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GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND CONTINUOUS DEPENDENCE ON A PARAMETER

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0. Introduction

The starting point of this investigation is the following theorem:

Let the functions $f_k(x, t)$, $k = 0, 1, 2, \dots$ be defined and continuous for $(x, t) \in G \times \langle 0, T \rangle$, $0 \in G \subset E_n$, G open, $f_k(x, t) \in E_n$.[1]) Let us denote by $x_k(t)$ the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_k(x, t) \,, \quad x(0) = 0 \tag{0,01}$$

and let us suppose that there is a unique solution $x_0(t)$ and this solution is defined for $0 \leqq t \leqq T$.

Theorem 0,1. *Let the following conditions be fulfilled:*

$$F_k(x, t) = \int_0^t f_k(x, \tau) \,\mathrm{d}\tau \to \int_0^t f_0(x, \tau) \,\mathrm{d}\tau = F_0(x, t) \tag{0,02}$$

uniformly with $k \to \infty$,

$f_k(x, t)$ *are equicontinuous functions of* x *for fixed* k *and* t. (0,03)

Then $x_k(t)$ *are defined on* $\langle 0, T \rangle$ *for* k *great enough and* $x_k(t) \to x_0(t)$ *uniformly with* $k \to \infty$.

Theorem 0,1 is contained in the paper [2] of the author and Z. Vorel in a little more general form.[2]) An analogous theorem was formerly proved in [1] by Krasnoselskij and Krejn under the additional assumption that $f_k(x, t)$ are uniformly bounded.

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[1]) By $\langle 0, T \rangle$ we denote the closed interval from 0 to T. E_n denotes the n-dimensional Euclidian space.

[2]) See [2], Theorem 1.

Let us apply this theorem to an elementary example

$$\frac{dx}{dt} = xk^{1-\alpha}\cos kt + k^{1-\beta}\sin kt = f_k(x, t), \quad x(0) = 0,\ k = 1, 2, 3, \dots,$$

$$\frac{dx}{dt} = 0 = f_0(x, t)\,, \quad x(0) = 0\,. \tag{0,04}$$

The question arises for which $\alpha \leqq 1$ and $\beta \leqq 1$, $x_k(t) \to 0$ almost uniformly with $k \to \infty$. According to Theorem 0,1 $x_k(t) \to 0$ almost uniformly for $0 < \beta \leqq 1$, $\alpha = 1$ (according to the theorem of Krasnoselskij and Krejn for $\alpha = 1 = \beta$). By direct computation we obtain

$$x_k(t) = \exp\{k^{-\alpha}\sin kt\}\int_0^t \exp\{-k^{-\alpha}\sin k\tau\}\,k^{1-\beta}\sin k\tau\,d\tau\,, \qquad k = 1, 2, 3, \dots.$$

As

$$\int_0^t \exp\{-k^{-\alpha}\sin k\tau\}\,k^{1-\beta}\sin k\tau\,d\tau =$$

$$= k^{-\beta}(1 - \cos kt) - k^{1-\alpha-\beta}\int_0^t \sin^2 k\tau\,d\tau + O(k^{1-2\alpha-\beta})\,,$$

it follows that $x_k(t) \to 0$ almost uniformly with $k \to \infty$ if $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$. (If $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta = 1$ then $x_k(t) \to -\frac{1}{2}t$ almost uniformly). This fact seems to be significant and our purpose is to discover theoretic reasons of such convergence effects.

Let us emphasize another fact in connection with Theorem 0,1. It follows from the proof of Theorem 0,1, that $\{x_k(t)\}$ is a sequence of equicontinuous functions, if (0,03) holds, if $\int_0^t f_k(x, \tau)\,d\tau = F_k(x, t) \to F_0(x, t)$ uniformly while $F_0(x, t)$ need not admit the representation $F_0(x, t) = \int_0^t f_0(x, \tau)\,d\tau$ with $f_0(x, t)$ continuous. Is it possible to associate with $F_0(x, t)$ a generalized differential equation

$$\frac{dx}{dt} = DF_0(x, t) \tag{0,05}$$

in such a way that $x_0(t)$ is a solution of (0,05) if there exists such a subsequence $\{k_j\}$ that $x_{k_j}(t) \to x_0(t)$ with $j \to \infty$? In such case the fact that $x_{k_j}(t) \to x_0(t)$ means nothing else than that the solution $x_k(t)$ depends continuously on the parameter k. This question may be answered affirmatively by means of methods which are analogous to those used in this paper. However, our object is to include in the theory of generalised differential equations the convergence effect of equations (0,04). Let us fix the numbers α, β, $0 < \alpha < 1$, $0 < \beta < 1$,

$\alpha + \beta < 1$. In this case condition (0,03) is not fulfilled. On the other hand the functions

$$F_k(x, t) = \int_0^t f_k(x, \tau)\, \mathrm{d}\tau = xk^{-\alpha} \sin kt + k^{-\beta}(1 - \cos kt)$$

fulfil the condition

$$|F_k(x, t_2) - F_k(x, t_1)| \leqq K|t_2 - t_1|^\gamma , \quad \gamma = \min(\alpha, \beta) ,$$

if $|x| \leqq 1$ and K does not depend on k.

In order to be able to formulate the chief results, we shall describe the way, in which we associate the generalized equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{0,06}$$

with $F(x, t)$.

