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A REMARK ON APPROXIMATION OF CONTINUOUS FUNCTIONS

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The author gives a direct geometrical proof of Haar's theorem on approximation of continuous functions and of the Čebyšev characterization of the polynomial of best approximation.

In his beautiful paper [2] on the application of Minkowskian geometry to the theory of approximation, A. HAAR has given a necessary and sufficient condition that the best approximation of any continuous function by means of linear combinations of n given continuous functions be unique. His proof is based on the following idea. Suppose we have n linearly independent continuous functions $x_1, \ldots, x_n$ defined on $\langle 0, 1 \rangle$. In E_n we consider the set K of all vectors $[\xi_1, \ldots, \xi_n]$ which fulfil the inequality

$$\max_{0 \leqq t \leqq 1} |\xi_1 x_1(t) + \ldots + \xi_n x_n(t)| \leqq 1 .$$

It is easy to see that K is a convex body in E_n in the sense of MINKOWSKI. The discussion of the best approximation by linear combinations of $x_1, \ldots, x_n$ is then reduced to the study of properties of the Minkowskian geometry defined by K. The proof, though simple and clear enough, is by no means a short one. It is especially the sufficiency of Haar's condition which requires more subtle considerations. Even modern proofs (see, e. g. [1]) devote to the sufficiency more than three pages.

It is the purpose of the present remark to give a simple proof of Haar's theorem using only the simplest geometrical notions.

Theorem 1. *Let T be a compact Hausdorff space and let us denote by $C(T)$ the space of all continuous functions on T with the usual norm. Let E be a given n-dimensional subspace of $C(T)$. The best approximation of every $x \in C(T)$ by means of elements of E is unique if and only if there does not exist an $e \in E$, $e \neq 0$, such that the equation $e(t) = 0$ has at least n distinct solutions.*

We begin with a simple

Lemma. *Let L be a p-dimensional subspace of $C(T)$. Let f be a linear functional on L of norm one. Then there exist p distinct points $t_i \in T$ and numbers λ_i such that $f = \sum_{i=1}^{p} \lambda_i t_i$ and $\sum_{i=1}^{p} |\lambda_i| = 1$. The equation $f = \sum \lambda_i t_i$ is, of course, taken to mean $\langle x, f \rangle = \sum \lambda_i x(t_i)$ for every $x \in L$.*

To prove this lemma, we note first that, with every point $t \in T$, we may associate a linear functional $\varphi(t)$ on L defined by the relation $\langle x, \varphi(t) \rangle = x(t)$ for every $x \in L$. Clearly φ is a continuous mapping of T into L'. At the same time, the norm of $\varphi(t)$ is at most one for every $t \in T$ and it is easy to see that the unit sphere of L' coincides with the closed symmetrical convex envelope of the set $\varphi(T)$. Indeed, suppose we have an $x' \in L'$, $|x'| \leqq 1$ such that x' does not belong to the closed symmetrical convex envelope of $\varphi(T)$. It follows[1]) that there exists a point $x \in L$ such that $\sup |\langle x, \varphi(T) \rangle| < \langle x, x' \rangle$. This is, however, a contradiction, since $|x'| \leqq 1$ and $\sup |\langle x, \varphi(T) \rangle| = |x|$. The mapping φ being continuous, $\varphi(T)$ is compact and it follows from a result of Carathéodory[2]) that every point f of the unit sphere of L' may be expressed in the form $\sum_{i=1}^{p} \lambda_i \varphi(t_i)$ with $\sum |\lambda_i| \leqq 1$. If the norm of f is one, $\sum |\lambda_i| < 1$ is impossible which concludes the proof.

Before going into the proof it is convenient to state Haar's condition in another equivalent form. We have the following equivalence. There exists in

[1]) If W is a closed convex subset of a finite-dimensional vector space and w a point outside W, there exists, according to a well-known theorem, a hyperplane separating W and w. Since x' does not belong to the closed symmetrical convex envelope of $\varphi(T)$, it does not belong to the closed convex envelope of the union of $\varphi(T)$ and $-\varphi(T)$. Acording to the separation theorem there exists a linear functional g on L' such that $\sup |g(\varphi(T))| < g(x')$. The space L' being finite dimensional, linear functionals on L' may be identified with elements of L.

