order q, and therefore $\bigcap_{p \in \Pi} pG \subseteq qG \subseteq M$.

[3]) The paper was already in the press when the author found that the assertions of Theorem 1 and Corollary 1 were mentioned already in the monograph of L. Fuchs, Abelian groups, Budapest 1958.

Theorem 2. *Let* $G = \sum_{\delta \in \Delta} G_\delta$ *be a direct decomposition of a group* G. *Then* $\mathbf{\Phi}(G) = \sum_{\delta \in \Delta} \mathbf{\Phi}(G_\delta)$.

From Theorems 1 and 2 we also obtain

Corollary 2. *If* G *is a* p*-primary group, then* $\mathbf{\Phi}(G) = pG$. *If* G *is a torsion group,* and $G = \sum_{p \in \Pi} G_p$ *its direct decomposition with* p*-primary components* G_p, *then* $\mathbf{\Phi}(G) = \sum_{p \in \Pi} pG_p$ *(some of the direct summands can, of course, be trivial).*

Remark 2. Theorem 1 can also be formulated in the following equivalent forms: a) *The Frattini subgroup* $\mathbf{\Phi}(G)$ *of a group* G *is the subgroup of those elements* $g \in G$ *for which the equation* (1,1) *is solvable in* G *for any* $p \in \Pi$.

b) *The Frattini subgroup* $\mathbf{\Phi}(G)$ *of a group* G *is the subgroup of those elements* $g \in G$ *the characteristics* (1,2) *whose fulfil* $k_i \neq 0$ *for all* $i = 1, 2, \ldots$

Remark 3. From the form of the Frattini subgroup $\mathbf{\Phi}(G)$ of an abelian group G we see readily that $\mathbf{\Phi}(G)$ *is fully invariant in the group* G (in contrary to the non-commutative case, see B. H. Neumann [9]). In general, even for a non-abelian group G the relation

$$\mathbf{\Phi}(G)\,\eta \subseteq \mathbf{\Phi}(G\eta)$$

subsists for every homomorphism η of that group G; equality is not always true (compare W. Gaschütz [5]). An example of the infinite cyclic group $G(\infty) = \{u\}$ and the natural homomorphism of this group with the kernel $\{p^2 . u\} \subset \{u\}$, $p \in \Pi$, shows that equality $\mathbf{\Phi}(G)\,\eta = \mathbf{\Phi}(G\eta)$ is not generally true even if G is abelian. Similarly we can see that the proper inclusion $G \supset K$ does not imply $\mathbf{\Phi}(G) \supset \mathbf{\Phi}(K)$ but only $\mathbf{\Phi}(G) \supseteq \mathbf{\Phi}(K)$. Of course, one can immediately deduce that $\mathbf{\Phi}(G/\mathbf{\Phi}(G)) = 0$ (compare Gaschütz [5]).

3. THE $\boldsymbol{\varphi}$-CLOSURE OF AN ABELIAN GROUP

It is the purpose of this section to prove the main theorem asserting that every abelian group G is the Frattini subgroup of a suitable abelian group; moreover, we shall prove that among all groups whose Frattini subgroups are isomorphic to G exists a minimal group, unique up to isomorphism, the $\boldsymbol{\varphi}$-closure of the group G. First of all we shall prove the following lemmas:

Lemma 4. *Let* G *be an abelian group and* $p \in \Pi$. *Then there exists a group* H_p *which satisfies the following conditions*:

(I) $pH_p = G$.

(II) *For every non-zero element* $h \in H_p$ *there exists a positive integer* n[4]) *such that* $n . h \neq 0$ *and* $n . h \in G$.

[4]) In fact, either $n = p$ or $n = 1$.

Proof. Let us embed G in a divisible closure $\bar{G}$: $G \subseteq \bar{G}$ (see § 1). The set of elements $h \in \bar{G}$ with the property $p \,.\, h \in G$ obviously forms a subgroup of the group $\bar{G}$; let us denote it by H_p. Now, H_p satisfies the conditions (I) (every equation of the form (1,1) with $g \in G$ is solvable in $\bar{G}$) and (II) (indeed this is true for every element of $\bar{G}$).

Lemma 5. *Let $G \subseteq A$ be abelian groups and $\bar{A}$ a divisible closure of A; let $p \in \Pi$. Denote by A_p^* the subgroup of the group $\bar{A}$ consisting of all elements $\bar{a} \in \bar{A}$ such that $p \,.\, \bar{a} \in G$.*

a) *If $pA \supseteq G$, then $A_p^* \subseteq A$.*

b) *If $pA = G$, then $A_p^* = A$.*

Proof. First we shall prove the proposition a). Let a^* be an arbitrary element of the group A_p^*; then, $p \,.\, a^* = g^*, g^* \in G$. According to our assumption there exists an element $a \in A$ such that $p \,.\, a = g^*$. Thus $p \,.\, (a^* - a) = 0$; since $a^* - a \in \bar{A}$, necessarily there exists a positive integer n with

$$n \,.\, (a^* - a) \neq 0 \,, \quad n \,.\, (a^* - a) \in A \,.$$

Hence it follows that $(n, p) = 1$ and consequently $a^* - a \in A$. We obtain $a^* \in A$, *i. e.* $A_p^* \subseteq A$.