If $U(\tau, t)$ is a real-valued function defined for $\tau_* \leqq \tau \leqq \tau^*$, $\tau_* \leqq t \leqq \tau^*$ we define the set of major functions and the set of minor functions and in a way known from the theory of the Perron integral we define the integral of $\mathrm{D}U$ from τ_1 to τ_2 and denote it by $\int_{\tau_1}^{\tau_2} \mathrm{D}U$. It turns out, that if $U(\tau, t) = f(\tau) \,.\, \varphi(t)$ and if $\varphi(t)$ is a function of bounded variation, then $\int_{\tau_1}^{\tau_2} \mathrm{D}U$ exists if and only if $\int_{\tau_1}^{\tau_2} f(\tau)\, \mathrm{d}\varphi(\tau)$ exists in the sense of Perron and that $\int_{\tau_1}^{\tau_2} \mathrm{D}U = \int_{\tau_1}^{\tau_2} f(\tau)\, \mathrm{d}\varphi(\tau)$. In section 1 we give two equivalent definitions of $\int_{\tau_1}^{\tau_2} \mathrm{D}U$ and prove some fundamental theorems about this concept of integral.

In section 2 we define the generalised differential equations. Let $F(x, t)$ be defined for $x \in G \subset E_n$, $t \in \langle 0, T\rangle$, $F(x, t) \in E_n$. We say that the function $x(\tau)$ defined for $\tau \in \langle \tau_1, \tau_2\rangle \subset \langle 0, T\rangle$ is a solution of (0,06), if

$$x(\tau_4) = x(\tau_3) + \int_{\tau_3}^{\tau_4} \mathrm{D}F(x(\tau), t) , \quad \tau_3, \tau_4 \in \langle \tau_1, \tau_2\rangle \tag{0,07}$$

($F(x(\tau), t)$ is a vector function with components $F_1, \ldots, F_n$ and $\int_{\tau_3}^{\tau_4} \mathrm{D}F$ is a vector with components $\int_{\tau_3}^{\tau_4} \mathrm{D}F_1, \ldots, \int_{\tau_3}^{\tau_4} \mathrm{D}F_n$). If $\frac{\partial F}{\partial t} = f(x, t)$ is continuous and if $y(\tau)$ is continuous, then

$$\int_{\tau_3}^{\tau_4} \mathrm{D}F(y(\tau), t) = \int_{\tau_3}^{\tau_4} f(y(\tau), \tau)\, \mathrm{d}\tau$$

and every solution of (0,06) is at the same time a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau)$ and conversely.

In section 3 we find some sufficient conditions for the existence of $\int\limits_{\tau_1}^{\tau_2} DU$. In section 4 we develop the theory of generalized differential equations.

By $\mathbf{F} = \mathbf{F}(G \times \langle 0, T\rangle, K_2\eta^{\beta_1}, K_1\eta^{\alpha_1}, \sigma), \alpha_1, \beta_1, \sigma, K_1, K_2 > 0$, we denote the set of functions $F(x, t)$ which are defined for $(x, t) \in G \times \langle 0, T\rangle$, $F(x, t) \in E_n$ and fulfil the conditions

$$\|F(x, t_2) - F(x, t_1)\| \leqq K_2|t_2 - t_1|^{\beta_1}, \text{ if } x \in G,\ t_1, t_2 \in \langle 0, T\rangle,\ |t_2 - t_1| \leqq \sigma,$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\, K_1|t_2 - t_1|^{\alpha_1},$$

if $x_1, x_2 \in G$, $t_1, t_2 \in \langle 0, T\rangle$, $\|x_2 - x_1\| \leqq 2K_2\sigma^{\beta_1}$, $|t_2 - t_1| \leqq \sigma$. Let us suppose that $F(x, t) \in \mathbf{F}$ and that $\alpha_1 + \beta_1 > 1$. $x(\tau)$ is said to be a regular solution of (0,06) on $\langle \tau_1, \tau_2\rangle \subset \langle 0, T\rangle$, if (0,07) holds for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2\rangle$ and if there exists such a function $\sigma_2(\tau_0) > 0$, $\tau_0 \in \langle \tau_1, \tau_2\rangle$ that

$$\|x(\tau_4) - x(\tau_3)\| \leqq 2K_2|\tau_4 - \tau_3|^{\beta_1},\ \tau_3, \tau_4 \in \langle \tau_0 - \sigma_2(\tau_0), \tau_0 + \sigma_2(\tau_1)\rangle \cap \langle \tau_1, \tau_2\rangle .$$

We prove: 1. If $x_0 \in G$, $\tau_0 \in \langle 0, T\rangle$, then (0,06) has a solution $x(\tau)$ regular in a neighbourhood of τ_0, $x(\tau_0) = x_0$.

2. If $F_k(x, t) \in \mathbf{F}$, $k = 0, 1, 2, \ldots$, $F_k \to F_0$ uniformly with $k \to \infty$ and if there is a unique solution $x_0(\tau)$ of

$$\frac{dx}{d\tau} = DF_0(x, t)\,, \quad x(0) = x_0 \in G\,, \tag{0,07}$$

which is regular on $\langle 0, T\rangle$, then for k great enough solutions $x_k(\tau)$ of

$$\frac{dx}{d\tau} = DF_k(x, t)\,, \quad x(0) = x_0 \in G\,, \tag{0,08}$$

exist which are regular on $\langle 0, T\rangle$. These solutions need not be unique but $x_k(\tau) \to x_0(\tau)$ uniformly in any case.

If we wish to apply these results to equations (0,04), we put $\alpha_1 = \alpha$, $\beta_1 = \min(\alpha, \beta)$, and obtain that $x_k(t) \to x_0(t)$ almost uniformly if $0 < \alpha_1 \leqq 1$, $0 < \beta_1 \leqq 1$, $\alpha_1 + \beta_1 > 1$, i. e. if $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$, $\alpha > \frac{1}{2}$. Naturally every solution of (0,04) has a continuous derivative and is therefore regular.[3]) However, our theory does not explain the convergence of solutions of (0,04) fully, as we obtain the additional condition $\alpha > \frac{1}{2}$.

In section 5 we give applications of this theory to linear equations and to a convergence result in the classical theory of differential equations.

