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PREDICTING A STATIONARY PROCESS WHEN THE CORRELATION FUNCTION IS CONVEX

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By the method worked out in the paper [1] it is proved, that, in the case of a convex correlation function, it suffices to base the linear prediction on the last observation only, because the relative reduction of the residual variance, attainable by making use of any number of preceding observations, cannot exceed 50%.

1. Introduction and summary

Let us consider a wide-sense stationary process $\{x_t, -\infty < t < \infty\}$, and suppose, that its mean value μ, variance σ^2 and correlation function R_τ are known. The best linear prediction of the state of the process at the moment $t + \Delta$, say Pred $x_{t+\Delta}$, based on the single observation x_t at the moment t, viz.

$$\text{Pred}\, x_{t+\Delta} = \mu + R_\Delta(x_t - \mu)\,, \quad \Delta > 0\,, \tag{1}$$

is known to possess the residual variance

$$\mathsf{D}\{x_{t+\Delta} - \text{Pred}\, x_{t+\Delta}\} = \sigma^2(1 - R_\Delta^2)\,. \tag{2}$$

This variance may be reduced by making use of a certain number of preceding observations, i. e. by putting

$$\text{Pred}\, x_{t+\Delta} = \mu + \sum_{i=1}^{n} c_i(x_{t_i} - \mu)\,, \quad t_1 < \ldots < t_n = t < t + \Delta\,, \quad n \geqq 1\,. \tag{3}$$

We know, however, that if the correlation function is exponential, $R_\tau = e^{-a\tau}$ ($a > 0$), then the last observation contains all "linear" information, so that no reduction is possible in this way. The theorem below shows, that a somewhat weaker result is valid for all other convex correlation functions, namely to the effect, that the residual variance of (3) cannot be less than half of the right side of (2). In fact, the right side of (2) is only $1 + R_\Delta$ times greater, than the lower bound indicated by (4).

2. Theorem

Let $\{x_t, -\infty < t < \infty\}$ *be a wide-sense stationary process whose correlation function* R_τ *is convex. Let* Pred $x_{t+\Delta}$ *be any linear prediction of* $x_{t+\Delta}$ *defined by* (3). *Then*

$$\mathsf{D}\{x_{t+\Delta} - \text{Pred } x_{t+\Delta}\} \geqq \sigma^2(1 - R_\Delta) , \tag{4}$$

where $\sigma^2 = \mathsf{D}x_t$ *denotes the variance of* x_t.

Proof. Let us first suppose, that R_τ is continuous and $R_\infty = 0$. In [1] it is proved that such a convex correlation function is possessed by the process $x_t^0 = x_t^0(\lambda, \varphi, \ldots, y_{-1}, y_0, y_1, \ldots)$ defined by

$$x_t^0 = y_{\left[\frac{t}{\lambda} + \varphi\right]} , \quad -\infty < t < \infty , \tag{5}$$

where $\left[\frac{t}{\lambda} + \varphi\right]$ denotes the integral part of $\frac{t}{\lambda} + \varphi$, and $\lambda, \varphi, \ldots, y_{-1}, y_0, y_1, \ldots$ are mutually independent random variables; the distribution function of λ is given by

$$F(\lambda) = \begin{cases} \int\limits_0^\lambda \tau \, \mathrm{d}R'(\lambda) = \tau \, R'(\lambda) - R(\lambda) + 1 , & \lambda < 0 , \\ 0 & , \ \lambda \leqq 0 , \end{cases} \tag{6}$$

φ is distributed rectangularly over (0, 1), and $\ldots, y_{-1}, y_0, y_1, \ldots$ have each the same but otherwise arbitrary distribution with the mean value μ and the variance σ^2. As the validity of the inequality (4) for given instants $t_1 < \ldots < t_n = t < t + \Delta$ and constants $c_1, \ldots, c_n$ depends only on the correlation function R_τ, it suffices to prove it for any particular process possessing this correlation function, e. g. for the process x_t^0 as defined above. This is the main idea of our proof.

Let us choose an arbitrary $n \geqq 1$ and instants $t_1 < t_2 < \ldots < t_n = t < t_{n+1} = t + \Delta$, and keep them fixed in the course of all further considerations. Under the condition that

$$\left[\frac{t_i}{\lambda} + \varphi\right] = k_i , \quad i = 1, \ldots, n + 1 , \tag{7}$$

we may write

$$x_{t+\Delta}^0 - \text{Pred } x_{t+\Delta}^0 = y_{k_{n+1}} - \mu - \sum_{i=1}^n c_i(y_{k_i} - \mu) . \tag{8}$$