[2]) The result of Carathéodory referred to is the following: Let E be a p-dimensional vector space and let M be a compact subset of E. The closed convex envelope K of M consists of all vectors of the form $\sum_{i=1}^{p+1} \lambda_i m_i$, where $m_i \in M$, $\lambda_i \geqq 0$ and $\sum_{i=1}^{p+1} \lambda_i = 1$. If k is a point of the boundary of K, it may be expressed in the form $k = \sum_{i=1}^{p} \lambda_i m_i$ where $m_i \in M$, $\lambda_i \geqq 0$ and $\sum_{i=1}^{p} \lambda_i = 1$. Now suppose we have a compact set B and a point x of its closed symmetrical convex envelope S. There exists a number $\delta \geqq 1$ such that δx belong to the boundary of S. Since S clearly coincides with the closed convex envelope of the union of B and $-B$, we may express δx in the form $\delta x = \sum_{i=1}^{p} \omega_i \varepsilon_i b_i$, where $b_i \in B$, $\varepsilon_i = \pm 1$, $\omega_i \geqq 0$ and $\sum_{i=1}^{p} \omega_i = 1$. If we put $\lambda_i = \frac{\omega_i \varepsilon_i}{\delta}$, we have $x = \sum_{i=1}^{p} \lambda_i b_i$ and $\sum_{i=1}^{p} |\lambda_i| \leqq 1$. For a simple proof of these results, using the definition of convexity only, see a recent paper of the author's [3].

E a nonzero element e with at least n distinct zero points $t_1, \ldots, t_n$ if and only if there exists a nonzero linear combination $f = \sum_{i=1}^{n} \lambda_i t_i$ which vanishes on E. Indeed, if $e_1, \ldots, e_n$ is a basis of E, both conditions express the fact that $\det e_i(t_j) = 0$.

Proof of Haar's theorem. Suppose first that the best approximation is not always unique. Then there exist points $x_0 \in C(T)$, $e_0 \in E$ and a nonzero $e \in E$ such that, for every ε small enough in absolute value, the point $e_0 + \varepsilon e$ is the best approximation of x_0.

Let us denote by L the linear span of E and x_0, so that L has $n + 1$ dimensions. If we put $w_0 = x_0 - e_0$, it follows that there exists a linear functional f of norm one on L vanishing on E and assuming the value $|w_0|$ on w_0. It follows from our lemma that f may be expressed in the form $\sum_{i=1}^{n+1} \lambda_i t_i$ where $t_i \in T$ and $\sum_{i=1}^{n+1} |\lambda_i| = 1$.

We shall distinguish two cases:

1^0 We have $\lambda_i \neq 0$ for every i. Since

$$|w_0| = \langle w_0, f \rangle = \sum_{i=1}^{n+1} \lambda_i w_0(t_i) \quad \text{and} \quad \sum_{i=1}^{n+1} |\lambda_i| = 1\,,$$

we have $|w_0(t_i)| = |w_0|$ for every i. Since $|w_0 + \varepsilon e| = |w_0|$ for small ε, it follows that $e(t_i) = 0$ for every i.

2^0 We have $\lambda_i = 0$ for some i. We may clearly suppose that $\lambda_{n+1} = 0$. In this case, $\sum_{i=1}^{n} \lambda_i t_i$ is a nonzero linear combination of n points vanishing on E.

The sufficiency of Haar's condition is thus proved. On the other hand, suppose that Haar's condition is not fulfilled. Then there exists a nonzero point $e_0 \in E$, $|e_0| \leqq 1$ which vanishes in n distinct points $t_i \in T$ and a nonzero linear combination $f = \sum_{i=1}^{n} \lambda_i t_i$ which vanishes on E. We may clearly suppose that $\sum_{i=1}^{n} |\lambda_i| = 1$. Now choose an arbitrary $a \in C(T)$ such that $|a| \leqq 1$ and $a(t_i) = \operatorname{sign} \lambda_i$ whenever $\lambda_i \neq 0$. Let us define the function x_0 by the relation $x_0(t) = a(t)(1 - |e_0(t)|)$. We have clearly $x_0(t_i) = a(t_i) = \operatorname{sign} \lambda_i$ for $\lambda_i \neq 0$, $|x_0| = 1$ and $|x_0(t)| + |e_0(t)| \leqq 1$ for every $t \in T$. It follows that $|x_0 - e_0| \leqq \max_{t \in T} (|x_0(t)| + |e_0(t)|) \leqq 1$. We have, however, $|x_0 - e| \geqq 1$ for every $e \in E$. Indeed, if $|x_0 - e| < 1$ for some e, it would follow

