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BOUNDEDNESS IN UNIFORM SPACES AND TOPOLOGICAL GROUPS

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In this paper the boundedness is defined for sets in arbitrary uniform spaces and some properties of bounded sets in uniform spaces and topological groups are studied. Then some results are aplied to the study of groups of homomorphic mapping of locally bounded groups.

In metric spaces bounded sets are defined in an obvious way; boundedness is, however, not invariant even under uniformly continuous one-to-one mappings (since every metric space can be re-metrized to become bounded without changing its uniform structure). In this paper, a definition of the boundedness is given (see the definition 1.3) for sets in arbitrary uniform spaces which is invariant under uniformly continuous mappings. This definition is, in general, more restrictive than the usual one, but in normed linear spaces both definitions are equivalent. The main results of the first and the second section are the theorems 1.12, 1.13, 1.14, 1.18, 1.19, 2.5, 2.6, 2.8. In the third section groups of homomorphic mappings of locally bounded groups are studied and it is shown that the boundedness can replace some stronger conditions. Nevertheless, it is shown (example 3.8) that the theory of characters of locally bounded groups leads to results different from those well known for locally compact groups.

Now we are going to state some definitions and results we shall need in the sequel.

All definitions and notations for uniform spaces are taken from [1], where all assertions given in this introduction are proved. We recall only some basic definitions and point out where a different terminology is used in this paper or where a misunderstanding could arise due to the same terms used in different sense by different authors.

If P is a set, every subset of $P \times P$ is called a relation on P. If U is a relation on P, then we put $U^{-1} = \{(x, y) \in P \times P;\ (y, x) \in U\}$. If $U = U^{-1}$, we say that the relation U is symmetric. If U, V are relations on P, then $U \circ V$

is the set of all $(x, y) \in P \times P$ such that $(x, z) \in U$, $(z, y) \in V$ for some $z \in P$; U^n is defined, for every natural n, by putting $U^1 = U$, $U^n = U \circ U^{n-1}$. If U is a relation on P, $A \subset P$ a subset, then $U[A]$ is the set of all $x \in P$ such that $(x, y) \in U$ for some $y \in A$. Instead of $(U \circ V)[A]$ we write $U \circ V[A]$. It is easy to prove the following properties of relations:

1. $U \circ (V \circ W) = (U \circ V) \circ W$,
2. $U \circ V[A] = U[V[A]]$,
3. $U[\bigcup_\alpha A_\alpha] = \bigcup_\alpha U[A_\alpha]$,
4. $b \in V[a] \Leftrightarrow a \in V^{-1}[b]$.

A uniformity on a set P is a non-void family $\mathfrak{A}$ of relations on P such that

1. $x \in P$, $U \in \mathfrak{A} \Rightarrow (x, x) \in \mathfrak{A}$,
2. $U \in \mathfrak{A} \Rightarrow U^{-1} \in \mathfrak{A}$,
3. if $U \in \mathfrak{A}$, there exists $V \in \mathfrak{A}$ such that $V \circ V \subset U$,
4. $U \in \mathfrak{A}, V \in \mathfrak{A} \Rightarrow U \cap V \in \mathfrak{A}$,
5. $U \in \mathfrak{A}$, $U \subset V \subset P \times P \Rightarrow V \in \mathfrak{A}$.

The pair $(P, \mathfrak{A})$ is called a uniform space.

Since a uniformity generates a (unique) topology, all notions defined for topological spaces are meaningful for uniform spaces. If $(P, \mathfrak{A})$ is a uniform space, we can consider open or closed relations (in the topology on $P \times P$). Then the family of all open (all closed, all symmetric) relations belonging to a uniformity $\mathfrak{A}$ is a base for $\mathfrak{A}$. If $(P, \mathfrak{A})$ is a uniform space, $R \subset P$, we denote by $\mathfrak{A}_R$ the relative uniformity for R.

Our definition of bounded sets will not be equivalent with the usual definition in metric spaces. To avoid a misunderstanding, we shall say that a set A is bounded for a pseudo-metric ϱ, if ϱ is a bounded function on $A \times A$; if the boundedness is meant in this sense, then the pseudo-metric (a metric) in question will be always mentioned explicitly. We shall need the following theorem proved in [1]:

Theorem A. *Let U_n $(n = 0, 1, 2, \ldots)$ be a sequence of subsets of $P \times P$ such that $U_0 = P \times P$ and for each integer $n \geqq 0$ we have $x \in P \Rightarrow (x, x) \in U_n$, $U^3_{n+1} \subset U_n$. Then there exists a finite non-negative function d on $P \times P$ such that*

(a) $d(x, y) + d(y, z) \geqq d(x, z)$ *for each* $x, y, z \in P$,

(b) $U_n \subset \{(x, y); d(x, y) < 2^{-n}\} \subset U_{n-1}$ *for each natural* n.

If each U_n is symmetric, there exists a pseudo-metric d satisfying the condition (b).

This theorem will be used especially in the case that $(P, \mathfrak{A})$ is a uniform space and $U_n \in \mathfrak{A}$ are symmetric. Then d is a uniformly continuous pseudo-metric.

A corollary of this theorem is the assertion: A uniform space is metrizable if and only if it is Hausdorff and its uniformity has a countable base.

All definitions and notations for topological groups are taken from [5]. Let us remark that in a difference with [5] we say that a set U is a neighbourhood of an element x in a space G if x non $\in \overline{G \dot{-} U}$ (U need not be open), but, as in [5], we suppose that every topological group is a Hausdorff space. We shall consider only commutative groups, therefore we use the additive notation. If we consider only algebraic properties of topological groups, then a homomorphism is called an algebraic homomorphism etc. We say that a mapping is homomorphic if it is algebraically homomorphic and continuous, isomorphic if it is algebraically isomorphic and homeomorphic; a subgroup is an algebraic subgroup which is a closed subset.

Every topological group G is a uniform space with a base of the uniformity consisting of all sets $U' = \{(x, y) \in G \times G;\ x - y \in U\}$ for all neighbourhoods U of zero. It is easy to show that $U' \circ V' = (U + V)'$, $U'[A] = U + A$. Therefore all notions defined for uniform spaces are meaningful for topological groups, and it is clear how the operations on the relations are to be interpreted. Let ϱ be a pseudo-metric on a topological group G. We say that ϱ is an invariant pseudo-metric, if $\varrho(x + z, y + z) = \varrho(x, y)$ for arbitrary elements $x, y, z \in G$. We say that a group G is metrizable, if there exists an invariant metric generating the topology on G. If G is a metrizable (by a metric ϱ) group, H its subgroup, then G/H is metrizable by the metric $\varrho^*(X, Y) = \inf\limits_{\substack{x \in X \\ y \in Y}} \varrho(x, y)$, where $X, Y \in G/H$.

We shall need also the notion of a complete uniform space, but only in some theorems of the third section. We shall use the following theorems proved in [1].

Theorem B. *Let $(P, \mathfrak{A})$ be a uniform space, $(Q, \mathfrak{B})$ a Hausdorff complete uniform space, $A \subset P$; let f be a uniformly continuous mapping of A into Q. Then there exists a unique uniformly continuous mapping $\bar{f}$ of the set $\bar{A}$ into Q such that $\bar{f}_A = f$.*

Theorem C. *A uniform space is compact if and only if it is complete and totally bounded.*

Theorem D. *If G is a topological group, then there exists a complete group G_1 such that G is its algebraic subgroup, G is a subspace of the uniform space G_1 and $\bar{G} = G_1$. The group G_1 is determined uniquely to an isomorphism identical on G.*

1. Bounded sets in uniform spaces

In this section, we define chained and bounded sets in uniform spaces and consider their fundamental properties. In the following theorems, bounded sets are characterised either by means of uniformly continuous pseudo-metrics, metrics and real functions or by their properties in spaces, where some local conditions are fulfilled.