Now, the proposition b) follows immediately, since $pA = G$ implies $A \subseteq A_p^*$.

Remark 4. *Let $\bar{G}$ be a divisible closure of a group G. On the basis of Lemma 5 one can easily see that the group H satisfying $H \subseteq \bar{G}$ and $pH = G$ is unique.*

Lemma 6. *Let G and H_p be abelian groups satisfying the equality* (I). *Then the condition* (II) *is equivalent to*

(II$^{\mathrm{I}}$) *There exists no proper subgroup $H'_p \subset H_p$ such that*

$$pH'_p = G \,. \tag{3,1}$$

Proof. (II) $\Rightarrow$ (II$^{\mathrm{I}}$). Assume (II); any divisible closure $\bar{H}_p$ of the group H_p is also a divisible closure of the group G itself. The condition (II$^{\mathrm{I}}$) now follows by Lemma 5 (resp. Remark 4).

(II$^{\mathrm{I}}$) $\Rightarrow$ (II). We shall give an indirect proof. Let $h_0 \neq 0$ be an element of H_p such that $\{h_0\} \cap G = 0$. Embed the group H_p in a divisible closure $\bar{H}_p$:

$$G \subset H_p \subseteq \bar{H}_p \,.$$

Let $\bar{G}$ be a divisible closure of the group G, so that

$$G \subseteq \bar{G} \subset \bar{H}_p \,;$$

clearly, h_0 non $\in \bar{G}$. The set H'_p of those elements $\bar{g} \in \bar{G}$, for which $p \,.\, \bar{g} \in G$, forms a group satisfying obviously the relation (3,1); further, Lemma 5 implies that H'_p is a (proper) subgroup of H_p.

By similar arguments as in the proof of (II$^{\mathrm{I}}$) $\Rightarrow$ (II), Lemma 6, we easily verify

Lemma 7. *Let G and A be abelian groups such that $pA \supseteq G$, $p \in \Pi$. Then there exists a subgroup $H_p \subseteq A$ satisfying* (I) *and* (II).

Theorem 3. *Let G be an abelian group and $p \in \Pi$. Then the following propositions hold*:

a) *Groups satisfying the conditions* (I) *and* (II) *are minimal in the class of groups with* (I). *If A is a group such that $pA = G$, then there exists a subgroup $H_p \subseteq A$ satisfying the conditions* (I) *and* (II), *i. e. $pH'_p \neq G$ already holds for every proper subgroup $H'_p \subset H_p$.*

b) *A group is defined by the conditions* (I) *and* (II) *uniquely up to isomorphism. If H_p and K_p are different groups with the properties* (I) *and* (II) (*where, eventually, instead of H_p write K_p*), *then there exists an isomorphism between H_p and K_p which is an extension of the identical isomorphism of the group G.*

Proof. The proposition a) is an easy consequence of Lemmas 6 and 7.

Thus, we shall only prove the proposition b). Divisible closures $\overline{H}_p$ and $\overline{K}_p$ of the groups H_p and K_p respectively, are also divisible closures of the group G. Now, an isomorphism exists between the groups $\overline{H}_p$ and $\overline{K}_p$ which is an extension of the identical isomorphism of the group G; according to Lemma 5 this isomorphism carries H_p onto K_p, as desired.

Remark 5. If G is a torsion-free group one can readily see that there exists a torsion-free group H_p satisfying (I); condition (I) determines such a group uniquely (up to isomorphism), since (I) implies (II) for torsion-free groups. Furthermore, it is easily shown that if G is finite then the group H_p of Lemma 4 is also finite. In general, there holds the equality $\mathrm{r}_D(H_p) = \mathrm{r}_D(G)$ for groups G and H_p with properties (I) and (II).

Now we can formulate the main theorem.