Owing to the independence of the random variables $\lambda, \varphi, \ldots, y_{-1}, y_0, y_1, \ldots$, the conditional distribution of $\{y_{k_1}, \ldots, y_{k_{n+1}}\}$ under the condition (7) will be identical with the non-conditional distribution, namely, the variance of $y_{k_{n+1}}$ will equal σ^2 and $y_{k_{n+1}}$ will be independent of $\{y_{k_1}, \ldots, y_{k_n}\}$ when $k_1 \leqq \ldots \leqq k_n < k_{n+1}$. Consequently, denoting the conditional variance under the condition A by $\mathsf{D}\{. \mid A\}$, we have

$$\mathsf{D}\left\{x^0_{t+\Delta} - \operatorname{Pred} x^0_{t+\Delta} \,\middle|\, \left[\frac{t_i}{\lambda} + \varphi\right] = k_i\,, \quad i = 1, \ldots, n+1\right\} =$$

$$= \mathsf{D}\{y_{k_{n+1}} - \mu - \sum_{i=1}^{n} c_i(y_{k_i} - \mu)\} = \mathsf{D}\{y_{k_{n+1}}\} + \mathsf{D}\{\sum_{i=1}^{n} c_i y_{k_i}\} \geqq \tag{9}$$

$$\geqq \mathsf{D}\{y_{k_{n+1}}\} = \sigma^2\,.$$

From the inequality (9) follows that

$$\mathsf{D}\{x^0_{t+\Delta} - \operatorname{Pred} x^0_{t+\Delta}\} \geqq \sum_{k_1 \leqq \ldots \leqq k_n < k_{n+1}} \mathsf{P}\left\{\left[\frac{t_i}{\lambda} + \varphi\right] = k_i\,, \quad i = 1, \ldots, n+1\right\}.$$

$$\mathsf{D}\left\{x^0_{t+\Delta} - \operatorname{Pred} x^0_{t+\Delta} \,\middle|\, \left[\frac{t_i}{\lambda} + \varphi\right] = k_i\,, \quad i = 1, \ldots, n+1\right\} \geqq$$

$$\geqq \sigma^2 \sum_{k_1 \leqq \ldots \leqq k_n < k_{n+1}} \mathsf{P}\left\{\left[\frac{t_i}{\lambda} + \varphi\right] = k_i\,, \quad i = 1, \ldots, n+1\right\} =$$

$$= \sigma^2 \mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right]\right\}. \tag{10}$$

The probability $\mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right]\right\}$ may be found by means of the conditional probability $\mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right] \,\middle|\, \lambda\right\}$ with respect to λ. As λ and φ are independent, the probability $\mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right] \,\middle|\, \lambda\right\}$ can for all $\lambda > 0$ be chosen equal to the ordinary probability of $\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right]$ when λ is fixed and φ is distributed rectangularly over (0,1). The latter probability, however, as may be easily seen, equals 1 or $\frac{\Delta}{\lambda}$ according to whether $\Delta \geqq \lambda$ or $\Delta < \lambda$, respectively. Hence, in accordance with (6),

$$\mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right]\right\} = \int_0^\infty \mathsf{P}\left\{\left[\frac{t}{\lambda} + \varphi\right] < \left[\frac{t+\Delta}{\lambda} + \varphi\right] \,\middle|\, \lambda\right\} \mathrm{d}F(\lambda) =$$

$$= \int_0^\Delta \mathrm{d}F(\lambda) + \int_\Delta^\infty \frac{\Delta}{\lambda}\, \mathrm{d}F(\lambda) = F(\Delta) + \Delta \int_\Delta^\infty \mathrm{d}R'(\tau) =$$

$$= \Delta R'(\Delta) - R(\Delta) + 1 - \Delta R'(\Delta) = 1 - R(\Delta)\,.$$

Inserting this result in (10) we get (4).

Let us complete our proof by examining an arbitrary convex correlation function. Every convex correlation function is continuous for $0 < \tau < \infty$,[1])

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[1]) LITTELWOOD, HARDY, POLYA, „Inequalities". The possible discontinuity may occur in the point $\tau = 0$ only.

non-negative and non-increasing (see [1]), so that there exist limits $0 \leqq R_\infty \leqq \leqq R_{0+} \leqq 1$.

Let us introduce mutually independent stationary processes $y_t^0, v_t, z_t = z$ such that

$$\begin{aligned} \mathsf{D}y_t &= \sigma^2(R_{0+} - R_\infty)\,, \quad -\infty < t < \infty\,, \\ \mathsf{D}v_t &= \sigma^2(1 - R_{0+})\,, \\ \mathsf{D}z &= \sigma^2 R_\infty\,, \\ \mathrm{Cov}\,\{y_t, y_{t+\tau}\} &= \sigma^2(R_\tau - R_\infty)\,, \\ \mathrm{Cov}\,\{v_t, v_{t+\tau}\} &= 0\,, \quad \tau \neq 0\,, \end{aligned} \tag{10}$$

and put $x_t^0 = y_t + v_t + z$, $-\infty < t < \infty$. Process x_t^0 is obviously stationary and has the variance σ^2 and correlation function R_τ. Furthermore, owing to the independence of y_t, v_t and $z_t = z$, the relation

