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ON LIPSCHITZIAN MAPPINGS FROM THE PLANE INTO EUCLIDEAN SPACES

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In this paper the integrals associated with a Lipschitzian mapping from the plane into the r-dimensional Euclidean space are studied. Some theorems are proved, which are in a certain relation to the Stokes formula.

1. Notation. For every positive integer r, E_r is the Euclidean space of r dimensions. Denoting by p_i the i-th coordinate of the point $p \in E_r$, we define

$$|p| = \sqrt{\sum_{i=1}^{r} p_i^2}\,.$$

Further we put for $p, q \in E_r$

$$p \cdot q = \sum_{i=1}^{r} p_i q_i\,, \quad p \pm q = [p_1 \pm q_1, \ldots, p_r \pm q_r]\,.$$

If Φ is a mapping from the set $M \neq \emptyset$ into E_r, then Φ_i $(i = 1, \ldots, r)$ are functions on M such that $\Phi(\mathfrak{z}) = [\Phi_1(\mathfrak{z}), \ldots, \Phi_r(\mathfrak{z})]$ for every $\mathfrak{z} \in M$. Further we put

$$\|\Phi\|_M = \sup_{\mathfrak{z} \in M} |\Phi(\mathfrak{z})|\,.$$

If A is a subset of E_2, we denote by $\bar{A}$ (resp. A^0) the closure (resp. the interior) of A.

Let now a, b be real numbers, $a < b$, and let g be a mapping from $\langle a, b\rangle$ into E_r. We denote by $v(a, b, g)$ the least upper bound of all the sums $\sum_{i=1}^{n} |g(t^i) - g(t^{i-1})|$, where $a = t^0 < t^1 < \ldots < t^n = b$ is an arbitrary subdivision of $\langle a, b\rangle$.

Let f be a continuous mapping from $\langle a, b\rangle$ into E_2, $f(\langle a, b\rangle) = C$, and suppose that Φ, Ψ are continuous mappings from C into E_r. Then we define

$$\int_f \Phi \, \mathrm{d}\Psi = \sum_{i=1}^{r} \int_a^b \Phi_i(f(t)) \, \mathrm{d}\Psi_i(f(t))\,,$$

provided that the Stieltjes integrals[1]) on the right exist. If we put

$$t^{n,k} = a + k\frac{b-a}{n} \quad (k = 0, \ldots, n)\,, \quad \Delta^{n,k} = \Psi(f(t^{n,k})) - $$
$$- \Psi(f(t^{n,k-1})) \quad (k = 1, \ldots, n)\,,$$

then

$$\int_f \Phi \, \mathrm{d}\Psi = \lim_{n\to\infty} \sum_{k=1}^{n} \Phi(f(t^{n,k}))\,.\,\Delta^{n,k}\,, \tag{1}$$

under the hypothesis that $\int_f \Phi \, \mathrm{d}\Psi$ exists in the sense of preceding definition. If $v(a, b, \Psi(f)) < \infty$, then $\int_f \Phi \, \mathrm{d}\Psi$ exists and, as it is easy to see from (1),

$$\left|\int_f \Phi \mathrm{d}\Psi\right| \leqq \|\Phi\|_C \, v(a, b, \Psi(f))\,. \tag{2}$$

If at least one of the symbols $\int_f \Phi \, \mathrm{d}\Psi$, $\int_f \Psi \, \mathrm{d}\Phi$ has a meaning, then both these symbols are meaningful and the following relation is valid:

$$\int_f \Phi \, \mathrm{d}\Psi + \int_f \Psi \, \mathrm{d}\Phi = \Phi(f(b))\,.\,\Psi(f(b)) - \Phi(f(a))\,.\,\Psi(f(a))\,.$$