1.1. Definition. Let $(P, \mathfrak{A})$ be a uniform space. Let $U \in \mathfrak{A}$, $A \subset P$. We say that A is *U-chained*, if for each two points $a \in A$, $b \in A$ there exists a finite sequence of points $x_i \in A$, $i = 1, \ldots, n + 1$ such that $x_1 = a$, $x_{n+1} = b$, $(x_i, x_{i+1}) \in U$ for $i = 1, \ldots, n$. We say that A is *chained*, if it is U-chained for every $U \in \mathfrak{A}$.

1.2. Remark. In the above definition it is sufficient to take U from some base of the uniformity $\mathfrak{A}$ only, and to consider the subspace $(A, \mathfrak{A}_A)$ instead of $(P, \mathfrak{A})$.

It is easy to show that every connected uniform space is chained and every compact chained space is connected. We do not prove these assertions for we shall not need them. It will be quite sufficient for us to know that every topological linear space is chained.

1.3. Definition. Let $(P, \mathfrak{A})$ be a uniform space. We say that a set $A \subset P$ is *bounded* (more precisely: bounded in $(P, \mathfrak{A})$), if for every $U \in \mathfrak{A}$ there exists a finite set $K \subset P$ and a natural number n such that $A \subset U^n[K]$; if $n = 1$ may be put (for every U), then we say that A is totally bounded.

1.4. Remark. A) Clearly, every totally bounded set is bounded. It is well known that every compact set is totally bounded. B) In the definition 1.3, U may be taken from any given base of the uniformity $\mathfrak{A}$ (instead from $\mathfrak{A}$). C) Clearly, $A \subset P$ is totally bounded in $(P, \mathfrak{A})$ if and only if it is totally bounded in $(A, \mathfrak{A}_A)$. On the contrary, boundedness of a set A depends essentially on the space $(P, \mathfrak{A})$; if $(P, \mathfrak{A}) \supset (R, \mathfrak{A}_R)$ are uniform spaces, then $A \subset R$ is bounded in P whenever it is bounded in R, but it is easy to show that the converse does not hold, in general. D) In the sequel, a set is called simply "bounded" if it is bounded in the whole space under consideration.

1.5. Remark. In the definition 1.3 we may require $K \subset A$.

Proof. The case $A = \emptyset$ is clear; suppose that $A \neq \emptyset$. Let $U \in \mathfrak{A}$ be given, let $V \subset \mathfrak{A}$ be symmetric and $V^2 \subset U$. Then there exist $a_1, \ldots, a_n \in P$ and a natural number n such that $A \subset V^n[a_1, \ldots, a_k] = \bigcup_{i=1}^{k} V^n[a_i]$. We may suppose that $V^n[a_i] \cap A \neq \emptyset$ for each i. Choose $b_i \in V^n[a_i] \cap A$. Then

$$V^n[a_i] \subset V^n[(V^{-1})^n[b_i]] = V^n \circ V^n[b_i] = U^n[b_i]\,,$$

hence

$$A \subset V^n[a_1, \ldots, a_k] \subset U^n[b_1, \ldots, b_k]\,.$$

1.6. Remark. If the space P is chained, $a \in P$ is an arbitrary point, we may put $K = (a)$ in the definition of the boundedness.

Proof. Let $U \in \mathfrak{U}$ be given; there exist $a_1, \ldots, a_k \in P$ and a natural number n such that $A \subset U^n[a_1, \ldots, a_k]$. For $i = 1, \ldots, k$ there exist natural numbers r_i such that $a_i \in U^{r_i}[a]$. Put $r = \max_{i=1,\ldots,k} r_i$. Then

$$A \subset U^n[U^r[a]] = U^{n+r}[a] .$$

1.7. Remark. Boundedness of sets in locally convex topological linear spaces has been defined by J. v. Neumann in [4] and A. Kolmogorov in [2]. It is easy to prove the equivalence of Neumann's and Kolmogorov's definitions. By means of the remark 1.6 the equivalence of our definition with Neumann's definition can be proved in this special case.

1.8. Theorem. *A set in a normed linear space is bounded if and only if it is bounded in the usual sense (i. e. as a metric space).*

1.9. Theorem. *Let A, B be (totally) bounded sets in a uniform space $(P, \mathfrak{U})$, $C \subset A$. Then the sets C, $\bar{A}$, $A \cup B$ are also (totally) bounded.*

Proofs of both these theorems are easy and may be left to the reader. Theorem 1.8 will be used especially for sets of real numbers.

1.10. Theorem. *Let $(P, \mathfrak{U}_1)$, $(P_2, \mathfrak{U}_2)$ be uniform spaces, let f be a uniformly continuous mapping of P_1 into P_2. If $A \subset P_1$ is (totally) bounded, so is $f(A)$.*

Proof. Let $U \in \mathfrak{U}_2$ be given; there exists $V \in \mathfrak{U}_1$ such that $(x, y) \in V \Rightarrow$ $\Rightarrow (f(x), f(y)) \in U$. If $A \subset P_1$ is bounded, there exists a finite set K and a natural number n such that $V^n[K] \supset A$. If $z \in f(A)$, there exists $y \in A$ such that $f(y) = z$. As $y \in V^n[K]$, there exist $y = y_1, \ldots, y_n \in P_1$, $y_{n+1} \in K$ so that $(y_i, y_{i+1}) \in V$ for $i = 1, \ldots, n$. Then $(f(y_i), f(y_{i+1})) \in U$ and $z \in U^n[f(K)]$. Putting $n = 1$ we have the proof for the total boundedness.

1.11. Theorem. *Let a non-void uniform space be a cartesian product of an arbitrary system of uniform spaces: $(P, \mathfrak{U}) = \mathsf{X}_{\alpha \in M} (P_\alpha, \mathfrak{U}_\alpha)$. Then a set $A \subset P$ is (totally) bounded if and only if its projection into every coordinate space P_α is (totally) bounded in P_α.*

Proof. The necessity follows immediately from theorem 1.10, because the projection of P onto P_α is a uniformly continuous mapping. We shall prove the sufficiency. As $A \subset \mathsf{X}_{\alpha \in M} A_\alpha$, it is sufficient to prove the (total) boundedness of $A' = \mathsf{X}_{\alpha \in M} A_\alpha$. Let $U \in \mathfrak{U}$ be given; there exist $\alpha_1, \ldots, \alpha_l \in M$ and $U_{\alpha_i} \in \mathfrak{U}_\alpha$ for $i = 1, \ldots, l$ so that

$$U \supset \{(x, y) \in P \times P ;\quad (x_{\alpha_i}, y_{\alpha_i}) \in U_{\alpha_i} \quad \text{for} \quad i = 1, \ldots, l\} .$$

To each $i = 1, \ldots, l$, there exists a finite set $K_{\alpha_i} \subset P_{\alpha_i}$ and a natural number m_{α_i} such that

$$U_{\alpha_i}^{m_{\alpha_i}}[K_{\alpha_i}] \supset A_{\alpha_i} .$$

Put $m = \max_{i=1,\ldots,l} m_{\alpha_i}$. For each $\alpha \neq \alpha_i$ ($i = 1, \ldots, l$) let $K_\alpha \subset P_\alpha$ be an arbitrary set containing only one point. The set $\mathsf{X}_{\alpha \in M} K_\alpha = K$ is finite. We shall prove the relation $U^m[K] \supset A'$ for each coordinate α. This is clear, if $\alpha \neq \alpha_i$. For each α_i we have

$$U^m_{\alpha_i}[K_{\alpha_i}] \supset U^{m_{\alpha_i}}_{\alpha_i}[K_{\alpha_i}] \supset A_{\alpha_i}^{\cdot}$$

and the boundedness is proved. If all sets A_α are totally bounded, we can choose K_{α_i} such that $m_{\alpha_i} = 1$ may be put, and therefore $m = 1$.

1.12. Theorem. *Let $(P, \mathfrak{A})$ be a uniform space. A set $A \subset P$ is bounded if and only if it is bounded for every uniformly continuous pseudo-metric defined on P.*

Proof. The necessity follows from theorem 1.8 and 1.10. If we choose $a \in P$ and ψ is a uniformly continuous pseudo-metric, then $\psi(a, x)$ is a uniformly continuous mapping of P into the set of all real numbers.