Theorem 4. *Let G be an abelian group. Then there exists a group H satisfying the conditions* (II) (*with H instead of H_p*) *and*

$$\text{(III)} \qquad \Phi(H) = G.$$

Proof. In fact, let us embed the group G in a divisible closure $\overline{G}$; let H_p be the uniquely defined subgroups of $\overline{G}$ for which

$$pH_p = G \quad (\text{for every } p \in \Pi) .$$

Putting $H = \{H_p\}_{p \in \Pi}$ we have obviously $G \subseteq H \subseteq \overline{G}$. Since $G \subseteq H_p \subseteq H$ for every $p \in \Pi$, we obtain $G \subseteq pH$ for every $p \in \Pi$, i. e.

$$(3,2) \qquad \Phi(H) = \bigcap_{p \in \Pi} pH \supseteq G .$$

Now we shall prove the converse inclusion. In fact, the quotient group H/G is a direct sum of the p-primary (elementary, in fact) groups H_p/G and therefore the relation

$$(3,3) \qquad H_{p_0} \cap \{H_p\}_{p \in \Pi_0} = G , \quad \text{where} \quad \Pi_0 = \Pi \setminus (p_0)$$

holds for every $p_0 \in \Pi$. Every element $h \in H$ can be expressed formally in the form of an infinite sum

$$h = \sum_{q \in \Pi} h_q\,, \quad h_q \in H_q\,, \tag{3,4}$$

where $h_q = 0$ for all but a finite number of $q \in \Pi$. The expression (3,4) is not unique; if

$$h = \sum_{q \in \Pi} h'_q\,, \quad h'_q \in H_q\,,$$

is another expression of the element $h \in H$, then (3,3) implies

$$h'_q - h_q \in G \quad \text{for every} \quad q \in \Pi\,. \tag{3,5}$$

Thus, an element h^* of the group pH can be written in the form

$$h^* = p\,.\,h = \sum_{q \in \Pi} p\,.\,h_q = \sum_{q \in \Pi} h^*_q\,, \quad h^*_q \in H_q \quad \text{and} \quad h^*_p \in G \tag{3,6}$$

for every $p \in \Pi$. The relations (3,5) and (3,6) then immediately show that every element of $\mathbf{\Phi}(H) = \bigcap_{p \in \Pi} pH$ belongs to G; this assertion and the relation (3,2) together imply the desired equality (III). The validity of (II) is, of course, obvious.

Lemma 8. *Let $G \subseteq A$ be abelian groups and $\bar{A}$ a divisible closure of A. Let us denote by A^* the subgroup of the group $\bar{A}$ of those elements $\bar{a} \in \bar{A}$ that $n\,.\,\bar{a} \in G$ for a suitable square-free positive integer n.*

a) *If $\mathbf{\Phi}(A) \supseteq G$, then $A^* \subseteq A$.*

b) *If there exists for each element $a \in A$ a positive integer t such that $t\,.\,a \in G$ and if $\mathbf{\Phi}(A) = G$, then $A^* = A$.*[5]

Proof. First, one can readily see that A^* is indeed a subgroup of the group $\bar{A}$. In order to prove the proposition a) we recall the following relation

$$G \subseteq \mathbf{\Phi}(A) \subseteq pA \subseteq \bar{A} \tag{3,7}$$

which holds by Theorem 1 for every $p \in \Pi$. Thus, in view of Lemma 5, every element $\bar{a} \in \bar{A}$ such that $p\,.\,\bar{a} \in G$ holds for some $p \in \Pi$ belongs to A.

Now, let a^* be an arbitrary element of A^*; consequently, there exists a square-free positive integer n such that $n\,.\,a^* = g$, $g \in G$. Let

$$n = p_1 p_2 \ldots p_m\,,$$

where p_i are different primes. Put

$$r_i = p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_m \quad \text{for} \quad i = 1, 2, \ldots, m\,.$$

It follows that $(r_1, r_2, \ldots, r_m) = 1$, *i. e.* there exist integers $s_1, s_2, \ldots, s_m$ such that $\sum_{i=1}^{m} s_i r_i = 1$. Since, according to the above consideration, $n\,.\,a^* = p_i\,.\,(r_i\,.\,a^*) \in G$ and therefore

$$r_i\,.\,a^* = a_i\,, \quad a_i \in A \quad \text{for} \quad i = 1, 2, \ldots, m\,,$$

[5]) Especially, if $\mathbf{\Phi}(A) = G$ and if A satisfies the condition (II) (with A in place of H_p) then $A^* = A$.

we obtain the equality

$$a^* = \sum_{i=1}^{m} s_i r_i \cdot a^* = \sum_{i=1}^{m} s_i \cdot a_i ;$$

consequently $a^* \in A$, *i. e.* $A^* \subseteq A$.

In order to prove the statement b) it is therefore sufficient to show that $A^* \supseteq A$. Assume the contrary, that this inclusion is not true, *i. e.* that there exists an element $a \in A$ and a positive integer t such that

$$t = q^2 . t_0 \quad \text{for some} \quad q \in \Pi \tag{3,8}$$

and

$$t . a \in G , \quad \text{but} \quad qt_0 . a \text{ non } \in G . \tag{3,9}$$

We shall now prove that the element $qt_0 . a$ belongs to $\mathbf{\Phi}(A)$: First, obviously,

$$qt_0 . a \in qA . \tag{3,10}$$

Let p be an arbitrary prime, $p \neq q$; then there exist integers r, s such that

$$rp + sq = 1 . \tag{3,11}$$

Further, by (3,7) there exists an element $a_p \in A$ for which

$$p . a_p = st . a . \tag{3,12}$$

It follows then for the element $rqt_0 . a + a_p \in A$ by (3,12), (3,8) and (3,11) that

$$p . (rqt_0 . a + a_p) = prqt_0 . a + sq^2t_0 . a = prqt_0 . a + (1 - pr)\, qt_0 . a = qt_0 . a ,$$

and hence

$$qt_0 . a \in pA \quad \text{for every} \quad p \in \Pi , \quad p \neq q . \tag{3,13}$$