Finally let us note, that the results of section 4 admit an interpretation in terms of the theory of distributions. If $F(x, t) \in \mathbf{F}$, $\alpha_1 + \beta_1 > 1$, let us put $f(x, t) = \dfrac{\partial F}{\partial t}$, where the derivative is taken in the sense of the theory of

[3]) If $\beta_1 = 1$ it may prove necessary to enlarge K_2.

distributions, that means $f(x, t)$ is a (special) distribution for fixed x. If $y(\tau) \in G$ for $\tau \in \langle \tau_1, \tau_2 \rangle$ and if

$$\|y(\tau_4) - y(\tau_3)\| \leqq K|\tau_4 - \tau_3|^\gamma , K > 0, \gamma > 1 - \alpha_1, \quad \tau_4, \tau_3 \in \langle \tau_1, \tau_2 \rangle , \quad (0{,}09)$$

then $g(\zeta) = \int_{\tau_0}^{\zeta} \mathrm{D}F(y(\tau), t)$ exists for τ_0, $\zeta \in \langle \tau_1, \tau_2 \rangle$ and is continuous (see Theorems 3,1 and 1,3,6) and we may define

$$f(y(t), t) = \frac{\mathrm{d}}{\mathrm{d}t} g(t) .$$

$y(t)$ is said to be a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = f(y(t), t) , \quad (0{,}10)$$

if (0,09) holds and if the distributions on both sides of (0,10) are equal. It can be shown, that $y(t)$ is a solution of (0,10) if and only if $y(\tau)$ is a regular solution of (0,06).

1. The Generalized Perron Integral

1,1. Definition of the integral. Let $\tau_* < \tau^*$. Let us denote by $\mathbf{S} = \mathbf{S}\langle \tau_*, \tau^* \rangle$ the system of sets $S \subset E_2$ having the following property: for every $\tau, \tau_* \leqq \leqq \tau \leqq \tau^*$, there exists such a $\delta = \delta(\tau) > 0$ that $(\tau, t) \in S$ if $\tau - \delta(\tau) \leqq t \leqq \leqq \tau + \delta(\tau), \tau_* \leqq t \leqq \tau^*$. Let the function $U(\tau, t)$ be defined on some $S \in \mathbf{S}$, real-valued, finite.

Definition 1,1,1. *A real-valued finite function $M(\tau)$, $\tau^* \leqq \tau \leqq \tau^*$ is a major function of U, if there exists such a set $S_1 \subset S$, $S_1 \in \mathbf{S}$ that*

$$(\tau - \tau_0)(M(\tau) - M(\tau_0)) \geqq (\tau - \tau_0)(U(\tau_0, \tau) - U(\tau_0, \tau_0)) \qquad (1{,}1{,}1)$$

for $(\tau_0, \tau) \in S_1$. The set of major functions of U is denoted by $\mathbf{M}(U)$. A function $m(\tau)$, $\tau_* \leqq \tau \leqq \tau^*$ is called a minor function of U, if $-m(\tau) \in \mathbf{M}(-U)$. The set of minor functions of U is denoted by $\mathbf{m}(U)$.

Definition 1,1,2. *Let $\mathbf{M}(U) \neq \emptyset \neq \mathbf{m}(U)$. The lower bound of numbers $M(\tau^*) - M(\tau_*)$ where $M(\tau) \in \mathbf{M}(U)$, is called the upper Perron integral of* DU *from τ_* to τ^* and denoted by $^{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U$. Similarly the upper bound of numbers $m(\tau^*) - m(\tau_*)$ where $m(\tau) \in \mathbf{m}(U)$ is called the lower Perron integral of* DU *and denoted by $_{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U$.*

Obviously $^{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U = - {}_{-}\int_{\tau_*}^{\tau^*} \mathrm{D}(-U)$.

Note 1,1,1. Naturally the symbol DU alone has no meaning, only the upper (lower) integral of DU is defined.

Our next task is to justify the terms upper and lower integral. In order to prove that ${}^{-}\!\!\int \geqq {}_{-}\!\!\int$ let us introduce the following concept:

Definition 1,1,3. *Let us denote by A a finite sequence of numbers $(\alpha_0, \tau_1, \alpha_1, \tau_2, \alpha_2, \ldots, \tau_k, \alpha_k)$, $\tau_* = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \tau^*$, $\alpha_0 \leqq \tau_1 \leqq \alpha_1 \leqq \tau_2 \leqq \alpha_2 \leqq \ldots \ldots \leqq \tau_k \leqq \alpha_k$. A is called a subdivision of $\langle \tau_*, \tau^* \rangle$ subordinate to $S \in \mathbf{S}$ if $(\tau_j, t) \in S$ for $\alpha_{j-1} \leqq t \leqq \alpha_j$, $j = 1, 2, \ldots, k$. The set of all subdivisions A of $\langle \tau_*, \tau^* \rangle$ subordinate to S is denoted by $\mathbf{A}(S)$.*

Lemma 1,1,1. $\mathbf{A}(S) \neq \emptyset$.

