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# ON THE CLOSED GRAPH THEOREM

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A theorem on closed subspaces of a product $E \times F$ of locally convex spaces is proved which contains as special cases both the closed graph theorem and the open mapping theorem.

In the present remark, which forms a continuation of the author's analysis of the open mapping theorem [1], [2], we present a result concerning closed subspaces of a Cartesian product $E \times F$ of two convex spaces. This result includes both the open mapping and closed graph theorems and elucidates the substance of both by eliminating inessential complications. The formalism of the theory of duality enables us to give a proof which is even simpler than the proofs of both results mentioned. For a discussion of the notions used below, consult [1] and [2].

## 1. Terminology and notation

We use the term "convex space" instead of "locally convex Hausdorff topological vector space over the real field". Let $E$, $F$ be two convex spaces. The points of the Cartesian product $E \times F$ will be denoted $[x, y]$, where $x \in E$, $y \in F$. For every $[x, y] \in E \times F$, we put $P_E[x, y] = x$, $P_F[x, y] = y$. The space $(E \times F)'$ consists of all couples $[x', y']$ where $x' \in E'$, $y' \in F'$, the scalar product being defined by

$$\langle [x, y], [x', y'] \rangle = \langle x, x' \rangle + \langle y, y' \rangle .$$

Let $G$ be a fixed linear subspace of $E \times F$. For every $A \subset E$ let

$$G_F(A) = P_F(P_E^{-1}(A) \cap G) , \quad G(A) = P_E^{-1}(A) \cap G .$$

## 2. A property of closed subspaces of $E \times F$

We intend to prove the following result:

**Theorem.** *Let $E$, $F$ be two convex topological linear spaces, let $G$ be a closed subspace of $E \times F$ such that $P_F(G) = F$. Let $E$ be $B$-complete. Suppose that the* *following condition is fulfilled: For every neighbourhood of zero $U$ in $E$, the closure of $G_F(U)$ is a neighbourhood of zero in $F$. Then $G_F(U)$ itself is a neighbourhood of zero in $F$ for every $U$.*

Proof. Let us denote by $Z$ the annihilator of $G$ in $E' \times F'$. Let us denote by $Q$ the set of all $x' \in E'$ such that $[x', y'] \in Z$ for a suitable $y' \in F'$. Let $X$ be the set of all $x \in E$ such that $[x, 0] \in G$. We intend to show now that $Q^{00} = X^0$. If $x' \in Q$ and $x \in X$, we have $[x, 0] \in G$ and $[x', y'] \in Z$ for some $y'$ whence $\langle x, x' \rangle = \langle [x, 0], [x', y'] \rangle = 0$. It follows that $Q \subset X^0$. On the other hand take an $x \in Q^0$ and let us show that $x \in X$. Suppose not. We have then $[x, 0]$ non $\in G$. The set $G$ being closed, there exists an $[x', y'] \in Z$ such that $\langle [x, 0], [x', y'] \rangle \neq 0$. Since $[x', y'] \in Z$ we have $x' \in Q$ so that $\langle x, x' \rangle = 0$. This is, however, a contradiction since $\langle x, x' \rangle = \langle [x, 0], [x', y'] \rangle \neq 0$. We have thus $Q^0 \subset X$; this, together with $Q \subset X^0$, gives $Q^{00} = X^0$.

Let $U$ be a fixed neighbourhood of zero in $E$ and let us denote by $W(U)$ the set of all $x' \in E'$ such that $[x', y'] \in Z$ for a suitable $y' \in G_F(U)^0$. Clearly $W(U) \subset Q$. Let us show first that, for every $y' \in G_F(U)^0$, there exists an $x' \in E'$ such that $[x', y'] \in Z$. To see that, take an arbitrary $y' \in G_F(U)^0$. Since $G_F(U)^0 \subset G_F(0)^0$, we have $y' \in G_F(0)^0$. If $x \in P_E(G)$, $y_1 \in G_F(x)$ and $y_2 \in G_F(x)$, we have $y_1 - y_2 \in G_F(0)$, whence

$$\langle y_1, y' \rangle = \langle y_2 + (y_1 - y_2), y' \rangle = \langle y_2, y' \rangle .$$