The relation (3,13) together with (3,10) imply in view of Theorem 1 that the element $qt_0 . a$ belongs to $\mathbf{\Phi}(A)$, in contradiction to (3,9) (for $G = \mathbf{\Phi}(A)$ by our assumption). This completes the proof of the lemma.

Remark 6. The assumption concerning the existence of the integer t in Lemma 8 b) cannot be omitted (in the related proposition b) of Lemma 5 the existence of such an integer already follows from the assumption $pA = G$): Let $G = R^+$ and $A = R^+ + G(\infty)$, where R^+ is the additive group of all rational numbers and $G(\infty)$ the infinite cyclic group. In fact, $\mathbf{\Phi}(A) = R^+ = G$, but $A^* = R^+ \neq A$.

Remark 7. *Let $\overline{G}$ be a divisible closure of a group G.* In view of Lemma 8 b) it is easy to see that *the group H satisfying $H \subseteq \overline{G}$ and $\mathbf{\Phi}(H) = G$ is unique. The group H is, in fact, precisely the subgroup of those elements $\overline{g} \in \overline{G}$ for which there exist square-free integer n such that $n . \overline{g} \in G$.*

Lemma 9. *Let G and H be abelian groups satisfying the equality* (III). *Then the condition* (II) (*with H instead of H_p*) *is equivalent to the following*

(II^{III}) *There exists no proper subgroup $H' \subset H$ such that*

$$\Phi(H') = G\,. \tag{3,14}$$

Proof. (II) $\Rightarrow$ (II^{III}). This follows immediately on the basis of Lemma 8 (resp. Remark 7) in the same way as the assertion (II) $\Rightarrow$ (II^{I}) of Lemma 6.

(II^{III}) $\Rightarrow$ (II). Let us follow again a similar line as in the proof of Lemma 6. Let $h_0 \in H$ be a non-zero element such that $\{h_0\} \cap G = 0$. Embed the group H in a divisible closure $\overline{H}$ and consider a divisible closure $\overline{G}$ of the group G such that $G \subseteq \overline{G} \subseteq \overline{H}$. Repeating the construction from the proof of Theorem 4 we obtain the group H' satisfying $H' \subseteq \overline{G}$ and $\Phi(H') = G$.[6]) Since h_0 non $\in \overline{G}$, it follows that h_0 non $\in H'$ and thus H' is in view of Lemma 8 a proper subgroup of the group H with property (3,14), as desired.

Similar consideration yield

Lemma 10. *Let G and A be abelian groups such that $\Phi(A) \supseteq G$. Then there exists a subgroup $H \subseteq A$ satisfying* (III) *and* (II) (*with H instead of H_p*).

Now, the assertions of Lemmas 8, 9 and 10 (resp. Remark 7) immediately imply the following appendix to Theorem 4.

Appendix to Theorem 4. a) *Groups satisfying the conditions* (III) *and* (II) *are minimal in the class of groups with* (III). *If A is a group such that $\Phi(A) = G$, then there exists a subgroup $H \subseteq A$ satisfying the conditions* (III) *and* (II) (*with H instead of H_p*), *i. e. $\Phi(H') \neq G$ holds already for each proper subgroup $H' \subset H$.*

b) *A group is defined by the conditions* (III) *and* (II) *uniquely up to isomorphism. If H and K are different groups with the properties* (III) *and* (II) (*with H or K instead of H_p*), *then there exists an isomorphism between H and K which is an extension of the identical isomorphism of the group G.*

The group defined by a given group G in this manner (*unique up to isomorphism*) *is said to be the φ-closure of the group G; we shall denote it by $\varphi(G)$.*[7])

Theorem 4 implies directly

Corollary 3. *The φ-closure of a torsion (resp. torsion-free) group is a torsion (resp. torsion-free) group also. The φ-closure of a finite group is finite. The D-rank of a group G and that of its φ-closure $\varphi(G)$ are equal, $r_D(\varphi(G)) = r_D(G)$. If $r_D(G) < \aleph_0$ and $\Phi(H) = G$ with $r_D(H) = r_D(G)$ is fulfilled for a group H, then $H = \varphi(G)$.*

We shall conclude this paragraph with the following theorem concerning the φ-closures of abelian groups.