$$1 = \langle x_0, f \rangle = \langle x_0 - e, f \rangle \leqq |x_0 - e|\, |f| < 1\,,$$

which is a contradiction. It follows that both points 0 and e_0 are best approximations of x_0.

We conclude this remark with a few words concerning the characterization of the polynomial of best approximation. First of all, let us have a compact Hausdorff space T and an n-dimensional subspace $E \subset C(T)$ fulfilling Haar's condition. Let $x \in C(T)$, x non $\in E$ be given; the best approximation $e \in E$ of x is thus uniquely determined. We assert now that *the equation* $|x(t) - e(t)| = |x - e|$ *is fulfilled for al least* $n + 1$ *distinct points* t. In fact, we know that there exist $n + 1$ points $t_1, \ldots, t_{n+1}$ and real numbers $\lambda_1, \ldots, \lambda_{n+1}$ with $\sum_{i=1}^{n+1} |\lambda_i| = 1$ such that, for $f = \sum_{i=1}^{n+1} \lambda_i t_i$ we have $\langle x - e, f \rangle = |x - e|$ and $\langle E, f \rangle = 0$. From the first equation it follows that $|x(t_i) - e(t_i)| = |x - e|$ whenever $\lambda_i \neq 0$. We have, however, $\lambda_i \neq 0$ for every i since, in the contrary case, E would not fulfil Haar's condition (see the section 2^0 of the preceding proof).

A more precise result is hardly to be expected in the general case. It may be shown on examples that there may be exactly $n + 1$ points t_i where $|x(t_i) - e(t_i)| = |x - e|$ and $x(t_i) - e(t_i) = |x - e|$ for every i.

Let us consider now the case where T is a compact interval $\langle a, b \rangle$. In this case the above method yields a particularly simple proof of the classical theorem of Čebyšev. We intend to show that the theorem of Čebyšev as well as the related result of de la Vallée-Poussin are both immediate consequences of the following lemma:

Let $T = \langle a, b \rangle$ *and let* E *an* n*-dimensional subspace of* $C(T)$ *fulfilling Haar's condition. If* $t_1 < t_2 < \ldots < t_{n+1}$ *are given points of* T, *there exists exactly one (apart from a scalar factor) nonzero linear combination* $f = \sum_{i=1}^{n+1} \lambda_i t_i$ *vanishing on* E. *All numbers* λ_i *are different from zero and they alternate in sign.*

Proof. The space E being n-dimensional, the $n + 1$ functionals t_i are linearly dependent on E. There exists a nonzero linear combination $f = \sum_{i=1}^{n+1} \lambda_i t_i$ such that $\langle E, f \rangle = 0$. The assumption that $\lambda_i = 0$ for some i would lead to a contradiction with our assumption concerning E. First of all, let $n = 1$ and let e be a nonzero element of E. It follows from Haar's condition that $e(t)$ is different from zero on the whole of T. Suppose now that $\lambda_1 e(t_1) + \lambda_2 e(t_2) = 0$. We have sign $e(t_1) =$ sign $e(t_2)$ so that λ_1 and λ_2 cannot be of the same sign. Suppose now that $n > 1$ and that there is an index p such that λ_p and λ_{p+1} are of the same sign. It follows that there exists an i such that at least one of the numbers λ_{i-1} or λ_{i+1} is of the same sign as λ_i. Clearly we may suppose that $\lambda_i > 0$. Choose now two positive numbers α_{i-1} and α_{i+1} such that

$$\lambda_{i-1}\alpha_{i-1} + \lambda_{i+1}\alpha_{i+1} > 0 \, .$$

Now there exists an $e \in E$ such that $e(t_{i-1}) = \alpha_{i-1}$, $e(t_{i+1}) = \alpha_{i+1}$ and $e(t_j) = 0$ for every j different from $i - 1$, i, $i + 1$. Since $\langle E, f \rangle = 0$, we have, in particular, $\langle e, f \rangle = 0$.