We see thus that, for every $x \in P_E(G)$, the value of $y'$ is constant on the whole of $G_F(x)$. Let us denote it by $f(x)$. We have thus a linear form $f$ defined on $P_E(G)$. Clearly

$$|f(U \cap P_E(G))| = |\langle G_F(U), y' \rangle| \leqq 1$$

so that $f$ is continuous on $P_E(G)$. There exists an $x' \in E'$ such that $\langle x, x' \rangle = f(x)$ for every $x \in P_E(G)$. Clearly $[-x', y'] \in Z$. Let us show further that $W(U)$ is $\sigma(E', E)$-closed in $E'$. Let $n$ be a natural number, $x_1, \ldots, x_n \in P_E(G)$, $\beta_1, \ldots, \beta_n$ arbitrary real numbers and $\varepsilon$ an arbitrary positive number. We denote by $K(x_1, \ldots, x_n; \beta_1, \ldots, \beta_n; \varepsilon)$ the set of all $y' \in G_F(U)^0$ which fulfill the inequalities

$$|\langle G_F(x_i), y' \rangle - \beta_i| \leqq \varepsilon$$

for $i = 1, 2, \ldots, n$. Suppose now that $x'_0 \in E'$ belongs to the $\sigma(E', E)$-closure of $W(U)$. It follows that, for every $x_1, \ldots, x_n \in P_E(G)$ and every $\varepsilon > 0$, the set

$$K(x_1, \ldots, x_n; \langle x_1, x'_0 \rangle, \ldots, \langle x_n, x'_0 \rangle; \varepsilon)$$

is nonvoid. The sets $K$ are $\sigma(E', E)$-closed in $G_F(U)^0$ and possess the finite intersection property. According to our assumption the set $G_F(U)^0$ is $\sigma(E', E)$-compact. It follows that there exists a $y'_0 \in G_F(U)^0$ which belongs to every $K$ so that

$$\langle x, x'_0 \rangle = \langle G_F(x), y'_0 \rangle$$

for every $x \in P_E(G)$. We have thus $x'_0 \in W(U)$ so that $W(U)$ is closed in $E'$.

The inclusions

$$Q \cap U^0 \subset W(U) \subset Q$$

are obvious. We have then $Q \cap U^0 = W(U) \cap U^0$ so that $Q \cap U^0$ is closed in $E'$ for every $U$. The space $E$ being $B$-complete, we have $Q = Q^{00}$ whence $Q = X^0$.

(We may remark here that the assumption of $B_r$-completeness is sufficient if $X = (0)$.)

To complete the proof of our theorem it is sufficient to prove the inclusion $\overline{G_F(U)} \subset 2G_F(U)$ for every $U$. Let $y_0 \in \overline{G_F(U)}$ and suppose that $y_0$ non $\in 2G_F(U)$. There exists an $x_0$ such that $[x_0, y_0] \in G$. It follows that the sets $2U$ and $x_0 + X$ are disjoint. There exists an $x' \in U^0$ such that $\langle x_0 + X, x' \rangle = \beta > 1$. Hence $x' \in X^0$. Since $X^0 = Q$ we have $[x', y'] \in Z$ for some $y' \in F'$. We have thus

$$|\langle G_F(U), y' \rangle| = |\langle U \cap P_E(G), x' \rangle| \leqq 1$$

and $\langle y_0, y' \rangle = -\beta$. This is a contradiction since $y_0 \in \overline{G_F(U)}$. The proof is complete.

**Corollary.** We have seen during the proof of the preceding theorem that the assertion remains valid under the weaker assumption that $E$ is $B_r$-complete if $P_E(G \cap P_F^{-1}(0)) = (0)$.

**Theorem 2.** *Let $E$ be a $B$-complete convex space and let $F$ be a convex space. Let $f$ be a linear mapping of a subspace $E_0 \subset E$ onto $F$, the graph of which is closed in $E \times F$. If $f$ is nearly open, then $f$ is open.*

Proof. This is an immediate consequence of the main theorem. Take $G$ as the graph of $f$; it is then sufficient to note that $G_F(U) = f(U)$ for every $U$.