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[6]) Alternatively, H' is the subgroup of $\overline{G}$ of all those elements $\overline{g} \in \overline{G}$ that are solutions of equations $n \,.\, x = g$, where $g \in G$ and n is a square-free integer (see Remark 7).

[7]) In contradistinction to the Frattini subgroup $\Phi(G)$ of a group G being an unique subgroup of G, $\varphi(G)$ is determined up to isomorphism only; thus, the equality $H = \varphi(G)$ means precisely that H satisfies the conditions (III) and (II) (with H instead of H_p).

Theorem 5. a) $\boldsymbol{\Phi}(\boldsymbol{\varphi}(G)) = G$.

b) *There holds* $\boldsymbol{\varphi}(\boldsymbol{\Phi}(G)) \subseteq G$ *up to isomorphism.*[8])

c) *If* $H \subseteq G$, *then* $\boldsymbol{\varphi}(H) \subseteq \boldsymbol{\varphi}(G)$ *up to isomorphism.*[8])

d) *If* $G = \sum_{\delta \in \Delta} G_\delta$, *then* $\boldsymbol{\varphi}(G) \cong \sum_{\delta \in \Delta} \boldsymbol{\varphi}(G_\delta)$.

Proof. The statement a) is trivial. On the basis of Lemma 10 and Appendix to Theorem 4 we deduce easily also b) and c).

We shall prove d). Let $\overline{G} = \sum_{\delta \in \Delta} \overline{G}_\delta$ be a divisible closure of the group G, where $\overline{G}_\delta$ are divisible closures of the groups G_δ. Let H_δ be the $\boldsymbol{\varphi}$-closure of the group G_δ for which $H_\delta \subseteq \overline{G}_\delta$, $\delta \in \Delta$. In view of Theorem 2, G is the Frattini subgroup of the group $\sum_{\delta \in \Delta} H_\delta$. Since $\sum_{\delta \in \Delta} H_\delta \subseteq \overline{G}$, according to Appendix to Theorem 4 (resp. Remark 7) we conclude that $\sum_{\delta \in \Delta} H_\delta$ is a $\boldsymbol{\varphi}$-closure of the group G.[9])

4. SOME APPLICATIONS

A) By the *descending Frattini series* ($\boldsymbol{\Phi}$-series) of a group G we shall mean the descending series

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_\alpha \supset \ldots \supset G_\tau ,$$

where $G_\alpha = \boldsymbol{\Phi}(G_{\alpha-1})$ for isolated ordinals α, $G_\alpha = \bigcap_{\beta < \alpha} G_\beta$ for a limit α and τ is the least ordinal number such that $\boldsymbol{\Phi}(G_\tau) = G_\tau$. According to Corollary 1 G_τ is divisible (G_τ is obviously the maximal divisible subgroup of the group G); thus, the $\boldsymbol{\Phi}$-series of G ends at the trivial group if and only if G is reduced. We shall refer to the ordinal τ as the $\boldsymbol{\Phi}$-*length* of the group G. Let us observe that $\boldsymbol{\Phi}(G_\alpha / G_{\alpha+1}) = 0$ for every $\alpha < \tau$. An element $g \in G$ belongs to G_n (for a positive integer n) if and only if $k_i \geqq n$ is fulfilled in its characteristic (1,2) for each index i; further, $g \in G_\omega$[10]) is equivalent to the proposition that $k_i = \infty$ in (1,2) for each i.

Consequently, the $\boldsymbol{\Phi}$-length of a torsion-free group G is $\leqq \omega$; a necessary and sufficient condition for $G_\omega = 0$ is that G is reduced. Especially, for torsion-free groups R of rank 1 (i. e. for non-zero subgroups of the additive group R^+ of all rational numbers) we easily deduce:

α) $\boldsymbol{\Phi}(R) = 0$ *if and only if the type corresponding to the group* (i. e. the class of all characteristics (1,2) differing one from another only for a finite number of

[8]) *I. e.* $\boldsymbol{\varphi}(\boldsymbol{\Phi}(G))$ (resp. $\boldsymbol{\varphi}(H)$) is isomorphic to a subgroup of the given group G (resp. of $\boldsymbol{\varphi}(G)$) and this isomorphism is, moreover, an extension of the identical isomorphism of $\boldsymbol{\Phi}(G)$ (resp. of H).

[9]) Of course, it is possible to verify the statement d) directly by making use of the preceding assertions a), b) and c).

[10]) By ω we denote the first infinite ordinal.

components k_i that are different from ∞) *consists of characteristics* (1,2) *with* $k_i = 0$ *for an infinite number of indices i.*

β) *If* $\mathbf{\Phi}(R_1) \cong \mathbf{\Phi}(R_2) \neq 0$, *then* $R_1 \cong R_2$.