We shall prove the sufficiency. Let a set $A \subset P$ be given. Let $U \in \mathfrak{A}$ be symmetric. We define a relation on P: $x \sim y$ means that there exists a natural number n such that $x \in U^n[y]$. The relation $\sim$ is reflexive, symmetric and transitive. Therefore it defines a decomposition $P = \bigcup_{\alpha \in M} P_\alpha$. According to theorem A there exists a uniformly continuous pseudo-metric ψ on P such that

$$\{(x, y);\ \psi(x, y) < 1\} \subset U\,.$$

By means of the pseudo-metric ψ we shall construct another pseudometric, defining it first on each set P_α. Each P_α is U-chained. Let $x, y \in P_\alpha$. Let us consider U-chains connecting x with y, i. e. finite sequences $x = x_1, x_2, \ldots, \ldots, x_n, x_{n+1} = y$ such that $(x_i, x_{i+1}) \in U$ for $i = 1, \ldots, n$. The number $\sum_{i=1}^{n} \psi(x_i, x_{i+1})$ will be called the length of this chain. Put $\varphi(x, y) = \inf \sum_{i=1}^{n} \psi(x_i, x_{i+1})$ where the infimum is taken over all U-chains connecting x with y. It is sufficient to take the infimum only over all irreducible chains i. e. such that (x_i, x_{i+j}) non $\in U$ for any $i \geqq 1$ and any $j > 1$. It is easy to prove that φ is a finite non-negative function on $\bigcup_{\alpha \in M} (P_\alpha \times P_\alpha)$ and it is a pseudo-metric on each P_α. It is uniformly continuous, because it is equal to ψ on U.

Now we shall extend the pseudo-metric onto the whole space P. First we define a function μ on the set M. Put

$$M_A = \{\alpha \in M;\ P_\alpha \cap A \neq \emptyset\}\,.$$

If M_A is finite, we put $\mu(\alpha) = 1$ for each $\alpha \in A$. If M_A is infinite, there exists an infinite sequence $\{\alpha_n\}_{n=1}^{\infty}$ where $\alpha_n \in M_A$, $\alpha_m \neq \alpha_n$ if $m \neq n$, and we put $\mu(\alpha_n) = n$, $\mu(\alpha) = 1$ for $\alpha \in M_A$, $\alpha \neq \alpha_n$. Choose $v_\alpha \in P_\alpha$ for each $\alpha \in M$. If $x \in P_\alpha$, $y \in P_\beta$, we put

$$\varrho(x, y) = \varphi(x, y) \quad \text{if} \quad \alpha = \beta\,,$$
$$\varrho(x, y) = \varphi(x, v_\alpha) + \varphi(y, v_\beta) + \mu(\alpha) + \mu(\beta) \quad \text{if} \quad \alpha \neq \beta\,.$$

It is easy to prove that ϱ is a pseudo-metric on P. We shall prove only the triangle inequality. Suppose that $x \in P_\alpha$, $y \in P_\beta$, $z \in P_\gamma$. There are only three possible cases:

1. $\alpha = \beta = \gamma$; then $\varrho = \varphi$ and all is clear,

2. $\alpha \neq \beta \neq \gamma$; then we have

$$\varrho(x, z) \leqq \varphi(x, v_\alpha) + \varphi(z, v_\gamma) + \mu(\alpha) + \mu(\gamma) < \varphi(x, v_\alpha) + \varphi(y, v_\beta) + \\ + \mu(\alpha) + \mu(\beta) + \varphi(y, v_\beta) + \varphi(z, v_\gamma) + \mu(\beta) + \mu(\gamma) = \varrho(x, y) + \varrho(y, z) ,$$

3. $\alpha \neq \beta = \gamma$; then $v_\beta = v_\gamma$ and we have

$$\varphi(v_\beta, y) + \varphi(y, z) \geqq \varphi(v_\beta, z) ,$$
$$\varphi(x, v_\alpha) + \varphi(v_\beta, y) + \mu(\alpha) + \mu(\beta) + \varphi(y, z) \geqq \\ \geqq \varphi(x, v_\alpha) + \varphi(v_\beta, z) + \mu(\alpha) + \mu(\beta) ,$$
$$\varrho(x, y) + \varrho(y, z) \geqq \varrho(x, z) .$$

If $\varrho(x, y) < 1$, then x and y belong to the same class P_α and $\varrho(x, y) = \varphi(x, y)$, therefore ϱ is also uniformly continuous.

Now let A be bounded for the pseudo-metric ϱ. Then M_A must be finite and $\varphi(x, v_\alpha)$ is a bounded function for $x \in A \cap P_\alpha$. Then $a = \max_{\alpha \in M} (\sup_{x \in A \cap P_\alpha} \varphi(x, v_\alpha)) < \infty$. Let $m > a$ be a natural number. Consider an element $\alpha \in M_A$. For each point $x \in A \cap P_\alpha$ there exists an irreducible U-chain connecting x with v_α. Therefore, there exists a sequence $x = x_1, x_2, \ldots, x_{n+1} = v_\alpha$ such that $(x_i, x_{i+1}) \in U$, $\sum_{i=1}^{n} \psi(x_i, x_{i+1}) < m$. As this chain is irreducible, the inequality $\psi(x_i, x_{i+1}) < \frac{1}{2}$ cannot hold for two consecutive indices i. It means that $\psi(x_i, x_{i+1}) < \frac{1}{2}$ holds at most for $\frac{1}{2}(n + 1)$ indices i and therefore $\psi(x_i, x_{i+1}) \geqq \frac{1}{2}$ holds at least for $\frac{1}{2}(n - 1)$ indices. Then the length of the chain is at least $\frac{1}{4}(n - 1)$, which implies $n < 4m + 1$. Therefore $A \subset U^{4m+1}[v_\alpha; \alpha \in M_A]$ and the boundedness of the set A is proved.

1.13. Theorrm. *Let $(P, \mathfrak{A})$ be a metrizable uniform space. A set $A \subset P$ is bounded if and only if it is bounded for every metric generating the same uniformity as $\mathfrak{A}$.*

Proof. The necessity follows from theorem 1.12. The proof of the sufficiency is the same as in the theorem 1.12, if we take ψ as a metric defining the same uniformity as $\mathfrak{A}$; this is possible, since the unifority $\mathfrak{A}$ has a countable base and $(P, \mathfrak{A})$ is a Hausdorff topological space. Then ϱ is a uniformly continuous metric and generates the same uniformity as ψ, because if $\varrho(x, y) < 1$ or $\psi(x, y) < 1$ then $\varrho(x, y) = \psi(x, y)$.

1.14. Theorem. *Let $(P, \mathfrak{A})$ be a uniform space. A set $A \subset P$ is bounded if and only if for every real function f uniformly continuous on P the set $f(A)$ is a bounded set of real numbers.*

Proof. The necessity follows from theorems 1.10, 1.8. If the set $A \subset P$ is not bounded, there exists, according to theorem 1.12, a uniformly continuous pseudo-metric ψ on P, for which A is not bounded. If we choose $a \in P$, then $f(x) = \psi(x, a)$ is a uniformly continuous real function on P and $f(A)$ is not bounded.

1.15. Remark. Let P be a completely regular topological space. The family of all continuous pseudo-metrics generates a uniformity on P (see [1]). The topology generated by this uniformity coincides with the given topology. From theorem 1.14 it follows that $A \subset P$ is bounded in this uniformity if and only if A is relatively pseudo-compact, i. e. every continuous real function on P is bounded on A (see [3]).

1.16. Definition. We say that a uniform space $(P, \mathfrak{U})$ has a property V *uniformly locally*, if there exists $U \in \mathfrak{U}$ such that $U[x]$ has the property V for every $x \in P$.