Proof: Let $\delta(\tau)$ be such a positive function that $(\tau, t) \in S$ for $\tau_* \leqq \tau \leqq \tau^*$, $\tau - \delta(\tau) \leqq t \leqq \tau + \delta(\tau)$. From the system of intervals $(\tau - \delta(\tau), \tau + \delta(\tau))$ we can extract a finite minimal system of intervals $J_j = (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$, $j = 1, 2, \ldots, r$, which covers $\langle \tau_*, \tau^* \rangle$ (i. e.: if we remove any arbitrary of the intervals J_j, then the new system does not cover $\langle \tau_*, \tau^* \rangle$). As the numbers τ_j are different, let $\tau_1 < \tau_2 < \ldots < \tau_r$. It follows that $J_j \cap J_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, r - 1$, $J_j \cap J_i = \emptyset$ for $|i - j| > 1$, $i, j = 1, 2, \ldots, r$, $J_j \neq J_j \cap J_{j+1} \neq J_{j+1}$ for $j = 1, 2, \ldots, r - 1$, $\tau_* \in J_1$, $\tau^* \in J_r$. Hence we find that there exist such numbers $\tau_* = \alpha_0 < \alpha_1 < \ldots < \alpha_r = \tau^*$ that $\alpha_j \in J_j \cap J_{j+1}$, $\tau_j \leqq \alpha_j \leqq \tau_{j+1}$, $j = 1, 2, \ldots, r - 1$. The sequence $(\alpha_0, \tau_1, \alpha_1, \ldots, \tau_r, \alpha_r)$ belongs to $\mathbf{A}(S)$ as $(\tau_{j+1}, t) \in S$ for $\alpha_j \leqq t \leqq \alpha_{j+1}$, $\alpha_j \leqq \tau_{j+1} \leqq \alpha_{j+1}$, $j = 1, 2, \ldots, r - 1$.

We now pass to the following

Lemma 1,1,2. If $\mathbf{M}(U) \neq \emptyset \neq \mathbf{m}(U)$, *then* ${}_{-}\!\!\int_{\tau_*}^{\tau^*} \mathrm{D}U \leqq {}^{-}\!\!\int_{\tau_*}^{\tau^*} \mathrm{D}U$.

Proof. Let us choose $M(\tau) \in \mathbf{M}(U)$, $m(\tau) \in \mathbf{m}(U)$. Assuming that (1,1,1) holds for $(\tau_0, \tau) \in S_1$ and that analogously

$$(\tau - \tau_0)(m(\tau) - m(\tau_0)) \leqq (\tau - \tau_0)(U(\tau_0, \tau) - U(\tau_0, \tau_0)) \tag{1,1,2}$$

holds for $(\tau_0, \tau) \in S_2$, let us choose a subdivision $A \in \mathbf{A}(S_1 \cap S_2)$. As $(\tau_j, t) \in S_1 \cap S_2$ for $\alpha_{j-1} \leqq t \leqq \alpha_j$, we get, according to (1,1,1)

$$M(\alpha_j) - M(\tau_j) \geqq U(\tau_j, \alpha_j) - U(\tau_j, \tau_j)\,,$$
$$M(\tau_j) - M(\alpha_{j-1}) \geqq U(\tau_j, \tau_j) - U(\tau_j, \alpha_{j-1})$$

and

$$M(\alpha_j) - M(\alpha_{j-1}) \geqq U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\,.$$

Analogously

$$m(\alpha_j) - m(\alpha_{j-1}) \leqq U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})$$

and consequently

$$M(\alpha_j) - M(\alpha_{j-1}) \geqq m(\alpha_j) - m(\alpha_{j-1})\,.$$

Adding for $j = 1, 2, \ldots, r$ we get

$$M(\tau^*) - M(\tau_*) \geqq m(\tau^*) - m(\tau_*) \text{ and } {}^{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U \geqq {}_{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U .$$

The proof of lemma (1,1,2) is complete.

Definition 1,1,4. *If* $\mathbf{M}(U) \neq \emptyset \neq \mathbf{m}(U)$, ${}^{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U = {}_{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U$, *the generalized Perron integral* $\int_{\tau_*}^{\tau^*} \mathrm{D}U$ *of* $\mathrm{D}U$ *from* τ_* *to* τ^* *is defined by means of the equation* $\int_{\tau_*}^{\tau^*} \mathrm{D}U = {}_{-}\int_{\tau_*}^{\tau^*} \mathrm{D}U$. *In this case the function* U *is called integrable (in the sense of Perron) on* $\langle \tau_*, \tau^* \rangle$ *and the set of integrable functions is denoted by* $\mathfrak{P}$, $\mathfrak{P}\langle \tau_*, \tau^* \rangle$ *respectively.*

Note 1,1,2. Let $f(\tau)$ be a real-valued finite function and let us put $U(\tau, t) = f(\tau)\, t$, $\tau_* \leqq \tau \leqq \tau^*$, $-\infty < t < \infty$. It follows that the integral $\int_{\tau_*}^{\tau^*} \mathrm{D}U$ exists if and only if the Perron integral $\int_{\tau_*}^{\tau^*} f(\tau)\, \mathrm{d}\tau$ exists (in the usual sense) and that in this case both integrals are equal.

It is well known that the Perron integral $\int_{\tau_*}^{\tau^*} f(\tau)\, \mathrm{d}\tau$ is defined in a very similar manner to $\int_{\tau_*}^{\tau^*} \mathrm{D}U$. A finite function $M(\tau), \tau_* \leqq \tau \leqq \tau^*$ is called a major function of f, if $\underline{\mathrm{D}}M(\tau) \geqq f(\tau)$ for $\tau_* \leqq \tau \leqq \tau^*$, where $\underline{\mathrm{D}}M(\tau)$ is the lower derivative of $M(\tau)$. By $\mathbf{H}(f)$ is denoted the set of major functions of f. It is almost obvious that

1. $\mathbf{M}(U) \subset \mathbf{H}(f)$, 2. if $M(\tau) \in \mathbf{H}(f)$, then $M(\tau) + \varepsilon\tau \in \mathbf{M}(U)$ for every $\varepsilon > 0$.

An analogous situation holds for the minor functions, whence the assertion follows immediately.

The same assertion holds also for the Perron integral $\int_{\tau_*}^{\tau^*} f(\tau)\, \mathrm{d}\varphi(\tau)$ where $\varphi(\tau)$ is of bounded variation (we put $U(\tau, t) = f(\tau) \,.\, \varphi(t)$).