γ) *Let us denote by* $\mathfrak{R}$ *the class of all (non-isomorphic) torsion-free groups of rank* 1 *and by* $\mathfrak{R}_0$ *the subclass of all groups of* $\mathbf{\Phi}$*-length* 1. *Thus, by* α) *and* β) $\mathbf{\Phi}$ *is a correspondence which carries the set* $\mathfrak{R} \cup (0)$ *onto itself; moreover, the correspondence* $\mathbf{\Phi}$ *is one-to-one between* $(\mathfrak{R} \setminus \mathfrak{R}_0) \cup (0)$ *and* $\mathfrak{R} \cup (0)$.

δ) *The* $\mathbf{\Phi}$*-length of a group R is equal to zero if and only if* $R \cong R^+$. *A necessary and sufficient condition for a group R to have a finite non-zero* $\mathbf{\Phi}$*-length is that the type* $\mathfrak{r}$ *corresponding to the group R satisfies the following condition: There exists a positive integer N such that* $k_i \leqq N$ *is true for an infinite number of indices i in each characteristic* (1,2) *belonging to* $\mathfrak{r}$.

ε) *In the contrary case the* $\mathbf{\Phi}$*-length of a group R is equal to* ω:

$$R = R_0 \supset R_1 \supset R_2 \supset \ldots \supset R_n \supset \ldots \supset R_\omega = 0\,.$$

All the groups R_n $(n = 1, 2, \ldots)$ *are isomorphic if and only if the type corresponding to the group R consists of characteristics* (1,2) *with* $k_i = \infty$ *for only a finite number of indices i.*

For p-primary reduced groups, of course, the concepts of $\mathbf{\Phi}$-series and the $\mathbf{\Phi}$-length coincide with the concepts of Ulm's series and the length of the p-primary group (see *e. g.* I. KAPLANSKY [6]).

B) Consider the ascending series

$$G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq \ldots \subseteq G_\omega\,, \tag{4,1}$$

where $G_n = \boldsymbol{\varphi}(G_{n-1})$ and $G_\omega = \bigcup_{n<\omega} G_n$. If (1,2) is the characteristic of an element $g \in G_n$ in the group G_n, then evidently

$$(k_1 + m,\ k_2 + m,\ \ldots,\ k_i + m,\ \ldots)$$

is the characteristic of this element g in the group G_{n+m} (of course, $\infty + m = \infty$). Hence, G_ω is necessarily a divisible group, *i. e.* $\boldsymbol{\varphi}(G_\omega) = G_\omega$. It is easy to see that G_ω is the divisible closure of the group G. We deduce further that the $\boldsymbol{\varphi}$-series (4,1) of a group G is either strictly ascending (and its $\boldsymbol{\varphi}$-length is ω) or G is divisible (and then, clearly, $G = G_n = G_\omega$ for $n = 1, 2, \ldots$).

C) Let us apply this to the study of generating systems of abelian groups. If $\mathfrak{G}$ is an irreducible generating system of a group G (for terminology we refer to [4]), then necessarily $\mathfrak{G} \cap \mathbf{\Phi}(G) = \emptyset$. Moreover, any linear combination of the form

$$k_1 \cdot g_1 + k_2 \cdot g_2 + \ldots + k_n \cdot g_n$$

where $g_i \in \mathfrak{G}$ $(i = 1, 2, \ldots, n)$ and $k_i = \pm 1$ at least for one index i, does not lie in $\mathbf{\Phi}(G)$.

On the basis of the preceding consideration we can easily prove, that *a group* $G = D + A$, *where* D *is a divisible group and* A *a group possessing a generating system* $\mathfrak{A}$ *with the property*

$$\mathrm{m}(\mathfrak{A}) < \mathrm{m}(D) \tag{4,2}$$

has no irreducible generating system. In fact, we see immediately that $\boldsymbol{\Phi}(G) \supseteq D$. If $\mathfrak{G}$ is an arbitrary (infinite) generating system of the group G, then there exists a proper subset $\mathfrak{G}_0 \subset \mathfrak{G}$ such that $\{\mathfrak{G}_0\} \supseteq A$ holds. Let g^* be an element of $\mathfrak{G} \setminus \mathfrak{G}_0$:

$$g^* = d^* + a^* , \quad d^* \in D , \quad a^* \in A .$$

For suitable integers k_i and elements $g_i^{(0)} \in \mathfrak{G}_0$ ($i = 1, 2, \ldots, n$) we thus have the relation

$$a^* = k_1 \,.\, g_1^{(0)} + k_2 \,.\, g_2^{(0)} + \ldots + k_n \,.\, g_n^{(0)} ,$$

i. e.

$$d^* = g^* - k_1 \,.\, g_1^{(0)} - k_2 \,.\, g_2^{(0)} - \ldots - k_n \,.\, g_n^{(0)} ;$$

consequently, according to our preceding consideration, $\mathfrak{G}$ is not irreducible (for the coefficient by g^* equals 1 and $d^* \in \boldsymbol{\Phi}(G)$).