Suppose now that $f(a) = f(b)$ and that $\int_f \Phi \, \mathrm{d}\Psi$ exists. Then also $\int_f \Psi \, \mathrm{d}\Phi$ exists and

$$\int_f \Phi \, \mathrm{d}\Psi = -\int_f \Psi \, \mathrm{d}\Phi\,. \tag{3}$$

Further let be $p, q \in E_r$ and put

$$\Phi^1 = \Phi - p\,, \quad \Psi^1 = \Psi - q\,.^{2)} \tag{4}$$

According to (1) we have $\int_f \Phi \, \mathrm{d}\Psi = \int_f \Phi \, \mathrm{d}\Psi^1$ and, by symmetry, $\int_f \Phi \, \mathrm{d}\Psi^1 = -\int_f \Psi^1 \, \mathrm{d}\Phi = -\int_f \Psi^1 \, \mathrm{d}\Phi^1$, so that

$$\int_f \Phi \, \mathrm{d}\Psi = \int_f \Phi^1 \, \mathrm{d}\Psi^1\,. \tag{5}$$

By the word "interval" we always mean a closed non-degenerate interval in E_2. Given an interval I, we denote by $u(I)$ the perimeter-length of I. Further, let H_I be a continuous mapping from $\langle 0, 1\rangle$ into E_2 defining parametrically a simple description of the boundary of I (in positive sense). Of course, $H_I(0) = H_I(1)$, and, as it is easy to see,

$$v(0, 1, H_I) = u(I)\,. \tag{6}$$

The Lebesque measure in E_2 will be denoted by μ. The words "a. e. (almost every, almost everywhere)" are always to be taken with respect to μ.

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[1]) Vide, e. g., [2], chap. X, § 7, pp. 415–422.

[2]) That means $\Phi^1(\mathfrak{z}) = \Phi(\mathfrak{z}) - p$, $\Psi^1(\mathfrak{z}) = \Psi(\mathfrak{z}) - q$ for every $\mathfrak{z} \in C$.

The terms "constant, number, function" mean a finite real constant, number, function respectively.

2. Definition. *Let be* $\emptyset \neq K \subset E_2$ *and let* Ψ *be a mapping from* K *into* E_r. *We say that* Ψ *is Lipschitzian on* K *if there exists a constant* λ *such that*

$$|\Psi(\mathfrak{z}^1) - \Psi(\mathfrak{z}^2)| \leqq \lambda|\mathfrak{z}^1 - \mathfrak{z}^2| \tag{7}$$

for every pair of points $\mathfrak{z}^1$, $\mathfrak{z}^2$ *in* K.

3. Remark. Let K be an interval and let Φ, Ψ be continuous mappings from K into E_r. Suppose that Ψ fulfils (7) for every pair of points $\mathfrak{z}^1$, $\mathfrak{z}^2$ in K. Then we have for any interval $I \subset K$ (compare (6))

$$v(0, 1, \Psi(H_I)) \leqq \lambda v(0, 1, H_I) = \lambda\, u\,(I) < \infty\,, \tag{8}$$

so that the symbol $\int_{H_I} \Phi \, \mathrm{d}\Psi$ is available.

4. Notation. In the sequel we often use the following licence. For $\mathfrak{z} \in E_2$, we write $\mathfrak{z} = [x, y]$ instead of $[\mathfrak{z}_1, \mathfrak{z}_2]$.

If f is a continuous mapping from $\langle a, b\rangle$ into E_2 and if φ, ψ are continuous functions on $C = f(\langle a, b\rangle)$ such that $\varphi(x, y) = x$(resp. $\psi(x, y) = y$) for every $[x, y] \in C$, we write $\int_f x \, \mathrm{d}\psi$ (resp. $\int_f \varphi \, \mathrm{d}y$) instead of $\int_f \varphi \, \mathrm{d}\psi$. The symbols $\int_f y \, \mathrm{d}\psi$, $\int_f \varphi \, \mathrm{d}x$, $\int_f x \, \mathrm{d}y$ etc. have a similar meaning.