Note 1,1,3. If $\tau_* = \tau^*$ we put $\int_{\tau_*}^{\tau^*} \mathrm{D}U = 0$. If $\tau_* > \tau^*$, then we put $\int_{\tau_*}^{\tau^*} \mathrm{D}U = - \int_{\tau^*}^{\tau_*} \mathrm{D}U$.

1,2. An equivalent definition of $\int_{\tau_*}^{\tau^*} \mathrm{D}U$. Let the function $U(\tau, t)$ be defined for $(\tau, t) \in S_1 \in \mathbf{S}$. To every subdivision $A \in \mathbf{A}(S_1)$ there corresponds the number

$$B(A) = \sum_{j=1}^{k} [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] .$$

Definition 1,2,1. *The function $U(\tau, t)$ belongs to the set $\mathfrak{B}$ if to every $\varepsilon > 0$ there exists such a set $S \in \mathbf{S}$, $S \subset S_1$, that $|B(A_1) - B(A_2)| < \varepsilon$ if $A_1, A_2 \in \mathbf{A}(S)$.*

The following lemma is obvious

Lemma 1,2,1. *If $U \in \mathfrak{B}$, then there exists a unique number, which will be denoted by $(b) \int_{\tau_*}^{\tau^*} \mathrm{D}U$, with the following property: to every $\varepsilon > 0$ there exists such a $S \in \mathbf{S}$ that $|(b) \int_{\tau_*}^{\tau^*} \mathrm{D}U - B(A)| < \varepsilon$ for $A \in \mathbf{A}(S)$.*

We show, that Definition 1,2,1 and Lemma 1,2,1 give an equivalent definition of $\int_{\tau_*}^{\tau^*} \mathrm{D}U$.

Theorem 1,2,1. *α) $\mathfrak{P} = \mathfrak{B}$. β) If $U \in \mathfrak{P}$, then* $(\mathrm{b}) \int_{\tau_*}^{\tau^*} \mathrm{D}U = \int_{\tau_*}^{\tau^*} \mathrm{D}U$.

Proof. Let $U \in \mathfrak{P}$, $\varepsilon > 0$. Let us choose $M(\tau) \in \mathbf{M}(U)$, $m(\tau) \in \mathbf{m}(U)$ in such a way that

$$M(\tau^*) - M(\tau_*) - \tfrac{1}{2}\varepsilon < \int_{\tau_*}^{\tau^*} \mathrm{D}U < m(\tau^*) - m(\tau_*) + \tfrac{1}{2}\varepsilon \qquad (1,2,1)$$

that (1,1,1) holds for $(\tau_0, \tau) \in S_1$ and that (1,1,2) holds for $(\tau_0, \tau) \in S_2$. Let us choose a subdivision $A \in \mathbf{A}(S_1 \cap S_2)$. Similarly as in the proof of Lemma 1,1,2 we get

$$M(\alpha_j) - M(\alpha_{j-1}) \geqq U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1}) \geqq m(\alpha_j) - m(\alpha_{j-1}) .$$

Hence $M(\tau^*) - M(\tau_*) \geqq B(A) \geqq m(\tau^*) - m(\tau_*)$. According to (1,2,1) we have

$$\int_{\tau_*}^{\tau^*} \mathrm{D}U + \tfrac{1}{2}\varepsilon > B(A) > \int_{\tau_*}^{\tau^*} \mathrm{D}U - \tfrac{1}{2}\varepsilon \quad \text{for} \quad A \in \mathbf{A}(S_1 \cap S_2) .$$

It follows that $\mathfrak{P} \subset \mathfrak{B}$ and that β) holds. It remains to prove that $\mathfrak{B} \subset \mathfrak{P}$.

Let $U \in \mathfrak{B}$. Let us choose such a $S \in \mathbf{S}$ that

$$|B(A_1) - B(A_2)| < \tfrac{1}{2}\varepsilon \quad \text{for} \quad A_1, A_2 \in \mathbf{A}(S) . \qquad (1,2,2)$$

For $\tau_* < \tau \leqq \tau^*$ let A_τ denote a subdivision of $\langle \tau_*, \tau \rangle$ subordinate to S. The set of subdivisions of $\langle \tau_*, \tau \rangle$ subordinate to S, is denoted by $\mathbf{A}(S)\langle \tau_*, \tau \rangle$. Let us put

$$\left.\begin{aligned} M(\tau) &= \sup_{A_\tau} B(A_\tau) , \\ m(\tau) &= \inf_{A_\tau} B(A_\tau) , \end{aligned}\right\} \text{ where } A_\tau \in \mathbf{A}(S)\langle \tau_*, \tau \rangle, \tau_* < \tau \leqq \tau^* ,$$

$$M(\tau_*) = 0 = m(\tau_*) .$$

As (1,2,2) holds and as according to Lemma 1,1,1 a subdivision $A^{(\tau)} \in \mathbf{A}(S) \langle \tau, \tau^* \rangle$ exists for $\tau < \tau^*$, it follows that $M(\tau), m(\tau)$ are finite and that $|M(\tau^*) - m(\tau^*)| \leqq \tfrac{1}{2}\varepsilon$.

Let us choose a $\tau_0, \tau_* \leqq \tau_0 \leqq \tau^*$ and let $\delta(\tau_0) > 0$ be such a number that $(\tau_0, \tau) \in S$ for $\tau \in \langle \tau_0 - \delta(\tau_0), \tau_0 + \delta(\tau_0) \rangle \cap \langle \tau_*, \tau^* \rangle$. According to the definition of $M(\tau)$ we have

$$M(\tau) \geqq M(\tau_0) + U(\tau_0, \tau) - U(\tau_0, \tau_0) \quad \text{for} \quad \tau_0 \leqq \tau \leqq \tau_0 + \delta(\tau_0)\,,$$
$$M(\tau) \leqq M(\tau_0) - [U(\tau_0, \tau_0) - U(\tau_0, \tau)] \quad \text{for} \quad \tau_0 - \delta(\tau_0) \leqq \tau \leqq \tau_0\,.$$

This signifies $M(\tau) \in \mathbf{M}(U)$. Analogously $m(\tau) \in \mathbf{m}(U)$. Theorem 1,2,1 is proved.