1.17. Lemma. *Let $(P, \mathfrak{U})$ be a uniform space. let $U \in \mathfrak{U}$ be such that $U \circ U[x]$ is totally bounded for each $x \in P$. Let $A \subset P$ be totally bounded. Then $U[A]$ is also totally bounded.*

Proof. There exist $a_1, \ldots, a_k \in P$ so that $A \subset U[a_1, \ldots, a_k]$. Then

$$U[A] \subset U \circ U[a_1, \ldots, a_k] = U \circ U[a_1] \cup \ldots \cup U \circ U[a_k]$$

and we see that $U[A]$ is a subset of a totally bounded set.

1.18. Theorem. *Let a uniform space $(P, \mathfrak{U})$ be uniformly locally totally bounded. Then every set bounded in P is totally bounded.*

Proof. Let $U \in \mathfrak{U}$ be such that $U[x]$ is totally bounded for each $x \in P$. Choose $V \in \mathfrak{U}$ so that $V^2 \subset U$. Then $V \circ V[x]$ is for each x also totally bounded. According to lemma 1.17 we can prove by induction that $V^n[x]$ is totally bounded for each natural number n. If $A \subset P$ is bounded, there exist points $a_1, \ldots, a_k \in P$ and a natural number m so that $A \subset V^m[a_1, \ldots, a_k]$. Since $V^m[a_1, \ldots, a_k]$ is a union of a finite number of totally bounded sets, A is totally bounded.

1.19. Theorem. *Let a uniform space $(P, \mathfrak{U})$ be uniformly locally compact. Then a subset of P is compact if it is closed and bounded in P. If the space $(P, \mathfrak{U})$ is uniformly locally bounded and every closed and bounded subset of P is compact, then $(P, \mathfrak{U})$ is uniformly locally compact.*

Proof. Let $U \in \mathfrak{U}$ be such that $U[x]$ is compact for each $x \in P$. Let a set $A \subset P$ be bounded and closed. According to theorem 1.18, the set A is totally bounded. There exist points $a_1, \ldots, a_k \in P$ such that $A \subset U[a_1, \ldots, a_k]$. Then A is a closed subset of a compact set, therefore it is compact.

Let $U \in \mathfrak{U}$ be such that $U[x]$ is bounded for each $x \in P$. Choose $V \in \mathfrak{U}$ closed so that $V \subset U$. Then $V[x]$ is bounded, closed and therefore it is compact for each x.

Corollary. A subset of a uniformly locally compact Hausdorff uniform space is compact if and only if it is bounded and closed.

The following theorem shows a case, when the boundedness in a uniform space implies the boundedness in a subspace. We shall need it in the third section.

1.20. Theorem. *Let $(P, \mathfrak{U})$ be a uniform space, let $(R, \mathfrak{U}_R)$ be its subspace, $\bar{R} = P$. If a set $A \subset R$ is bounded in P, it is also bounded in R.*

Proof. Let $U \in \mathfrak{U}_R$ be given. Then exists $V \in \mathfrak{U}$ such that $U = V \cap (R \times R)$. Choose a symmetric $W \in \mathfrak{U}$ such that $W^3 \subset V$. The set A is bounded in P, there exists a finite set $K \subset A$ and a natural number n so that $A \subset W^n[K]$, i. e., for any $x \in A$, there exists $a \in K$ and a sequence $x = x_1, \ldots, x_{n+1} = a$ such that $(x_i, x_{i+1}) \in W$ for $i = 1, \ldots, n$. Choose now for $i = 2, \ldots, n$ points $y_i \in W[x_i] \cap R$; this is possible since $\bar{R} = P$. Put $y_1 = x$, $y_{n+1} = a$. Then $y_i \in W \circ W \circ W^{-1}[y_{i+1}]$ for $i = 1, \ldots, n$, which implies $(y_i, y_{i+1}) \in V$ and since all $y_i \in R$, we have $(y_i, y_{i+1}) \in U$ and $A \subset U^n[K]$. We see that A is bounded in R.

2. Bounded sets in topological groups

The theorems of the first section hold also for bounded sets in topological groups. In this section, however, we shall try to characterize bounded sets by invariant pseudo-metrics or metrics and homomorphic mappings into metrizable topological groups.

2.1. Theorem. *If ϱ is an invariant pseudo-metric on a group G, then the function $r(x) = \varrho(x, 0)$ is*

(1) *finite, non-negative and $r(0) = 0$,*

(2) *even: $r(x) = r(-x)$,*

(3) *subadditive: $r(x + y) \leqq r(x) + r(y)$.*

If ϱ is a metric, then also

(4) $x \neq 0 \Rightarrow r(x) > 0$.

On the other hand, if a function $r(x)$ defined on a group G has properties (1), (2) *and* (3), *then $\varrho(x, y) = r(x - y)$ is an invariant pseudo-metric on the group G, and if also* (4) *holds, then ϱ is an invariant metric.*

If G is a topological group, then ϱ is continuous if and only if r is continuous at the point zero (and then r is continuous on the whole group G).

The proof is easy and may be left to the reader.

2.2. Definition. We say that a function r is a pseudo-norm on a topological group G, if it has the properties (1), (2) and (3) of theorem 2.1. If (4) also holds, we say that the function r is a norm.

We shall often construct a pseudo-norm instead of an invariant pseudo-metric, we shall consider on a group a topology defined by a norm etc.

2.3. Theorem. *Let G be a group, $\{U_n\}_{n=0}^{\infty}$ a sequence of subsets of the group G such that $U_0 = G$, $0 \in U_n$, $3U_{n+1} \subset U_n$ for each integer $n \geqq 0$. Then, there exists a finite, non-negative and subadditive function d on the group G such that*

$$\text{(i)} \qquad U_n \subset \{x; d(x) < 2^{-n}\} \subset U_{n-1}$$

for each natural number n. If each U_n is symmetric, then there exists a pseudo-norm satisfying the condition (i).

Proof (in the way used in [1]). First we define another function f on G. Put $f(x) = 2^{-n}$ for $x \in U_{n-1} \dot{-} U_n$, $f(x) = 0$ for $x \in \bigcap_{n=0}^{\infty} U_n$. Now we put $d(x) = \inf \sum_{i=1}^{m} f(x_i)$, where the infimum is taken over all finite sequences $x_1, \ldots, x_m$ such that $\sum_{i=1}^{m} x_i = x$. The function d is obviously finite, non-negative and subadditive; if each U_n is symmetric, it is a pseudo-norm. From $d(x) \leqq f(x)$ we have $U_n \subset \{x; d(x) < 2^{-n}\}$. To complete the proof we shall show, by induction, that $f(x) \leqq 2 \sum_{i=1}^{m} f(x_i)$ whenever $\sum_{i=1}^{m} x_i = x$. The assertion is clear for $m = 1$. Suppose that $m > 1$ and $\sum_{i=1}^{m} f(x_i) = a$. Let k be the largest integer such that $\sum_{i=1}^{k-1} f(x_i) \leqq \frac{1}{2}a$; then also $\sum_{i=k+1}^{m} f(x_i) \leqq \frac{1}{2}a$ (for $k = m$ this sum is void). By the induction hypothesis $f(\sum_{i=1}^{k-1} x_i) \leqq a$, $f(\sum_{i=k+1}^{m} x_i) \leqq a$ and obviously $f(x_k) \leqq a$. Let n be the smallest integer such that $2^{-n} \leqq a$. We may suppose that $n > 2$, for the case $a \geqq \frac{1}{4}$ is evident. Then $\sum_{i=1}^{k-1} x_i \in U_{n-1}$, $x_k \in U_{n-1}$, $\sum_{i=k+1}^{m} x_i \in U_{n-1}$ and therefore $x \in U_{n-2}$, which implies $f(x) \leqq 2^{-n+1} \leqq 2a$. Now if $d(x) < 2^{-n}$, there exist x_i such that $\sum_{i=1}^{m} x_i = x$ and $\sum_{i=1}^{m} f(x_i) < 2^{-n}$. Hence $f(x) < 2^{-n+1}$, $f(x) \leqq 2^{-n}$ and $x \in U_{n-1}$. The proof is complete.