This reduces to

$$\lambda_{i-1}\alpha_{i-1} + \lambda_i e(t_i) + \lambda_{i+1}\alpha_{i+1} = 0 .$$

It follows that $e(t_i) < 0$. Since $e(t_{i-1})$ and $e(t_{i+1})$ are positive, there exist points $s_{i-1} \in (t_{i-1}, t_i)$ and $s_{i+1} \in (t_i, t_{i+1})$ with $e(s_{i-1}) = e(s_{i+1}) = 0$. This is a contradiction with Haar's condition.

Theorem 2. *Let $T = \langle a, b\rangle$ and let E be an n-dimensional subspace of $C(T)$ fulfilling Haar's condition. Let $x \in C(T)$ be given. The following condition is sufficient and necessary for a point $e \in E$ to be the best approximation of x:*

There exist $n+1$ points $t_1 < t_2 < \ldots < t_{n+1}$ of T and a number ε with $|\varepsilon| = 1$ such that $x(t_i) - e(t_i) = (-1)^i \varepsilon |x - e|$.

Proof. Let e be the best approximation of x. We have then a functional $f = \sum_{i=1}^{n+1} \lambda_i t_i$ with $\sum_{i=1}^{n+1} |\lambda_i| = 1$ vanishing on E and such that $\langle x - e, f\rangle = |x - e|$. Hence $x(t_i) - e(t_i) = |x - e| \operatorname{sign} \lambda_i$ whenever $\lambda_i \neq 0$. If the t_i are arranged in increasing order, it follows from the preceding lemma that the λ_i are different from zero and alternate in sign. The rest is easy.

The second part of the theorem is a consequence of the following result of de la Vallée-Poussin.

Let $T = \langle a, b\rangle$ and let E be an n-dimensional subspace of $C(T)$ fulfilling Haar's condition. Let $x_0 \in C(T)$ and $e_0 \in E$ be given. Suppose there exist $n+1$ points $t_1 < t_2 < \ldots < t_{n+1}$, positive numbers ε_i and a number ε with $|\varepsilon| = 1$ such that

$$x_0(t_i) - e_0(t_i) = (-1)^i \varepsilon\varepsilon_i .$$

We have then

$$\min_{e \in E} |x_0 - e| \geqq \min_{1 \leqq i \leqq n+1} \varepsilon_i .$$

Proof. According to the preceding lemma there exist positive numbers $\lambda_1, \ldots, \lambda_{n+1}$ with $\sum_{i=1}^{n+1} \lambda_i = 1$ such that

$$f = \sum_{i=1}^{n+1} (-1)^i \varepsilon \lambda_i t_i$$

vanishes on E. For every $e \in E$, we have

$$|x_0 - e| \geqq \langle x_0 - e, f\rangle = \langle x_0 - e_0, f\rangle = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \geqq \min \varepsilon_i$$

which concludes the proof.

BIBLIOGRAPHY

[1] *Н. И. Ахиезер:* Теория аппроксимаций, Moskva-Leningrad 1947.

[2] *A. Haar:* Minkowskische Geometrie und Annäherung an stetige Functionen, Math. Ann. *78* (1918), 293—311.

[3] *V. Pták:* O absolutně konvexním obalu množiny v konečně dimensionálním vektorovém prostoru, Čas. pro pěst. mat., *83* (1958), No 3.

Резюме

ЗАМЕТКА ОБ АППРОКСИМАЦИИ НЕПРЕРЫВНЫХ ФУНКЦИЙ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага

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В статье дается прямое геометрическое доказательство теоремы Хаара о приближении непрерывных функций.

Теорема. *Пусть T — компактное хаусдорфово пространство; обозначим через $C(T)$ пространство всех непрерывных функций на T с обычной нормой. Пусть E — данное n-мерное подпространство пространства $C(T)$. Наилучшее приближение каждого $x \in C(T)$ при помощи элементов пространства E будет однозначно определенным тогда и только тогда, если не будет существовать ненулевой элемент $e \in E$, имеющий не менее чем n различных нулей.*