Note 1,2,1. Let $\mathbf{S}^*$ be the system of sets S^*_σ, $\sigma > 0$, where (τ, t) belongs to S^* if the following inequalities are fulfilled: $\tau_* \leqq \tau \leqq \tau^*$, $\tau - \sigma \leqq t \leqq \tau + \sigma$, $\tau_* \leqq t \leqq \tau^*$. Let us replace the system $\mathbf{S}$ in Definition 1,4,1 by the system $\mathbf{S}^*$. Again a Lemma analogous to Lemma 1,2,1 holds and the number which in this way corresponds to some functions $U(\tau, t)$ may be called the generalized Riemann integral. In fact if $U(\tau, t) = f(\tau)\, t$, we get the usual Riemann integral $\int_{\tau_*}^{\tau^*} f(\tau)\, \mathrm{d}\tau$. It follows from Theorem 1,2,1 that the Perron integral exists if the Riemann integral exists and that both integrals are equal. This assertion holds in the usual form and in the generalized form as well.

1,3. Fundamental theorems. The following theorems will be deduced from Theorem 1,2,1. It is easy to prove these theorems directly from Definition 1,1,4 as well.

Theorem 1,3,1. *If* $U \in \mathfrak{P}$, $\beta \in E_1$, *then* $\int_{\tau_*}^{\tau^*} \mathrm{D}\beta U = \beta \int_{\tau_*}^{\tau^*} \mathrm{D}U$.

Theorem 1,3,2. *If* $U_1, U_2 \in \mathfrak{P}$, *then* $U_1 + U_2 \in \mathfrak{P}$ *and*

$$\int_{\tau_*}^{\tau^*} \mathrm{D}(U_1 + U_2) = \int_{\tau_*}^{\tau^*} \mathrm{D}U_1 + \int_{\tau_*}^{\tau^*} \mathrm{D}U_2\,.$$

Theorem 1,3,3. *If* $\tau_* \leqq \tau_1 < \tau_2 \leqq \tau^*$, $U \in \mathfrak{P}\langle \tau_*, \tau^* \rangle$, *then* $U \in \mathfrak{P}\langle \tau_1, \tau_2 \rangle$.

Theorems 1,3,1 and 1,3,2 are obvious. Theorem 1,3,3 is proved by means of Lemma 1,1,1.

Theorem 1,3,4. *If* $\tau_* < \tau < \tau^*$, $U \in \mathfrak{P}\langle \tau_*, \tau \rangle$, $U \in \mathfrak{P}\langle \tau, \tau^* \rangle$, *then* $U \in \mathfrak{P}\langle \tau_*, \tau^* \rangle$ *and*

$$\int_{\tau_*}^{\tau} \mathrm{D}U + \int_{\tau}^{\tau^*} \mathrm{D}U = \int_{\tau_*}^{\tau^*} \mathrm{D}U. \tag{1,3,1}$$

Proof. Let $\varepsilon > 0$. Let us find such a set $S_1 \in \mathbf{S}$ that $|B(A_\tau) - B(\bar{A}_\tau)| < \frac{\varepsilon}{2}$ for $A_\tau, \bar{A}_\tau \in \mathbf{A}(S_1)\langle \tau^*, \tau \rangle$ and such a set $S_2 \in \mathbf{S}$ that $|B(A^{(\tau)}) - B(\bar{A}^{(\tau)})| < \frac{\varepsilon}{2}$ for $A^{(\tau)}, \bar{A}^{(\tau)} \in \mathbf{A}(S_2)\langle \tau, \tau^* \rangle$. Let Q_1 be a closed square the vertices of which are (τ_*, τ_*), (τ, τ_*), (τ, τ), (τ_*, τ) and let Q_2 be a closed square with the vertices (τ, τ), (τ^*, τ), (τ^*, τ^*), (τ, τ^*). We put $S = (S_1 \cup Q_1) \cup (S_2 \cap Q_2)$.

Let $A \in \mathbf{A}(S)$, $A = (\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k)$. There exists necessarily such an index r that either $\alpha_r = \tau$ or $\tau_r = \tau$. If $\alpha_j \neq \tau$, $j = 1, 2, \ldots, k$, then $\tau_r = \tau$ and

$\alpha_{r-1} < \tau_r < \alpha_r$. Let us divide the interval $\langle\alpha_{r-1}, \alpha_r\rangle$ into the intervals $\langle\alpha_{r-1}, \alpha'_r\rangle$ and $\langle\alpha'_r, \alpha_r\rangle$, where $\alpha'_r = \tau_r$ and let us consider the subdivision $\bar{A} = (\alpha_0, \tau_1, \alpha_1, \ldots \alpha_{r-1}, \tau_r, \alpha'_r, \tau_r, \alpha_r, \ldots, \tau_k, \alpha_k)$. In any case to each $A \in \mathbf{A}(S)$ there exist $A_\tau \in \mathbf{A}(S_1)\langle\tau_*, \tau\rangle$ and $A^{(\tau)} \in \mathbf{A}(S_2)\langle\tau, \tau^*\rangle$ in such a way that $B(A) = B(A_\tau) + B(A^{(\tau)})$. If we take another subdivision $\tilde{A} \in \mathbf{A}(S)$, then there exist $\tilde{A}_\tau, \tilde{A}^{(\tau)}$ in such a way that $B(\tilde{A}) = B(\tilde{A}_\tau) + B(\tilde{A}^{(\tau)})$. Consequently $|B(A) - B(\tilde{A})| \leq |B(A_\tau) - B(\tilde{A}_\tau| + |B(A^{(\tau)}) - B(\tilde{A}^{(\tau)})| < \varepsilon$ and $U \in \mathfrak{P}\langle\tau_*, \tau^*\rangle$; (1,3,1) is obvious.