2.4. Remark. If in theorem 2.3 G is a topological group and all U_n are symmetric neighbourhoods of zero, then the function d is a cotinuous pseudo-norm.

Corollary. A topological group is metrizable if and only if it has a countable complete system of neighbourhoods of zero.

2.5. Theorem. *Let G be a topological group. A set $A \subset G$ is bounded in G if and only if it is bounded for every continuous invariant pseudo-metric defined on G.*

Proof. The necessity is clear. We shall prove the sufficiency by the construction of a suitable continuous pseudonorm. Let a set $A \subset G$ be given. Let U be a symmetric neighbourhood of zero in G. Put $G_0 = \bigcup_{n=1}^{\infty} nU$. Then G_0 is an open subgroup. According to theorem 2.3 and 2.4, there exists such a continuous pseudo-norm d on G that

$$\{x; d(x) < 1\} \subset U .$$

First we shall construct another continuous pseudo-norm on G_0. For every $x \in G_0$ we put $s(x) = \inf \sum_{i=1}^{m} d(x_i)$, where the infimum is taken over all finite sequences $x_1, \ldots, x_m$ such that $x_i \in U$ for each i and $\sum_{i=1}^{m} x_i = x$. It is sufficient to take the infimum only over such sequences that $(x_i + x_j) \in U$ for any $i \neq j$. It is easy to prove that the function d is a pseudo-norm on G_0. As $s(x) = d(x)$ for $x \in U$, it is continuous at zero.

Now we shall extend the pseudo-norm s onto the whole group G. We select a single element from each coset of the subgroup G_0, we arrange them into a transfinite sequence $a_1, a_2, \ldots, a_\alpha, \ldots$ of the type ϑ. We may suppose that there exists $\tau \leqq \vartheta$ such that $(a_\alpha + G_0) \cap A \neq \emptyset$ exactly for all $\alpha < \tau$ and $a_\alpha \in A$ for $\alpha < \tau$. Let G_β be the subgroup generated by the subgroup G_0 and by the elements a_α for $\alpha < \beta$. Obviously, $G_1 = G_0$, $G_\vartheta = G$. On each group G_β we shall define by induction a pseudo-norm r_β so that $r_\beta(x) = r_\alpha(x)$ for arbitrary $\alpha < \beta$ and $x \in G_\alpha$. We shall prove only the subadditivity, other properties will be clear. Put $r_1 = s$. Let $\lambda > 1$ and let pseudo-norms r_μ on G_μ be defined for all $\mu < \lambda$. If λ is a limit number, then $G_\lambda = \bigcup_{\mu<\lambda} G_\mu$ and we put $r_\lambda(x) = r_\mu(x)$ for $x \in G_\mu$; the function r_λ is subadditive, because r_μ is subadditive for all $\mu < \lambda$. Let $\lambda = \mu + 1$. If $a_\mu \in G_\mu$, we put $r_\lambda = r_\mu$, as $G_\lambda = G_\mu$. Suppose that a_μ non $\in G_\mu$. Then every element $y \in G_\lambda$ can be written in the form

$$y = x + pa_\mu \tag{i}$$

where $x \in G_\mu$, p is an integer. Then we have to separate two cases

(1) pa_μ non $\in G_\mu$ for any $p \neq 0$.
Then the expression (i) is unique.

(2) There exists a smallest integer $m > 1$ such that $ma_\mu \in G_\mu$. Then the expression (i) is unique, if also $0 \leqq p < m$ holds. Now we consider two possibilities for λ:

a) $\lambda < \omega$ (i. e. λ is a natural number); we put in the case (1): $r_\lambda(x + pa_\mu) = r_\mu(x) + |p| \, . \, \lambda$. Proof of the subadditivity:

$$\begin{aligned} r_\lambda(x_1 + x_2 + (p_1 + p_2)\, a_\mu) &= r_\mu(x_1 + x_2) + |p_1 + p_2| \, . \, \lambda \leqq \\ &\leqq r_\mu(x_1) + r_\mu(x_2) + |p_1| \, . \, \lambda + |p_2| \, . \, \lambda = \\ &= r_\lambda(x_1 + p_1 a_\mu) + r_\lambda(x_2 + p_2 a_\mu) \, . \end{aligned}$$

in the case (2): $r_\lambda(x + pa_\mu) = r_\mu(x) + \max(\lambda, r_\mu(ma_\mu))$ for $0 < p < m$,

$r_\lambda(x) = r_\mu(x)$.

Proof of the subadditivity:

If $0 < p_1 < m$, $0 < p_2 < m$, then

$$r_\lambda(x_1 + x_2 + (p_1 + p_2)\, a_\mu) \leqq r_\mu(x_1 + x_2) + r_\mu(ma_\mu) + \max(\lambda, r_\mu(ma_\mu)) \leqq \\ \leqq r_\mu(x_1) + r_\mu(x_2) + 2\max(\lambda, r_\mu(ma_\mu)) = r_\lambda(x_1 + p_1a_\mu) + r_\lambda(x_2 + p_2a_\mu) .$$

If $0 < p_1 < m$, $p_2 = 0$, then

$$r_\lambda(x_1 + x_2 + p_1a_\mu) = r_\mu(x_1 + x_2) + \max(\lambda, r_\mu(ma_\mu)) \leqq \\ \leqq r_\mu(x_1) + r_\mu(x_2) + \max(\lambda, r_\mu(ma_\mu)) = r_\lambda(x_1 + p_1a_\mu) + r_\lambda(x_2) .$$

b) $\lambda > \omega$; we put

in the case (1): $r_\lambda(x + pa_\mu) = r_\mu(x) + |p|$,

in the case (2): $r_\lambda(x + pa_\mu) = r_\mu(x) + \max(1, r_\mu(ma_\mu))$ for $0 < p < m$

$r_\lambda(x) = r_\mu(x)$.

The subadditivity can be proved in the same way as for a), but we put 1 instead of λ (everywhere except indices). It is evident that $r_\lambda(x) = r_\mu(x)$ for $x \in G_\mu$. Put $r = r_\vartheta$. Then r is a pseudonorm on the whole group G. As $r(x) = s(x)$ on G_0 and G_0 is an open subgroup, it is also continuous at zero.

Now let the function r be bounded on the set A. We shall show that τ is finite. If not, then $a_\lambda \in A$ for each $\lambda < \omega$. Let l be such a natural number that $r(x) \leqq l$ for $x \in A$. If a_μ non $\in G_l$ for some $\mu < \omega$, then according to the definition of pseudo-norms r_ν for $l < \nu < \omega$ we have $r(a_\mu) > l$. It is not possible and therefore $a_n \in G_l$ for all $n < \omega$. Let $\lambda_1 < \lambda_2 < \ldots$ be the sequence of all natural numbers for which $G_{\lambda_j+1} \neq G_{\lambda_j}$. Then we have $G_{\lambda_j+1} = G_{\lambda_{j+1}}$ and obviously $G_l = G_{\lambda_k+1}$ for some k. From the expression (i) it follows by induction that each element $z \in G_l$ can be written in the form

$$z = y + \sum_{j=1}^{k} p_j a_{\lambda_j} , \tag{ii}$$

where $y \in G_0$ and p_j are integers. If necessary, let us bound the numbers p_j so that the expression (ii) is unique (see the cases (1), (2)). Then

$$r(z) = r(y + \sum_{j=1}^{k-1} p_j a_{\lambda_j}) + r(p_k a_{\lambda_k}) = \ldots = r(y) + \sum_{j=1}^{k} r(p_j a_{\lambda_j}) .$$

If $z \in A$, then $r(z) \leqq l$ and $r(p_j a_{\lambda_j}) \leqq l$ for each j. From the definition of $r_{\mu+1}$ for $\mu = \lambda_j$ in both cases (1), (2) it follows that for each j we have only a finite number of possibilities for the numbers p_j. As $a_n \in A$ for all natural n, then there exist two indices n_1, n_2 such that

$$a_{n_1} = y_1 + \sum_{j=1}^{k} p_j a_{\lambda_j} , \quad a_{n_2} = y_2 + \sum_{j=1}^{k} p_j a_{\lambda_j}$$

where $y_1 \in G_0$, $y_2 \in G_0$ (with the same integers p_j). Then $a_{n_1} - a_{n_2} \in G_0$ and a_{n_1} with a_{n_2} belong to the same coset. This contradiction proves that τ is finite.