5. Definition. *Let be* $\mathfrak{z}^0 \in E_2$, $\pi = [\pi_1, \ldots, \pi_r] \in E_r$ *and let* U *be a neighbourhood of the point* $\mathfrak{z}^0$ *in* E_2. *Let* φ *be a function on* U *and let* Ψ *be a mapping from* U *into* E_r. *Suppose that for* $\mathfrak{z} \in U$ *the following relation is true*

$$\varphi(\mathfrak{z}) - \varphi(\mathfrak{z}^0) = \sum_{k=1}^{r} \pi_k(\Psi_k(\mathfrak{z}) - \Psi_k(\mathfrak{z}^0)) + |\Psi(\mathfrak{z}) - \Psi(\mathfrak{z}^0)|\, z\,(\mathfrak{z})\,, \tag{9}$$

where z *is a function on* U *such that* $z(\mathfrak{z}) \to 0$ *for* $\mathfrak{z} \to \mathfrak{z}^0$. *Then we write*

$$\pi = \partial(\varphi, \Psi; \mathfrak{z}^0)\,.$$

For the special case that $r = 2$ *and that* Ψ *is the identity mapping we write* $\pi = \partial(\varphi; \mathfrak{z}^0)$; *the expression* $\pi_1(x - x^0) + \pi_2(y - y^0)$ *is then the ordinary total differential of* φ *at* $\mathfrak{z}^0$, *so that*

$$\pi_1 = \frac{\partial\varphi(\mathfrak{z}^0)}{\partial x}\,, \quad \pi_2 = \frac{\partial\varphi(\mathfrak{z}^0)}{\partial y}\,.$$

6. Lemma. *Let* U *be a neighbourhood of the point* $\mathfrak{z}^0 \in E_2$ *and let* Φ, Ψ *be continuous mappings from* U *into* E_r. *Suppose that* Ψ *is Lipschitzian on* U *and that the following relations are valid*

$$\pi^i = [\pi_1^i, \ldots, \pi_r^i] = \partial(\Phi_i, \Psi; \mathfrak{z}^0)\,, \tag{10}$$

$$\left[\frac{\partial\Psi_i(\mathfrak{z}^0)}{\partial x}, \frac{\partial\Psi_i(\mathfrak{z}^0)}{\partial y}\right] = \partial(\Psi_i; \mathfrak{z}^0) \tag{11}$$

$i = 1, \dots, r$). *Put*

$$D_{ik} = \begin{vmatrix} \frac{\partial \Psi_i(\mathfrak{z}^0)}{\partial x}, & \frac{\partial \Psi_i(\mathfrak{z}^0)}{\partial y} \\ \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial x}, & \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial y} \end{vmatrix}$$

and denote by K_δ *the square of center* $\mathfrak{z}^0$ *and of side-length* δ. *If we write*

$$\nu(K_\delta) = \int_{H_{K_\delta}} \Phi \, \mathrm{d}\Psi \,,^{3)}$$

then

$$\frac{\nu(K_\delta)}{\mu(K_\delta)} \to \sum_{\substack{i,k=1 \\ i<k}}^{r} (\pi_i^k - \pi_k^i) \, D_{ik} \tag{12}$$

for $\delta \to 0$.

Proof. Put $p = \Phi(\mathfrak{z}^0)$, $q = \Psi(\mathfrak{z}^0)$ and define the mappings Φ^1, Ψ^1 by (4). Further define the mapping $\Lambda = [\Lambda_1, \dots, \Lambda_r]$ by means of the relations

$$\Lambda_i = \Phi_i^1 - \sum_{k=1}^{r} \pi_k^i \Psi_k^1 \,, \quad i = 1, \dots, r \,. \tag{13}$$

(Clearly, Λ is a continuous mapping from U into E_r.) According to definition 5 (see (9), where we put $\varphi = \Phi_i$, $\pi_k = \pi_k^i$, $z = z_i$), we have by (10)

$$\Lambda_i(\mathfrak{z}) = |\Psi^1(\mathfrak{z})| \, z_i(\mathfrak{z}) \,, \quad \text{where} \quad z_i(\mathfrak{z}) \to 0 \quad \text{for} \quad \mathfrak{z} \to \mathfrak{z}^0 \,. \tag{14}$$

Fix now a $\delta_0 > 0$ such that $K_{\delta_0} \subset U$. We restrict the range of the variable δ to $0 < \delta < \delta_0$. The mapping Ψ^1 being Lipschitzian, it follows from (14) that, for $\mathfrak{z} \in K_\delta$,

$$|\Lambda_i(\mathfrak{z})| \leqq \delta s_i(\delta) \,, \quad s_i(\delta) \to 0 \quad \text{for} \quad \delta \to 0 \,.$$