The following theorem may be proved by means of the same method in a little more elaborate way.

Theorem 1,3,5. *Let the function $U(\tau, t)$ be defined on a set $S \in \mathbf{S}$, let $U \in \mathfrak{P}\langle\tau_*, \tau\rangle$ for $\tau_* < \tau < \tau^*$ and let there exist the finite limit*

$$\lim_{\tau\to\tau^*-} [\int_{\tau_*}^{\tau} \mathrm{D}U - U(\tau^*, \tau) + U(\tau^*, \tau^*)] = L\,.$$

Then $U \in \mathfrak{P}\langle\tau_*, \tau^*\rangle$ and $\int_{\tau_*}^{\tau^*} \mathrm{D}U = L$.

Theorem 1,3,6. *Let $U \in \mathfrak{P}\langle\tau_*, \tau^*\rangle$, $\tau_0 \in \langle\tau_*, \tau^*\rangle$. Then*

$$\lim_{\substack{\tau\to\tau_0\\ \tau\in\langle\tau_*,\tau^*\rangle}} [\int_{\tau_*}^{\tau} \mathrm{D}U - U(\tau_0, \tau) + U(\tau_0, \tau_0)] = \int_{\tau_*}^{\tau_0} \mathrm{D}U\,.$$

Specially $\int_{\tau_*}^{\tau} \mathrm{D}U$ depends continuously on τ at $\tau = \tau_0$ if and only if $U(\tau_0, t)$ depends continuously on t at τ_0.

Proof. Let $\varepsilon > 0$ and let us choose a set $S \in \mathbf{S}$ according to Definition 1,2,1. Let $(\tau_0, \tau) \in S$ for $\tau \in \langle\tau_0 - \delta, \tau_0 + \delta\rangle \cap \langle\tau_*, \tau^*\rangle$. Then $|\int_{\tau_0}^{\tau} \mathrm{D}U - [U(\tau_0, \tau) - U(\tau_0, \tau_0)]| \leq \varepsilon$ if $\tau \in \langle\tau_0 - \delta, \tau_0 + \delta\rangle \cap \langle\tau_*, \tau^*\rangle$. Hence the proof follows immediately.

Theorem 1,3,7. *Let the function $U(\tau, t)$ be defined on $S \in \mathbf{S}$ and let*

$$\lim_{\tau\to\tau_0} \frac{U(\tau_0, \tau) - U(\tau_0, \tau_0)}{\tau - \tau_0} = 0 \text{ for every } \tau_0 \in \langle\tau_*, \tau^*\rangle\,.$$

Then $U \in \mathfrak{P}$ and $\int_{\tau_*}^{\tau^*} \mathrm{D}U = 0$.

Proof. Let $\eta > 0$. Let us find such a set $S_1 \in \mathbf{S}$ that $\left|\frac{U(\tau_0, \tau) - U(\tau_0, \tau_0)}{\tau - \tau_0}\right| < \eta$ if $\tau \neq \tau_0$, $(\tau_0, \tau) \in S_1$. It follows that $|B(A)| < \eta\,.\,(\tau^* - \tau_*)$ for $A \in A(S)$. Hence the result follows immediately.

Let us introduce the following very simple concept: Functions $U_1(\tau, t)$, $U_2(\tau, t)$ are called almost identical, if there exists such a set $S \in \mathbf{S}$ that $U_1(\tau_0, \tau) - U_1(\tau_0, \tau_0) = U_2(\tau_0, \tau) - U_2(\tau_0, \tau_0)$ for $(\tau_0, \tau) \in S$. In this case we write $U_1 \doteqdot U_2$. The following lemma is obvious:

Lemma 1,3,1. *If $U_1 \doteqdot U_2$ and if $U_1 \in \mathfrak{P}$, then $U_2 \in \mathfrak{P}$ and $\int_{\tau_*}^{\tau^*} \mathrm{D}U_1 = \int_{\tau_*}^{\tau^*} \mathrm{D}U_2$.*

1,4. The vector function U. Let us finally assume, that the values of U belong to E_n. We may write $U = (U_1, \ldots, U_n)$ where U_j are real-valued.

Definition 1,4,1. *The function $U = (U_1, \ldots, U_n)$ is called integrable if U_i, $i = 1, 2, \ldots, n$ are integrable. In this case we write $U \in \mathfrak{P}$ and put*

$$\int_{\tau_*}^{\tau^*} \mathrm{D}U = (\int_{\tau_*}^{\tau^*} \mathrm{D}U_1, \int_{\tau_*}^{\tau^*} \mathrm{D}U_2, \ldots \int_{\tau_*}^{\tau^*} \mathrm{D}U_n) .$$

Analogously as in the scalar case we may introduce the sum $B(A)$ and prove the following assertion: $U \in \mathfrak{P}$ if and only if to every $\varepsilon > 0$ there is such a $S \in \mathbf{S}$ that $|B(A_1) - B(A_2)| < \varepsilon$ for $A_1, A_2 \in \mathbf{A}(S)$. Naturally Theorems 1,3,1 to 1,3,7 hold for vector functions U also.

2. Generalized Ordinary Differential Equations

2,1. Definition. Let the functions $F_j(x_1, \ldots, x_n, \tau, t)$ $j = 1, 2, \ldots n$ be defined for $(x_1, \ldots, x_n, \tau, t) \in R_{n+2} \subset E_{n+2}$ where R_{n+2} has the following property: to every point $(x_1, \ldots, x_n, \tau, \tau) \in R_{n+2}$ there exists ruch a $\delta > 0$ that $(x_1, \ldots, x_n, \tau, t) \in R_{n+2}$ for $|t - \tau| < \delta$.