**Theorem 3.** *Let $F$ be a convex space and $E$ a $B_r$-complete convex space. Let $f$ be a linear mapping of $F$ into $E$ the graph of which is closed in $E \times F$. If $f$ is nearly continuous, then $f$ is continuous.*

Proof. Take $G$ as the graph of $f$; it is then sufficient to note that $G_F(U) = f^{-1}(U)$ for every $U$. Clearly $P_E(G \cap P_F^{-1}(0)) = (f(0)) = (0)$ so that we may use the corollary to the main theorem.

## 3. Some remarks

In the proof of the main theorem, we have used the fact that, if $G \subset E \times F$ is closed, the annihilator of $G$ has a certain property. It is easy to see that this property is characteristic for closed $G$'s. Indeed, we have the following

**Lemma.** *Let $E$ and $F$ be two convex spaces, let $G$ be a subspace of $E \times F$ such that $P_F(G) = F$. Let $X$ be the subspace of those $x \in E$ for which $[x, 0] \in G$. Let $Q$ be the subspace of all $x' \in E'$ such that $[x', y']$ annihilates $G$ for a suitable $y' \in F'$. Then $G$ is closed if and only if $Q^0 \subset X$.*

Proof. Let $Q^0 \subset X$ and suppose that $[x_0, y_0]$ non $\in G$. There is an $x_1 \in E$ such that $[x_1, y_0] \in G$. It follows that $x_0 - x_1$ non $\in X$. It follows that $x_0 - x_1$ non $\in Q^0$ so that there exists an $x' \in Q$ with $\langle x_0 - x_1, x' \rangle \neq 0$. Since $x' \in Q$, there is an $y' \in F'$ such that

$$\langle G, [x', y'] \rangle = 0 .$$

We have

$$\langle [x_0, y_0], [x', y'] \rangle = \langle [x_1, y_0], [x', y'] \rangle + \langle [x_0 - x_1, 0], [x', y'] \rangle = \\ = \langle x_0 - x_1, x' \rangle \neq 0$$

so that $[x_0, y_0]$ does not belong to the closure of $G$.

The other implication is contained in the proof of the main theorem.

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## Резюме

# ТЕОРЕМА О ЗАМКНУТОМ ГРАФИКЕ


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В настоящей заметке автор продолжает свой анализ теоремы об открытом отображении и приводит результат, касающийся замкнутых подпространств декартова произведения двух локально выпуклых линейных пространств. Этот результат содержит как теорему об открытом отображении, так и теорему о замкнутом графике, и поясняет их сущность путем устранения несущественных усложнений формального характера. Доказательство этого результата проще, чем доказательства указанных двух теорем. Использованные в статье понятия и обозначения содержатся в [1] и [2].

Пусть $E$, $F$ — два локально выпуклых линейных пространства. Обозначим элементы декартова произведения $E \times F$ через $[x, y]$, где $x \in E$, $y \in F$. Для любого $[x, y] \in E \times F$ положим $P_E[x, y] = x$ и $P_F[x, y] = y$. Пусть

$G$ — фиксированное линейное подпространство в $E \times F$. Для каждого $A \subset E$ положим

$$G_F(A) = P_F(P_E^{-1}(A) \cap G)$$

(читатель без труда уяснит себе геометрический смысл $G_F$). Главный результат, который мы доказываем, можно сформулировать так:

**Теорема.** *Пусть $E$, $F$ — два локально выпуклых линейных пространства и пусть $G$ — замкнутое подпространство в $E \times F$ такое, что $P_F(G) = F$. Пусть $E$ является $B$-полным пространством. Предположим, что выполняется следующее условие: для каждой окрестности нуля $U$ в $E$ замыкание множества $G_F(U)$ является окрестностью нуля в $F$. Тогда и само $G_F(U)$ будет окрестностью нуля в $F$ для любого $U$.*

Доказательство основано на исследовании аннигилятора множества $G$. Из этой теоремы непосредственно вытекает теорема о замкнутом графике и теорема об открытом отображении, если в качестве $G$ взять график рассматриваемого отображения.