Writing $H_\delta = H_{K_\delta}$ we have for a suitable constant λ (compare (8))

$$v(0, 1, \Psi^1(H_\delta)) \leqq \lambda u(K_\delta) = 4\lambda\delta \,.$$

Hence we obtain (see (2))

$$\Big|\int_{H_\delta} \Lambda_i \, \mathrm{d}\Psi_i^1\Big| \leqq 4\lambda\delta^2 s_i(\delta) \,,$$

so that

$$\int_{H_\delta} \Lambda_i \, \mathrm{d}\Psi_i^1 = o(\delta^2) \,,^{4)} \quad i = 1, \dots, r \,.$$

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[3]) According to remark 3, this symbol is available for every sufficient small $\delta > 0$.

[4]) If $\varphi(\delta)$ is a function of the variable δ on $(0, \delta_0)$ we write, as usual, $\varphi(\delta) = o(\delta^2)$ to express the fact that $\frac{1}{\delta^2} \varphi(\delta) \to 0$ for $\delta \to 0$.

We have thus by (5)

$$\int_{H_\delta} \Phi \, \mathrm{d}\Psi = \int_{H_\delta} \Phi^1 \, \mathrm{d}\Psi^1 = \sum_{i=1}^{r} \int_{H_\delta} \Phi_i^1 \, \mathrm{d}\Psi_i^1 = (\text{see } (13)) = \sum_{i=1}^{r} \sum_{k=1}^{r} \pi_k^i \int_{H_\delta} \Psi_k^1 \, \mathrm{d}\Psi_i^1 + \\ + \sum_{i=1}^{r} \int_{H_\delta} \Lambda_i \, \mathrm{d}\Psi_i = \sum_{i,k=1}^{r} \pi_k^i \int_{H_\delta} \Psi_k^1 \, \mathrm{d}\Psi_i^1 + o(\delta^2) .$$

From the relations (compare (3))

$$\int_{H_\delta} \Psi_k^1 \, \mathrm{d}\Psi_i^1 = - \int_{H_\delta} \Psi_i^1 \, \mathrm{d}\Psi_k^1$$

we obtain $\int_{H_\delta} \Psi_i^1 \, \mathrm{d}\Psi_i^1 = 0$, and

$$\int_{H_\delta} \Phi \, \mathrm{d}\Psi = \sum_{\substack{i,k=1 \\ i<k}}^{r} (\pi_i^k - \pi_k^i) \int_{H_\delta} \Psi_i^1 \, \mathrm{d}\Psi_k^1 + o(\delta^2) . \tag{15}$$

Fix now the indices i, k. Using similar arguments it is easy to obtain from (11)

$$\int_{H_\delta} \Psi_i^1 \, \mathrm{d}\Psi_k^1 = \frac{\partial \Psi_i(\mathfrak{z}^0)}{\partial x} \int_{H_\delta} x \, \mathrm{d}\Psi_k + \frac{\partial \Psi_i(\mathfrak{z}^0)}{\partial y} \int_{H_\delta} y \, \mathrm{d}\Psi_k + o(\delta^2) . \tag{16}$$

Further we have

$$\int_{H_\delta} x \, \mathrm{d}\Psi_k = - \int_{H_\delta} \Psi_k \, \mathrm{d}x = - \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial x} \int_{H_\delta} x \, \mathrm{d}x - \\ - \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial y} \int_{H_\delta} y \, \mathrm{d}x + o(\delta^2) = \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial y} \mu(K_\delta) + o(\delta^2) . \tag{17}$$

In a similar way

$$\int_{H_\delta} y \, \mathrm{d}\Psi_k = - \int_{H_\delta} \Psi_k \, \mathrm{d}y = - \frac{\partial \Psi_k(\mathfrak{z}^0)}{\partial x} \mu(K_\delta) + o(\delta^2) . \tag{18}$$

It follows from (16)—(18)

$$\int_{H_\delta} \Psi_i^1 \, \mathrm{d}\Psi_k^1 = D_{ik} \mu(K_\delta) + o(\delta^2) .$$

Hence we derive at once with help of (15) the relation (12).