The function r is bounded on each set $A \cap (G_0 + a_\lambda)$. Then

$$a = \max_{\lambda < \tau} (\sup_{x \in A \cap (G_0 + a_\lambda)} s(x - a_\lambda)) < \infty .$$

Let $m > a$ be a natural number, $\lambda < \tau$. Then for each $x \in A \cap (G_0 + a_\lambda)$ there exist $x_1, \ldots, x_n \in U$ such that $\sum_{i=1}^{n} x_i = x - a_\lambda$, $\sum_{i=1}^{n} d(x_i) < m$ and $(x_i + x_j)$ non $\in U$ for any $i \neq j$. Therefore there exists at most one indice i such that $d(x_i) < \frac{1}{2}$ and then $m > \sum_{i=1}^{m} d(x_i) \geqq \frac{1}{2}(n - 1)$. It implies $n < 2m + 1$. We have $A \subset (2m + 1) U + \{a_\lambda; \lambda < \tau\}$ and the set A is bounded.

2.6. Theorem. *Let G be a metrizable topological group. A set $A \subset G$ is bounded if and only if it is bounded for every invariant metric defining the same topology on G.*

Proof. The necessity is clear. To prove the sufficiency we construct a suitable continuous norm in the same way as in the proof of theorem 2.5. We can take as U_n in theorem 2.3 a countable complete system of symmetric neighbourhoods of zero and then we get a function d, which is a norm. It is clear that the norm defines a topology which coincides with the given topology. If $\mathrm{d}(x) < 1$ or $r(x) < 1$, then $\mathrm{d}(x) = s(x) = r(x)$ and we see that all topologies are equivalent.

2.7. Lemma. *Let p be a continuous pseudo-norm on a topological group G. Then there exists a metrizable group H and a homomorphic mapping of G onto H such that $r(f(x)) = p(x)$ is a norm on the group H defining the topology on H.*

Proof. Put $G_0 = \{x \in G; p(x) = 0\}$. Obviously G_0 is an algebraic subgroup (even a subgroup). Put $H = G/G_0$. For $X \in H$ let $r(X) = p(x)$, where $x \in X$. The definition is independent on x, for if $y \in X$, then $p(x) \leqq p(y) + p(x - y) = p(y)$ and in the same way we get $p(y) \leqq p(x)$. The function r is a pseudo-norm on H and even a norm, as it is equal to zero only on G_0 and it defines a topology on H. If $x \in G$, we put $f(x) = X$, where X is such a coset of G_0 that $x \in X$. The mapping f has all properties required.

Let us remark that the mapping f constructed in the proof need not be the natural mapping of a group onto its factor group. We may take as G the additive group of real numbers with the discrete topology and put $p(x) = |x|$.

2.8. Theorem. *Let G be a topological group. A set $A \subset G$ is bounded if and only if the following assertion is true: If H is a metrizable group and f a homomorphic mapping of G into H, then $f(A)$ is bounded for the metric of the group H.*

Proof. The necessity is clear, we shall prove the sufficiency. Suppose that A is not bounded. Then there exists a continuous pseudo-norm p such that it is not a bounded function on A. We shall take f and H according to lemma 2.7. Then $f(A)$ is not bounded for the metric of the group H.

2.9. Definition. Let V be a property of subsets of a topological group G, which is invariant under uniform isomorphisms. We say that the group G has *locally* the property V, if there exists a neighbourhood of zero which has the property V.

2.10. Remark. If a group has locally a property V, then it also has this property uniformly locally. Therefore theorems 1.18 and 1.19 hold for topological groups without the word "uniformly". But they can be proved more simply by means of the following theorem.

2.11. Theorem. *Let G be a topological group, let $A \subset G$, $B \subset G$ be (totally) bounded resp. compact. Then $A + B$ is also (totally) bounded resp. compact.*

Proof. It is easy to show that the addition of two elements is a uniformly continuous mapping of $G \times G$ into G, the set $A \times B \subset G \times G$ is (totally) bounded resp. compact and so is its image in G.

Corrolary. Let G be a locally bounded topological group, let $A \subset G$ be a bounded set. Then there exists an open bounded set $V \subset G$ such that $A \subset V$.

3. Homomorphic mappings of locally bounded groups

In this section we shall examine some properties of the group of homomorphic mappings of a group into another. We shall often consider only locally bounded groups (they include locally compact groups as well as normed linear spaces); theorems proved for homomorphic mappings hold also for linear mapping of (real) topological linear spaces, because every homomorphic mapping of a topological linear space is also linear. Unfortunately local boundedness has also some disadvantages. There are examples of normed linear spaces containing a group which is not locally bounded in itself; therefore local boundedness of groups is not hereditary. Nevertheless, every factor group of a locally bounded group is locally bounded. Let us remark that if we replace the supposition of the normability of a linear space by the local boundedness, we do not get for locally convex topological linear spaces any new results since it is well known (cf. [2]) that locally bounded locally convex topological spaces are exactly those, the topology of which is generated by a norm.

Now we shall give the definition and some theorems concerning groups of homomorphic mappings.

3.1. Definition. Let G, H be topological groups. We shall denote by $[G \rightarrow H]$ the group of all homomorphic mappings of G into H, in which the operation of addition is introduced in the natural way and the topology is given so that a complete neighbourhood system of zero in $[G \to H]$ is the family of all sets

$$W(A, V) = \{f \in [G \to H]\,;\quad f(A) \subset V\}$$

where A are arbitrary bounded sets in G and V are arbitrary neighbourhoods of zero in H.

3.2. Remark. It is easy to prove that $[G \to H]$ is really a topological group. It is obviously sufficient to take as A only closed bounded sets. If G is locally compact, then the bounded and closed sets are exactly the compact sets and we may take as A all compact sets, which is the same as in the theory of characters in [5]. If the group G is locally bounded, it is sufficient to take as A bounded open sets or bounded neighbourhoods of zero (according to the corrolary of theorem 2.11).

3.3. Theorem. *Let G be a locally bounded group, let R be a metrizable (by a metric σ) group. Let U be a bounded neighbourhood of zero in G. If $f, g \in [G \to H]$, we put $\varrho_U(f, g) = \sup_{x \in U} \sigma(f(x), g(x))$. Then ϱ is a continuous invariant pseudo-metric on $[G \to R]$. If G is also U-chained, then ϱ_U is a metric defining the same topology as definition* 3.1.

Proof. The number $\varrho_U(f, g)$ is finite, as $f(U)$, $f(V)$ are bounded sets. The other properties of an invariant pseudo-metric are clear. Let us denote for each $a > 0$

$$V_a = \{x \in R;\ \sigma(x, 0) < a\}\,, \quad S_a = \{f \in [G \to R]\,;\quad \varrho_U(f, 0) < a\}\,.$$

The continuity follows from the relation $W(U, V_{\frac{a}{2}}) \subset S_a$ which holds for each $a > 0$.

Let G be U-chained, $f \neq 0$. Then there exists $x \neq 0$ such that $f(x) \neq 0$. Let $\sigma(f(a), 0) = 0$ for all $a \in U$. There exist $a_1, \ldots, a_n \in U$ such that $\sum_{i=1}^{n} a_i = x$ and

$$\sigma(f(x), 0) = \sigma(\sum_{i=1}^{n} f(a_i), 0) \leqq \sum_{i=1}^{n} \sigma(f(a_i), 0) = 0\,.$$

It is a contradiction with $\sigma(f(x), 0) > 0$. Therefore ϱ_U is a metric. Let $W(A, V_\varepsilon)$ be given, where $A \subset G$ is bounded and $\varepsilon > 0$. Then there exists n such that $nU \supset A$ and then

$$W(A, V_\varepsilon) \supset W(nU, V_\varepsilon) \supset W(U, V_{\frac{\varepsilon}{n}}) \supset S_{\frac{\varepsilon}{n}}$$

which proves our theorem.