$$\begin{aligned} \mathsf{D}\{x_{t+\Delta}^0 - \mathrm{Pred}\, x_{t+\Delta}^0\} &= \mathsf{D}\{y_{t+\Delta} - \mathrm{Pred}\, y_{t+\Delta}\} + \\ &+ \mathsf{D}\{v_{t+\Delta} - \mathrm{Pred}\, v_{t+\Delta}\} + \mathsf{D}\{z - \mathrm{Pred}\, z\} \end{aligned} \tag{12}$$

holds. The component y_t possesses the correlation function $\dfrac{R_\tau - R_\infty}{R_{0+} - R_\infty}$ which is clearly continuous and tends to 0 when $\tau \to 0$. Applying the first part of our proof, we may therefore write

$$\mathsf{D}\{y_{t+\Delta} - \mathrm{Pred}\, y_{t+\Delta}\} \geqq \sigma^2(R_{0+} - R_\infty)\left(1 - \frac{R_\Delta - R_\infty}{R_{0+} - R_\infty}\right) = \sigma^2(R_{0+} - R_\infty)\,. \tag{13}$$

Further, as the component v_t is a stationary process with uncorrelated component random variables, $v_{t+\Delta}$ is uncorrelated with every prediction of the form (3), and accordingly

$$\mathsf{D}\{v_{t+\Delta} - \mathrm{Pred}\, v_{t+\Delta}\} \geqq \mathsf{D}\{v_{t+\Delta}\} = \sigma^2(1 - R_{0+})\,. \tag{14}$$

Now, inserting (13), (14) and $\mathsf{D}\{z - \mathrm{Pred}\, z\} \geqq 0$ in (12), we can see that the inequality (4) is proved for all convex correlation functions.

R e m a r k 1. The result applies obviously also to processes with discontinuous parameter $t = \ldots, -1, 0, 1, \ldots$.

R e m a r k 2. The lower bound established in (4) is exact. Indeed, if we take the convex correlation function

$$r(\tau) = \begin{cases} 1 - \tau\,, & \tau < 1\,, \\ 0\,, & \tau \geqq 1\,, \end{cases}$$

and $\Delta < 1$, then we obtain for

$$\text{Pred}\, x_{t+\Delta} = \mu + \sum_{i=0}^{n} x_{t-i} \left[\frac{n+2}{n+1}\left(1 - \frac{\Delta}{n+2}\right) - \frac{i+1}{n+1}\right] - \sum_{i=0}^{n} x_{t+\Delta-i} \frac{n-i}{n+1}$$

the residual variance $\mathsf{D}\{x_{t+\Delta} - \text{Pred}\, x_{t+\Delta}\} = \sigma^2 \frac{n+2}{n+1} \Delta \left[1 - \frac{\Delta}{n+2}\right]$, which tends to $\sigma^2 \Delta = \sigma^2(1 - r_\Delta)$ as $n \to \infty$; for $\Delta > 1$ the right side of (4) is exactly atained when we put $\text{Pred}\, x_{t+\Delta} = \mu$.

REFERENCES

[1] *J. Hájek:* Линейная оценка средней стационарного случайного просесса с выпуклой корреляционной фуикцией, Czechoslovak mathematical journal, *6* (*81*), 1956, 94—117.

Резюме

ПРОГНОЗ СТАЦИОНАРНОГО ПРОЦЕССА С ВЫПУКЛОЙ КОРРЕЛЯЦИОННОЙ ФУНКЦИЕЙ

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Теорема. *Пусть дан стационарный случайный процесс* $\{x_t, -\infty < t < \infty\}$; *обозначим его среднее значение через* μ, *дисперсию через* σ^2 *и корреляционную функцию через* R_τ; *возьмем произвольный линейный прогноз состояния процесса в моменте* $t + \Delta$,

$$\text{Pred}\, x_{t+\Delta} = \mu + \sum_{i=1}^{n} c_i(x_{t_i} - \mu)\,, \quad t_1 < \ldots < t_n + t < t + \Delta,\ n \geqq 1\,. \tag{3}$$

Если корреляционная функция R_τ *выпукла, то имеет место*

$$\mathsf{D}\{x_{t+\Delta} - \text{Pred}\, x_{t+\Delta}\} \geqq \sigma^2(1 - R_\Delta)\,. \tag{4}$$

Нижняя граница для остаточной дисперсии, определяемая неравенством (4), лишь в $1 + R_\Delta$ раз меньше остаточной дисперсии (2) прогноза (1), основанного только на последнем наблюдении x_t. Итак, если корреляционная функция выпукла, то при образовании линейного прогноза можно ограничиться лишь последним наблюдением, так как использование любого числа дальнейших предыдущих наблюдений не может привести к существенному улучшению точности.