7. Notation. In the sequel (sections 8—10) the following notation will be kept. K is a fixed interval and Φ, Ψ are continuous mappings from K into E_r. The mapping Ψ is assumed to be Lipschitzian on K.

8. Lemma. *Let ν be an additive function of an interval on K.*[5]) *Suppose that there exists a constant β such that*

$$|\nu(I)| \leq \beta \mu(I) \tag{19}$$

for every interval $I \subset K$. Then there exists a signed measure ν^* [6]) *defined on Borel subsets in K such that the following relations are valid:*

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[5]) Cf. [4], chap. III, § 4, p. 61.

[6]) Cf. [1], chap. VI, § 28.

1. $\nu^*(I) = \nu(I)$ *for every interval* $I \subset K$,

2. $|\nu^*(B)| \leqq \beta\mu(B)$ *for every Borel set* $B \subset K$.

Proof. This lemma is an easy consequence of well-known theorems on measure theory. Vide, e. g., [4], chap. III, §§ 4, 5 and chap. II, theorem (7.4).

9. Lemma. *Put for any interval* $I \subset K$

$$\nu(I) = \int_{H_I} \Phi \, \mathrm{d}\Psi \,.^{7)}$$

Then ν *is an additive function of an interval on* K. *If the mapping* Φ *is also Lipschitzian on* K, *then there exists a constant* β *such that the relation* (19) *is valid for every interval* $I \subset K$.

Proof. The first part of this lemma is left to be proved by the reader. Suppose now that Φ is Lipschitzian on K and choose a constant λ such that, for every pair $\mathfrak{z}^1$, $\mathfrak{z}^2$ of points in K, $\lambda|\mathfrak{z}^1 - \mathfrak{z}^2|$ is a common upper bound for the numbers $|\Phi(\mathfrak{z}^1) - \Phi(\mathfrak{z}^2)|$, $|\Psi(\mathfrak{z}^1) - \Psi(\mathfrak{z}^2)|$. Let I be an arbitrary interval $(I \subset K)$, fix a $\mathfrak{z}^0 \in I$ and put $p = \Phi(\mathfrak{z}^0)$, $q = \Psi(\mathfrak{z}^0)$. If we define the mappings Φ^1, Ψ^1 by means of (4), we have $\|\Phi^1\|_I \leqq \frac{1}{2}\lambda u(I)$ and (compare (8)) $v(0, 1, \Psi^1(H_I)) \leqq \lambda u(I)$, so that we obtain by means of (5), (2)

$$|\nu(I)| = |\int_{H_I} \Phi^1 \, \mathrm{d}\Psi^1| \leqq \tfrac{1}{2}(\lambda u(I))^2 \,.$$

In particular, if I is a square, then the relation (19) holds for $\beta = 8\lambda^2$. If I is not a square then, to any $\varepsilon > 0$, we can find non-overlapping intervals $I_1, \ldots, I_{n+1}$ such that

$$\bigcup_{k=1}^{n+1} I_k = I \,, \quad u(I_{n+1}) < 4\varepsilon \,,$$

every I_k with $k \leqq n$ being a square. Hence it follows

$$|\nu(I)| \leqq \sum_{k=1}^{n} |\nu(I_k)| + |\nu(I_{n+1})| \leqq \sum_{k=1}^{n} 8\lambda^2\mu(I_k) + 8\lambda^2\varepsilon^2 \leqq 8\lambda^2(\mu(I) + \varepsilon^2) \,.$$

Since ε was an arbitrary positive number, it is sufficient to put $\beta = 8\lambda^2$ again to satisfy (19).