The assumption (4,2) cannot be weakened: In the paper [4] there is shown that the p-primary group $G(p^\infty) + \sum_{i=1}^{\infty} G_i(p)$, where $G(p^\infty)$ is Prüfer's group of the type p^∞ and $G_i(p)$ are cyclic groups of the order p ($i = 1, 2, \ldots$), possesses an irreducible generating system. In a similar manner we can prove that the torsion-free group $W = R^+ + \sum_{i=1}^{\infty} \{u_i\}$, where R^+ is the additive group of all rational numbers and $\sum_{i=1}^{\infty} \{u_i\}$ the free abelian group with the basis $u_1, u_2, \ldots, u_n, \ldots$, has an irreducible generating system: Let

$$r_2, r_3, \ldots, r_n, \ldots, (n + 1) \,.\, r_{n+1} = r_n \quad (n = 2, 3, \ldots)$$

be the familiar generating system and the defining relations of the group R^+; in the given group consider the set

$$\mathfrak{W} = (w_i)_{i=1,2,\ldots} ,$$

where

$$w_{3k-2} = r_{2k} + 3 \,.\, u_{2k-1} ,$$
$$w_{3k-1} = 2 \,.\, u_{2k-1} - (2k + 1) \,.\, u_{2k}$$

and

$$w_{3k} = r_{2k+1} + 2 \,.\, u_{2k}$$

for $k = 1, 2, \ldots$ It is easily shown that for $k = 1, 2, \ldots$

$$u_{2k} = - 6k(k + 1)(2k + 3) \,.\, w_{3k+3} - 12k(k + 1) \,.\, w_{3k+2} +$$
$$+ 8k(k + 1) \,.\, w_{3k+1} + (3k + 2) \,.\, w_{3k} + 3 \,.\, w_{3k-1} - 2 \,.\, w_{3k-2} ,$$
$$u_{2k-1} = (2k + 1) \,.\, w_{3k} + 2 \,.\, w_{3k-1} - w_{3k-2}$$

and

$$r_{2k} = w_{3k-2} - 3 \,.\, u_{2k-1} \quad (\text{resp. } r_{2k+1} = w_{3k} - 2 \,.\, u_{2k}) \,.$$

Hence $\mathfrak{W}$ is a generating system of the group W which is, evidently, irreducible (if the element w_{3k-2} resp. w_{3k-1}, or w_{3k} is omitted, then the remaining set generates a proper subgroup of W not containing the element u_{2k-1}, resp. u_{2k}).[11])

Finally, let us add the following

Remark 8. If $\mathfrak{G}$ is a generating system of a group G, then the set $\mathfrak{G} \setminus \mathbf{\Phi}(G)$ does not necessarily generate the whole group G (in the case that G is divisible the set $\mathfrak{G} \setminus \mathbf{\Phi}(G)$ is even void). Hence, it follows that Satz 18*, p. 122 in W. Specht [10] is not true; if the group $G/\mathbf{\Phi}(G)$ is finitely generated, then G need not have the same property (it is enough to consider the subgroup $W_n = R^+ + \sum_{i=1}^{n} \{u_i\}$ of the group W for a natural number n; obviously, $\mathbf{\Phi}(W_n) = R^+$).

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Резюме

ПОДГРУППЫ ФРАТТИНИ АБЕЛЕВЫХ ГРУПП

ВЛАСТИМИЛ ДЛАБ (Vlastimil Dlab), Хартум

Подгруппа Фраттини $\mathbf{\Phi}(G)$ группы G (вообще некоммутативной) определена как пересечение всех максимальных подгрупп этой группы, если G такими подгруппами обладает; иначе $\mathbf{\Phi}(G) = G$. Возможна также дру-

[11]) Thus, a slight modification of our consideration brings the following result: If $G = D + \sum_{i=1}^{\infty}\{u_i\}$, where D is a countable divisible group, then G possesses an irreducible generating system.

гая характеризация подгруппы $\mathbf{\Phi}(G)$ в терминах т. наз. „необразующих", т. е. тех элементов группы G, каждый из которых может быть удален из любой системы образующих группы G без нарушения свойства являться системой образующих всей группы; $\mathbf{\Phi}(G)$ — это именно подгруппа этих необразующих.

После вступительных замечаний исследует автор в § 2 структуру подгруппы Фраттини данной абелевой группы.[1])

Теорема 1.[2]) *Подгруппа Фраттини абелевой группы G имеет форму*

$$\mathbf{\Phi}(G) = \bigcap_{p \in \Pi} pG\,,$$

где Π — *множество всех простых чисел.*

Отсюда вытекает, в частности, дальнейшая характеризация полных абелевых групп (Следствие 1), из которой непосредственно получаем характеризацию при помощи систем образующих, приведенную в [4] (Заметка 1).

Следствие 1.[2]) *Абелева группа G является полной тогда и только тогда, если* $\mathbf{\Phi}(G) = G$.