3.4. Theorem. *Let G, G_1, H be topological groups; suppose that G or G_1 is locally bounded, H is complete, $G \subset G_1$, $\overline{G} = G_1$. Then $[G \to H]$ and $[G_1 \to H]$ are isomorphic.*

Proof. From the fundamental properties of the bounded sets and from theorem 1.20 it follows that both groups G and G_1 are locally bounded at the same time. We shall define a correspondence φ between the groups $[G \to H]$ and $[G_1 \to H]$. If $f \in [G_1 \to H]$, we put $\varphi(f) = f_G \in [G \to H]$. Obviously φ is an algebraic homomorphism and it is one-to-one (as f is continuous and $\overline{G} = G_1$). If $g \in [G \to H]$, then according to theorem B of the introduction the mapping g can be extended uniquely and uniformly continuously onto G_1. Let us denote by f this extended mapping. The set of those $(x, y) \in G_1 \times G_1$, for which $f(x) + f(y) - f(x + y) \neq 0$ is open and if it is not void, it has a non-void intersection with $G \times G$, but it is not possible. Therefore f is a homomorphism of G_1 into H and $\varphi(f) = f_G = g$. We see that φ is an algebraic isomorphism of $[G_1 \to H]$ onto $[G \to H]$.

Now we shall show that φ is a homeomorphic mapping. Let $W(A, V)$ be a neighbourhood of zero in $[G \to H]$. Then $W(A, V)$ is a neighbourhood of zero in $[G_1 \to H]$ as A is also bounded in G_1. On the other hand let $W(A, V)$ be a neighbourhood of zero in $[G_1 \to H]$, where A is bounded and open in G_1. Then according to theorem 1.20 $A \cap G$ is bounded in G. Choose in H a neighbourhood V_1 of zero such that $\overline{V}_1 \subset V$. Then if $f(A \cap G) \subset V_1$, then $f(A) \subset \subset f(\overline{A \cap G}) \subset \overline{V}_1 \subset V$ and therefore $W(A \cap G, V_1) \subset W(A, V)$.

3.5. Theorem. *Let G be a locally bounded, H a complete group. Then $[G \to H]$ is complete.*

Proof. Let $\{f_\alpha\}_{\alpha \in D}$ be a Cauchy net, $f_\alpha \in [G \to H]$. Let $x \in G$; then $\{f_\alpha(x)\}_{\alpha \in D}$ is a Cauchy net in H, therefore it is convergent. We may write $f_\alpha(x) \underset{\alpha}{\to} f(x)$, where f is a mapping of G into H. It is easy to prove that f is an algebraic homomorphism. Let us prove its continuity at zero. Let U be a bounded neighbourhood of zero in the group G, V a neighbourhood of zero in the group H. Let V_1 be such a neighbourhood of zero that $3V_1 \subset V$. The net $\{f_\alpha\}_{\alpha \in D}$ is Cauchy, then there exists $\gamma \in D$ such that $(f_\beta - f_\alpha)(U) \subset V_1$ for each $\beta \geqq \gamma$, $\alpha \geqq \gamma$ and therefore $f_\beta(x) - f_\alpha(x) \in V_1$ for each $x \in U$. Then for each $x \in U$ we have $f_\beta(x) \to f(x)$ and $f(x) - f_\alpha(x) \in \overline{V}_1 \subset 2V_1$. Let $U_1 \subset U$ be such a neighbourhood of zero in G that $f_\gamma(U_1) \subset V_1$. Then

$$f(U_1) \subset (f - f_\gamma)(U_1) + f_\gamma(U_1) \subset 2V_1 + V_1 \subset V .$$

We have proved that f is continuous. In the way of the proof we proved that $f - f_\alpha \in W(U, V)$ for $\alpha \geqq \gamma$ which proves that $f_\alpha \to f$ in the topology of the group $[G \to H]$.

3.6. Remark. It is possible to prove also other theorems as for example: the theorem on the associate homomorphism, theorems on the groups of homomorphic mappings of a factor group and of a direct sum of groups. They can be proved in the same way as in [5], it is sufficient to replace the compactness by the boundedness.

3.7. Remark. It is natural to ask whether the local boundedness of two groups G, H implies the local boundedness of $[G \to H]$. But here we do not get an analogous result. We shall give an example of a locally bounded group, the character group of which is not locally bounded. Let us remark that a character of a group is every homomorphic mapping into the group K, where K is the factor group of the additive group of real numbers by the subgroup of integers. We shall use the following assertion proved in [5]: All characters of the group K are given by $\alpha_k(x) = kx$, where k is an arbitrary integer (for different k different characters).

3.8. Example of locally bounded group, the character group of which is not locally bounded.

Let R be the additive group of real numbers (with the usual metric). For each natural n we denote by K_n the factor group of R by the subgroup generated by the element n. Let $\varkappa_n$ be the natural homomorphism of R onto K_n. The group K_n is metrizable as a factor group of a metrizable group. Let us denote by σ_n the metric on K_n generated by the metric on R. Evidently $K_1 = K$, K_n is isomorphic with K for each n; put $\sigma = \sigma_1$. If n is a natural number, $x \in K_n$, then there exists exactly one $a \in R$ such that $0 \leqq a < n$ and $\varkappa_n(a) = x$; we put $a = \varphi_n(x)$. Let us denote by ψ_n an isomorphism of K_n onto K, defined: if $x \in K_n$, then $\psi_n(x) = \varkappa_1\left(\frac{\varphi_n(x)}{n}\right)$. We know all characters of the group K (see 3.7). Therefore the characters of the group K_n are exactly all $\alpha_k^{(n)}$, defined $\alpha_k^{(n)}(x) = k\psi_n(x)$, where k is an arbitrary integer (for different k different characters). Let us remark that $\sigma(\psi_n,(x), 0) = \frac{1}{n}\,\sigma_n(x, 0)$ for each $x \in K_n$ and $\sigma(ky, 0) = |k| \,.\, \sigma(y, 0)$ for each $y \in K$ and integers k such that $|k| \,.\, \sigma(y, 0) \leqq \leqq \frac{1}{2}$.

Let G be a set of all bounded (in the usual sense) doublesequences $\{a_{m,n}\}_{m,n=1}^{\infty}$, where $a_{m,n} \in K_n$. We define the addition on G: if $a = \{a_{m,n}\}_{m,n=1}^{\infty}$, $b = \{b_{m,n}\}_{m,n=1}^{\infty}$, then $a + b = \{a_{m,n} + b_{m,n}\}_{m,n=1}^{\infty}$. We define an invariant metric on G: $\varrho(a, b) = \sup_{m,n} \sigma_n(a_{m,n}, b_{m,n})$. It is easy to show that G is a topological group, metrizable by the metric ϱ. The group G is chained and locally bounded, because $\{a \in G;\ \varrho(a, 0) < n\varepsilon\} = n\{a \in G;\ \varrho(a, 0) < \varepsilon\}$ for any $\varepsilon > 0$ and natural n.

Let $X = [G \to K]$ be the character group of the group G. Obviously $U = \{a \in G;\ \varrho(a, 0) < \frac{1}{2}\}$ is a bounded neighbourhood of zero in G and according to theorem 3.3

$$\tau(\beta, \gamma) = \sup_{x \in U} \sigma(\beta(x), \gamma(x)) \quad (\beta, \gamma \in X)$$

is an invariant metric, which generates the topology on X. For arbitrary natural numbers m, n let $L_{m,n} \subset G$ be the subgroup of all elements $y = \{y_{p,q}\}_{p,q=1}^{\infty}$, for which $y_{p,q} = 0$ if $(p, q) \neq (m, n)$. Obviously $L_{m,n}$ is isomorphic with K_n for each m, n. If we put $\alpha_k^{(m,n)}(x) = \alpha_k^{(n)}(x_{m,n})$ for $x \in G$, then $\alpha_k^{(m,n)} \in X$ for each natural m, n and each integer k. Then

$$\tau(\alpha_k^{(m,n)}, 0) = \sup_{x \in U} \sigma(\alpha_k^{(m,n)}(x), 0) = \sup_{|z|<\frac{1}{2}} \sigma\left(k\varkappa_1\left(\frac{z}{n}\right), 0\right) = \min\left(\frac{1}{2}, \frac{|k|}{2n}\right).$$

It is clear that $\alpha_1^{(m,n)} \neq \alpha_1^{(m',n')}$ for $(m, n) \neq (m', n')$.