10. Theorem. *Suppose that the mapping* Φ *is Lipschitzian on* K *and that*

$$\pi^i(\mathfrak{z}) = [\pi^i_1(\mathfrak{z}), \ldots, \pi^i_r(\mathfrak{z})] = \partial(\Phi_i, \Psi; \mathfrak{z}) \tag{20}$$

$(i = 1, \ldots, r)$ *for a. e.* $\mathfrak{z} \in K^0$. *Put (as far as the symbols involved are meaningful)*

$$D_{ik}(\mathfrak{z}) = \begin{vmatrix} \dfrac{\partial \Psi_i(\mathfrak{z})}{\partial x}, & \dfrac{\partial \Psi_i(\mathfrak{z})}{\partial y} \\ \dfrac{\partial \Psi_k(\mathfrak{z})}{\partial x}, & \dfrac{\partial \Psi_k(\mathfrak{z})}{\partial y} \end{vmatrix}, \tag{21}$$

$$\gamma(\mathfrak{z}) = \sum_{\substack{i,k=1 \\ i<k}}^{r} (\pi^k_i(\mathfrak{z}) - \pi^i_k(\mathfrak{z}))\, D_{ik}(\mathfrak{z}) \,. \tag{22}$$

[7]) Cf. remark 3.

Then the function γ is defined a. e. on K and the relation

$$\int_{H_K} \Phi \, \mathrm{d}\Psi = \int_K \gamma \, \mathrm{d}\mu \tag{23}$$

is true.

Proof. Since the functions Ψ_i are Lipschitzian on K, there is

$$\left[\frac{\partial \Psi_i(\mathfrak{z})}{\partial x}, \frac{\partial \Psi_i(\mathfrak{z})}{\partial y}\right] = \partial(\Psi_i; \mathfrak{z}) \tag{24}$$

$(i = 1, \ldots, r)$ for a. e. $\mathfrak{z} \in K^0$. (Cf. [3], theorem 1, p. 347.) Let us denote by $K(\mathfrak{z}, \delta)$ the square of center $\mathfrak{z}$ and of side-length δ. If the symbol ν has the same meaning as in lemma 9, we obtain by (20), (22) and by lemma 6 that

$$\lim_{\delta \to 0} \frac{\nu(K(\mathfrak{z}, \delta))}{\mu(K(\mathfrak{z}, \delta))} = \gamma(\mathfrak{z}) \tag{25}$$

for a. e. $\mathfrak{z} \in K^0$. Applying lemma 8 we extend ν to a signed Borel measure ν^* (note that ν^* is absolutely continuous with respect to μ) and put $\frac{\mathrm{d}\nu^*}{\mathrm{d}\mu} = \gamma^*$.[8]) (This derivative is to be taken in the sense of measure theory, so that γ^* is a Borel function on K such that

$$\nu^*(B) = \int_B \gamma^* \, \mathrm{d}\mu$$

for every Borel set $B \subset K$.) By theorem (6.3) from [4], chap. III, p. 118 there is

$$\lim_{\delta \to 0} \frac{\nu^*(K(\mathfrak{z}, \delta))}{\mu(K(\mathfrak{z}, \delta))} = \gamma^*(\mathfrak{z}) \tag{26}$$

for a. e. $\mathfrak{z} \in K$. Since $\nu^*(K(\mathfrak{z}, \delta)) = \nu(K(\mathfrak{z}, \delta))$, it follows from (25), (26) that $\gamma = \gamma^*$ a. e. on K so that, in particular, $\nu(K) = \nu^*(K) = \int_K \gamma^* \, \mathrm{d}\mu = \int_K \gamma \, \mathrm{d}\mu$.

11. Definition. *Let be $\emptyset \neq V \subset E_r$, $u^0 = [u_1^0, \ldots, u_r^0] \in V$, $\pi = [\pi_1, \ldots, \pi_r] \in E_r$ and let $\hat{\varphi}$ be a function on V. The expression*

$$\sum_{k=1}^{r} \pi_k(u_k - u_k^0) \tag{27}$$

is termed the differential of $\hat{\varphi}$ at u^0 with respect to V, if the relation

$$\hat{\varphi}(u) - \hat{\varphi}(u^0) = \sum_{k=1}^{r} \pi_k(u_k - u_k^0) + |u - u^0| \, w(u) \tag{28}$$

is valid, where $w(u^0) = 0$, $w(u) \to 0$ for $u \to u^0$, $u \in V$.