Для прямой суммы тогда справедлива

Теорема 2. *Пусть* $G = \sum_{\delta \in \Delta} G_\delta$ — *прямое разложение группы G. Тогда*

$$\mathbf{\Phi}(G) = \sum_{\delta \in \Delta} \mathbf{\Phi}(G_\delta)\,.$$

Следствие 2 специфицирует полученные результаты для специальных классов групп и замечание 2 дает эквивалентное выражение теоремы 1 при помощи понятия характеристики элемента группы. Из формы подгруппы Фраттини $\mathbf{\Phi}(G)$ группы G сразу вытекает, что $\mathbf{\Phi}(G)$ — вполне характеристическая в G и что $\mathbf{\Phi}(G/\mathbf{\Phi}(G)) = 0$.

Следующий § 3 посвящен вопросу существования группы, подгруппа Фраттини которой изоморфна данной группе G. Проблема решена при помощи конструкции групп H_p, обладающих свойством $pH_p = G$, где p — простое число (Теорема 3). Автор вводит понятие $\boldsymbol{\varphi}$-замыкания и формулирует решение в теореме 4 и добавлении к этой теореме.

Теорема 4. *Пусть G — абелева группа. Тогда существует группа, выполняющая следующие условия:*

(II) *Для каждого ненулевого элемента $h \in H$ существует натуральное число n такое, что $n \,.\, h \neq 0$ и $n \,.\, h \in G$ и*

(III) $\mathbf{\Phi}(H) = G$.

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[1]) В дальнейшем под группой всегда разумеется абелева группа с аддитивной записью.

[2]) Во время подготовки к печати автор заметил, что утверждения теоремы 1 и следствия 1 находятся уже в монографии *Л. Фукса*, Абелевы группы, Будапешт 1958.

Добавление к теореме 4. a) *Группы, выполняющие условия* (II) *и* (III), *являются минимальными в классе групп со свойством* (III). *Если* A *— группа такая, что* $\Phi(A) = G$, *то существует подгруппа* $H \subseteq A$, *выполняющая условия* (II) *и* (III), *т. е. для каждой собственной подгруппы* $H' \subset H$ *справедливо уже* $\Phi(H') \neq G$.

b) *Условиями* (II) *и* (III) *группа определена однозначно с точностью до изоморфизма. Если* H *и* K *— различные группы со свойствами* (II) *и* (III), *то между* H *и* K *существует изоморфизм, продолжающий тождественный автоморфизм группы* G.

Группу, определенную к данной группе G *таким способом* (*однозначно с точностью до изоморфизма*), *назовем* φ*-замыканием группы и обозначим ее через* $\varphi(G)$.

В следствии 3 показаны некоторые соотношения между D-рангами данной группы и ее φ-замыкания. Следующая теорема показывает некоторые свойства φ-замыкания группы G.

Теорема 5. a) $\Phi(\varphi(G)) = G$.

b) $\varphi(\Phi(G)) \subseteq G$ *справедливо с точностью до изоморфизма.*[3])

c) *Если* $H \subseteq G$, *то* $\varphi(H) \subseteq \varphi(G)$ *с точностью до изоморфизма.*[3])

d) *Если* $G = \sum_{\delta \in \Delta} G_\delta$, *то* $\varphi(G) \cong \sum_{\delta \in \Delta} \varphi(G_\delta)$.

Последний § 4 посвящен применению полученных результатов к изучению убывающих цепей Фраттини (которые в случае p-группы совпадают с ульмовскими цепями), возрастающих цепей Фраттини и к изучению систем образующих данной группы. При помощи понятия подгруппы Фраттини доказывается, напр., следующее утверждение: *Группа* $G = D + A$, *где* D *— полная группа и* A *— группа, обладающая системой образующих со свойством*

$$\text{(4,2)} \qquad \mathrm{m}(\mathfrak{A}) < \mathrm{m}(D)\,,^{4)}$$

не имеет неприводимую систему образующих.[5]) Одновременно показано на примере группы $W = R^+ + \sum_{i=1}^{\infty} \{u_i\}$, где R^+ — аддитивная групп рациональных чисел и $\sum_{i=1}^{\infty} \{u_i\}$ — свободная абелева группа счетного ранга, что предположения этого утверждения нельзя ослабумь.

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[3]) Т. е. $\varphi(\Phi(G))$ (или же $\varphi(H)$) изоморфна подгруппе данной группы G (или же $\varphi(G)$) и этот изоморфизм продолжает, сверх того, тождественный автоморфизм группы $\Phi(G)$ (или же H).

[4]) Символом $\mathrm{m}(\mathfrak{M})$ обозначена мощность множества $\mathfrak{M}$.

[5]) Система образующих группы G называется неприводимой, если всякое собственное подмножество не является уже системой образующих этой группы G (см. [4]).