Now suppose that $A = \left\{\beta \in X; \tau(\beta, 0) < \frac{1}{r}\right\}$, where $r > 1$ is a natural number, is bounded. Let $V = \left\{\beta \in X; \tau(\beta, 0) < \frac{1}{2r}\right\}$. Then

$$\alpha_k^{(m,n)} \in A \Leftrightarrow \frac{|k|}{2n} < \frac{1}{r} \Leftrightarrow |k| < \frac{2n}{r}$$

and analogously $\alpha_k^{(m,n)} \in V \Leftrightarrow |k| < \frac{n}{r}$. Obviously $\alpha_k^{(m,r)}$ non $\in V$ for any $k \neq 0$, but $\alpha_1^{(m,r)} \in A$ for all m. We shall show that if $\beta \in V$, then $\beta(L_{m,r}) = (0)$. Suppose that there exists $x \in L_{m,r}$ such that $\beta(x) \neq 0$; β as a character aplied only on $L_{m,r}$ must have a form $\beta(x) = k\,\psi_r(x_{m,r})$, where $k \neq 0$ is an integer. Then

$$\tau(\beta, 0) \geqq \sup_{x \in U \cap L_{m,r}} \sigma(\beta(x), 0) = \tau(\alpha_k^{(m,r)}, 0) \geqq \frac{1}{2r}.$$

This implies β non $\in V$ and we have a contradiction. We have an infinite sequence of characters $\{\alpha_1^{(m,r)}\}_{m=1}^{\infty}$. If we add to any $\alpha_1^{(m,r)}$ an arbitrary finite number of characters from V, we do not get another character from this sequence, for the characters from V are equal to zero on $L_{m,r}$ for all m. Therefore A is not bounded in X and X is not locally bounded.

3.9. Remark. The preceding example shows that the theory of characters of locally bounded groups cannot be developed quite analogously as for locally compact groups. The question arises whether the local boundedness of the character group of a locally bounded group can be deduced from some suitable stronger assumption. If a group G is locally totally bounded, we embed it (see theorem D of the introduction) into a complete group $G_1 = \bar{G}$, which is then locally compact (see theorem C). According to theorem 3.4 the groups $[G \to K]$, $[G_1 \to K]$ are isomorphic and as it is proved in [5], $[G_1 \to K]$ is locally compact.

It is clear if we know that the character group of a locally bounded group is locally totally bounded, then it is locally compact, because it is complete (see theorem 3.5). The local total boundedness is really quite a strong supposition, which does not include all usual cases. It would be interesting to find some weaker condition for the local boundedness of the character group.

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Резюме

ОГРАНИЧЕННОСТЬ В РАВНОМЕРНЫХ ПРОСТРАНСТВАХ И ТОПОЛОГИЧЕСКИХ ГРУППАХ

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Пусть $(P, \mathfrak{A})$ равномерное пространство.[1]) Множество $A \subset P$ назовем ограниченным в $(P, \mathfrak{A})$, если к любому $U \in \mathfrak{A}$ существует конечное множество $K \subset P$ и натуральное число n так, что $A \subset U^n[K]$. Если для каждого $U \in \mathfrak{A}$ можно положить $n = 1$, то множество A назовем вполне ограниченным в $(P, \mathfrak{A})$. Наше определение ограниченного множества в равномерном пространстве ново, в то время как понятие вполне ограниченного множества имеет обычный смысл. Пусть ϱ — псевдометрика на множестве P. Скажем, что $A \subset P$ ограничена по отношению к ϱ, если ϱ ограничена на $A \times A$ в обычном смысле. Понятие ограниченного множества в равномерном пространстве, образованном метрикой ϱ, отличается от понятия ограниченного множества по отношению к метрике ϱ. Однако, если P представляет собой нормированное линейное пространство, то оба понятия ограниченного множества совпадают. Если f равномерно непрерывное отображение равномерного пространства $(P_1, \mathfrak{A}_1)$ в равномерное пространство $(P_2, \mathfrak{A}_2)$ и если $A \subset P_1$ ограничено в $(P_1, \mathfrak{A}_1)$, то $f(A)$ ограничено в $(P_2, \mathfrak{A}_2)$. Множество будет ограниченным в декартовом произведении $\mathbf{X}\,(P_\alpha, \mathfrak{A}_\alpha)$ тогда и только тогда, если его проекция в P_α будет для каждого $\alpha \in M$ ограниченным множеством в $(P_\alpha, \mathfrak{A}_\alpha)$.

Теорема 1.12, 1.13, 1.14. *Пусть $(P, \mathfrak{A})$ — равномерное протсранство, пусть $A \subset P$. Следующие утверждения эквивалентны*:

(1) *A ограничено в $(P, \mathfrak{A})$.*

(2) *A ограничено по отношению к любой равномерно непрерывной псевдометрике.*

(3) *Каждая равномерно непрерывная действиетльная функция на $(P, \mathfrak{A})$ ограничена на A.*

[1]) В смысле определения, введнного в [1].

Мы скажем, что равномерное пространство $(P, \mathfrak{A})$ обладает равномерно локально свойством V, если существует $U \subset \mathfrak{A}$ так, что $U[x]$ имеет свойство V для каждого $x \in P$.

Теорема 1.18. *Пусть пространство $(P, \mathfrak{A})$ равномерно локально вполне ограничено. Тогда любое ограниченное в $(P, \mathfrak{A})$ множество является вполне ограниченным.*

Теорема 1.19. *Пусть $(P, \mathfrak{A})$ равномерно локально компактно. Если $A \subset P$ ограничено и замкнуто, то A компактно. Если $(P, \mathfrak{A})$ — равномерно локально ограбиченное пространство и если каждое его замкнутое ограниченное подмножество компактно, то $(P, \mathfrak{A})$ равномерно локально компактно.*

Подмножество топологической коммутативной группы G назовем ограниченным, если она является ограниченным множеством в равномерной структуре группы G.

Теорема 2.5, 2.6. *Пусть G — коммутативная топологическая группа. Множество $A \subset P$ ограничено в G тогда и только тогда, если оно ограничено по отношению к любой инвариантной псевдометрике группы G. Если G — метризуемая группа, то $A \subset P$ ограничено в G тогда и только тогда, если оно ограничено по отношению к любой инвариантной метрике, определяющей топологию группы G.*

Теорема 2.8. *Пусть G — коммутативная топологическая группа. Множество $A \subset G$ ограничено в G тогда и только тогда, если справедливо следующее утверждение: Если f — гомоморфное отображение G в метризуемую группу H, то $f(A)$ ограничено по отношению к метрике группы G.*

Пусть G и H — топологические коммутативные группы. Пусть $[G \to H]$ — группа всех гомоморфных отображений G в K с топологией, причем полной системой окрестностей нуля являются все множества вида

$$W(A, V) = \{f \in [G \to H]; f(A) \subset V\},$$

где A — ограниченное в G множество, V окрестность нуля в H.

Специальным выбором группы H можно получить группы характеров, так же, как и в [5]. Выводится ряд свойств группы $[G \to H]$ и показано, что локальная ограниченность может заменить некоторые более сильные условия. Пример 3.8 показывает, однако, что теория характеров локально ограниченных групп не приводит к результатам, аналогичным результатам, полученным для локально компактных групп; здесь построена локально ограниченная группа, группа характеров которой не является локально ограниченной.