12. Theorem. *Let Ψ be a Lipschitzian mapping from the interval K into E_r. Put $V = \Psi(K)$ and suppose that $\Gamma = [\Gamma_1, \ldots, \Gamma_r]$ is a Lipschitzian mapping from V into E_r. Further suppose that there exists a set $M \subset V$ with the following properties:*

[8]) Cf. [1], chap. VI, §§ 32, 31.

1. $\mu(\Psi^{-1}(M)) = 0$.

2. *At every point* $u^0 \in V - M$, *the expression* $\sum_{k=1}^{r} \tau_k^i(u^0)(u_k - u_k^0)$ *is the differential of* Γ_i *with respect to* V $(i = 1, \ldots, r)$.

We define the functions D_{ik} *by means of the relation* (21) *and put* (*as far as the symbols involved are meaningful*)

$$\gamma(\mathfrak{z}) = \sum_{\substack{i,k=1 \\ i<k}}^{r} (\tau_i^k(\Psi(\mathfrak{z})) - \tau_k^i(\Psi(\mathfrak{z}))) \, D_{ik}(\mathfrak{z}) \, . \tag{29}$$

Then the function γ *is defined a. e. on* K *and, for* $\Phi = \Gamma(\Psi)$, *the relation* (23) *is valid.*

Proof. Note first that the mapping Φ is Lipschitzian on K. Fix the index i and put $\hat{\varphi} = \Gamma_i$, $\varphi = \Phi_i$. Let $\mathfrak{z}^0$ be any point of $K^0 - \Psi^{-1}(M)$ and put $u^0 = \Psi(\mathfrak{z}^0)$, $\pi_k = \tau_k^i(u^0)$ $(k = 1, \ldots, r)$. Then the expression (27) is the differential of $\hat{\varphi}$ at u^0 with respect to V. If we put $u = \Psi(\mathfrak{z})$, $w(\Psi(\mathfrak{z})) = z(\mathfrak{z})$ in the relation (28), we obtain the relation (9) (note that the mapping Ψ is continuous and that $\hat{\varphi}(\Psi(\mathfrak{z})) = \varphi(\mathfrak{z})$), so that

$$[\tau_1^i(\Psi(\mathfrak{z}^0)) \, , \ldots, \tau_r^i(\Psi(\mathfrak{z}^0))] = \partial(\Phi_i, \Psi; \mathfrak{z}^0) \, .$$

Since $\mathfrak{z}^0$ was an arbitrary point of the set $K^0 - \Psi^{-1}(M)$, we see that all the assumptions from theorem 10 (where we write $\pi^i(\mathfrak{z}) = [\tau_1^i(\Psi(\mathfrak{z})), \ldots, \tau_r^i(\Psi(\mathfrak{z}))]$) are fulfilled and, consequently, the relation (23) is true.

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Резюме

ОБ ОТОБРАЖЕНИЯХ ПЛОСКОСТИ В ЕВКЛИДОВЫ ПРОСТРАНСТВА, УДОВЛЕТВОРЯЮЩИХ УСЛОВИЮ ЛИПШИЦА

ИОСЕФ КРАЛ (Josef Král), Прага

(Поступило в редакцию 14/XI 1957 г.)

Пусть E_r — r-мерное евклидово пространство. Если p — точка пространства E_r, то p_i обозначает i-тую координату точки p. Если Φ — отображение множества $M \neq \emptyset$ в пространство E_r, то Φ_i — такая функция на множестве M, что равенство $\Phi_i(\mathfrak{z}) = (\Phi(\mathfrak{z}))_i$ выполняется для всех $\mathfrak{z} \in M$ $(i = 1, \ldots, r)$. Для $p, q \in E_r$ полагаем

$$p \pm q = [p_1 \pm q_1, \ldots, p_r \pm q_r]\,, \quad |p| = \sqrt{\sum_{i=1}^{r} p_i^2}\,.$$

Пусть теперь f — непрерывное отображение отрезка $\langle a, b \rangle$ в плоскость и пусть Φ, Ψ — непрерывные отображения множества $f(\langle a, b \rangle)$ в пространство E_r. Тогда мы положим

$$\int_f \Phi \,\mathrm{d}\Psi = \sum_{i=1}^{r} \int_a^b \Phi_i(f(t)) \,\mathrm{d}\Psi_i(f(t))$$

в предположении, что существуют интегралы Стильтьеса в правой части этого равенства.

Словом „интервал" разумеется в дальнейшем двумерный замкнутый невырожденный интервал. Если I — интервал, то обозначим символом H_I непрерывное отображение отрезка $\langle 0, 1 \rangle$ в плоскость, определяющее простое описание границы интервала I в положительном направлении.

Символом μ мы обозначаем лебеговскую меру на плоскости.

Введём ещё следующее

Определение. *Пусть $\mathfrak{z}^0$ — точка на плоскости, $\pi = [\pi_1, \ldots, \pi_r]$ — точка из пространства E_r. Предположим, что в некоторой окрестности U точки $\mathfrak{z}^0$ определены функция φ и отображение Ψ, отображающее множество U в пространство E_r. Если для $\mathfrak{z} \in U$ выполняется равенство*

$$\varphi(\mathfrak{z}) - \varphi(\mathfrak{z}^0) = \sum_{k=1}^{r} \pi_k(\Psi_k(\mathfrak{z}) - \Psi_k(\mathfrak{z}^0)) + |\Psi(\mathfrak{z}) - \Psi(\mathfrak{z}^0)|\, z(\mathfrak{z})\,,$$

где $z(\mathfrak{z})$ — такая функция на множестве U, что $z(\mathfrak{z}) \to 0$ для $\mathfrak{z} \to \mathfrak{z}^0$, то мы пишем

$$\pi = \partial(\varphi, \Psi; \mathfrak{z}^0)\,.$$

В этих обозначениях справедлива следующая

Теорема. *Пусть K — интервал и пусть Φ, Ψ — отображения интервала K в пространство E_r, удовлетворяющие на K условию Липшица. Пусть, кроме того, соотношения*

$$\pi^i(\mathfrak{z}) = [\pi_1^i(\mathfrak{z}), \ldots, \pi_r^i(\mathfrak{z})] = \partial(\Phi_i, \Psi; \mathfrak{z}) \quad (i = 1, \ldots, r)$$

справедливы для почти всех $\mathfrak{z} \in K$. Определим функции D_{ik} соотношением (21) и функцию γ равенством (22) (коль скоро имеют смысл употребляемые символы).

Тогда функция γ определена почти всюду на K, и имеет место равенство (23).

Определение. *Пусть V — непустое подмножество пространства E_r, $\pi = [\pi_1, \ldots, \pi_r] \in E_r$ и $u^0 = [u_1^0, \ldots, u_r^0] \in V$. Пусть, далее, $\hat{\varphi}$ — функция на множестве V. Мы скажем, что выражение (27) является полным дифференциалом функции $\hat{\varphi}$ в точке u^0 по отношению к множеству V, если*

$$\frac{1}{|u - u^0|} \left| \hat{\varphi}(u) - \hat{\varphi}(u^0) - \sum_{k=1}^{r} \pi_k(u_k - u_k^0) \right| \to 0$$

для $u = [u_1, \ldots, u_r] \neq u^0$, $u \in V$, $u \to u^0$.

Из приведённой выше теоремы вытекает ещё следующее следствие:

Теорема. *Пусть Ψ — отображение интервала K в пространство E_r, удовлетворяющее на K условию Липшица. Пусть, далее, $\Gamma = [\Gamma_1, \ldots, \Gamma_r]$ — отображение множества $\Psi(K) = V$ в пространство E_r, удовлетворяющее условию Липшица на V. Предположим, что существует множество $M \subset V$, обладающее следующими свойствами:*

1. $\mu(\Psi^{-1}(M)) = 0$.

2. *В каждой точке $u^0 \in V - M$, выражение $\sum_{k=1}^{r} \tau_k^i(u^0)(u_k - u_k^0)$ является полным дифференциалом функции Γ_i по отношению к множеству V.*

Определим функции D_{ik} равенством (21) и функцию γ равенством (29), коль скоро имеют смысл употребляемые символы.

Тогда функция γ определена почти всюду на K, и для отображения $\Phi = \Gamma(\Psi)$ выполняется соотношение (23).