Definition 2,1,1. *The system of functions $x_1(\tau), \ldots, x_n(\tau)$, $\tau_3 < \tau < \tau_4$ is said to fulfil the system of differential equations*

$$\frac{\mathrm{d}x_j}{\mathrm{d}\tau} = \mathrm{D}F_j(x_1, \ldots, x_n, \tau, t) , \tag{2,1,1}$$

if $(x_1(\tau), \ldots, x_n(\tau), \tau, \tau) \in R_{n+2}$ for $\tau_3 < \tau < \tau_4$ and if

$$x_j(\tau_2) = x_j(\tau_1) + \int_{\tau_1}^{\tau_2} \mathrm{D}U_j \quad \tau_3 < \tau_1 < \tau_2 < \tau_4 , \tag{2,1,2}$$

where $U_j(\tau, t) = F_j((x_1(\tau), \ldots, x_n(\tau), \tau, t)$, $j = 1, 2, \ldots, n$.

Note 2,1,1: According to Theorem 1,3,4 the sense of Definition 2,1,1 does not alter if we require that (2,1,2) hold for every $\tau_2 \in (\tau_3, \tau_4)$ if $\tau_1 \in (\tau_3, \tau_4)$ is kept fixed.

Note 2,1,2: For systems (2,1,1), (2,1,2) we shall use the vector notations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, \tau, t) , \tag{2,1,3}$$

$$x(\tau_2) = x(\tau_1) + \int_{\tau_1}^{\tau_2} \mathrm{D}U \,, \tag{2,1,4}$$

$$x = (x_1, \ldots, x_n) \,, \quad F = (F_1, \ldots, F_n) \,, \quad U = (U_1, \ldots, U_n) \,.$$

Note 2,1,3: We do not define DF, only the solutions of equation (2,1,3) are defined.

Note 2,1,4: In the introduction we considered the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\overline{F}(x, t) \,, \tag{0,06}$$

where $\overline{F}(x, t)$ was defined on $G \times \langle 0, T \rangle$, $G \subset E_n$, G open. Let us put

$$\begin{aligned} F(x, \tau, t) &= \overline{F}(x, t) \quad \text{for} \quad (x, \tau, t) \in G \times E_1 \times \langle 0, T \rangle \,, \\ F(x, \tau, t) &= \overline{F}(x, 0) \quad \text{for} \quad (x, \tau, t) \in G \times E_1 \times (-1, 0) \,, \\ F(x, \tau, t) &= \overline{F}(x, T) \quad \text{for} \quad (x, \tau, t) \in G \times E_1 \times (T, T + 1) \,. \end{aligned}$$

In this sense the notation of the introduction is in accordance with the notation introduced in this section.

2,2. Comparison with the classical definition of differential equations. Let the function $f(x, \tau)$ be defined and continuous on an open set $G_{n+1} \subset E_{n+1}$. Let us consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau) \tag{2,2,1}$$

in the usual sense. By a solution of (2,2,1) we mean such a function $x(\tau)$ with a continuous derivative $\dot{x}(\tau)$ that (2,2,1) holds.

In connection with (2,2,1) let us consider the equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[f(x, \tau)t] \,, \tag{2,2,2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, \tau, t) \tag{2,2,3}$$

in the generalized sense, where $F(x, \tau, t) = \int_{\tau}^{t} f(x, \sigma) \, \mathrm{d}\sigma$, for such (x, τ, t) that the integral is defined. The function $F(x, \tau, t)$ is obviously defined on a set R_{n+2}, which fulfils the assumptions of section 2,1. If for example $G_{n+1} = G_n \times (\tau_1, \tau_2)$, $G_n \subset E_n$, let us consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_1(x, t) \,, \tag{2,2,4}$$

where $F_1(x, t) = \int_{\tau_0}^{t} f(x, \sigma) \, \mathrm{d}\sigma$, $\tau_0 \in (\tau_1, \tau_2)$. Every solution $x(\tau)$ of (2,2,3) is at the same time a solution of (2,2,4) and conversely, as $F(x(\tau), \tau, t) \doteq F_1(x(\tau), t)$.

The following theorem makes clear the relation of the classical notion of a differential equation to the generalized one.

Theorem 2,2,1. *Let $f(x, \tau)$ be continuous on an open set $G_{n+1} \subset E_{n+1}$. If $x(\tau)$ is a solution of* (2,2,1), *then $x(\tau)$ is a solution of* (2,2,2) *and* (2,2,3). *Conversely if $x(\tau)$ is a solution of either* (2,2,2) *or* (2,2,3), *then $x(\tau)$ is a solution of* (2,2,1).

Proof. Let $x(\tau) = (x_1(\tau), \ldots, x_n(\tau))$ be a solution of (2,2,1). We find that for every $\varepsilon > 0$

$$\begin{aligned} x_j(\tau) + \varepsilon\tau &\in \mathbf{M}(f_j(x_1(\tau), \ldots, x_n(\tau), \tau) \,.\, t)\,, \\ x_j(\tau) - \varepsilon\tau &\in \mathbf{m}(f_j(x_1(\tau), \ldots, x_n(\tau), \tau) \,.\, t)\,, \\ x_j(\tau) + \varepsilon\tau &\in \mathbf{M}(F_j(x_1(\tau), \ldots, x_n(\tau), \tau, t))\,, \\ x_j(\tau) - \varepsilon\tau &\in \mathbf{m}(F_j(x_1(\tau), \ldots, x_n(\tau), \tau, t))\,, \end{aligned}$$

and the first part of Theorem 2,2,1 is proved.

Let $x(\tau)$ be a solution of (2,2,2). We have

$$x(\tau_2) = x(\tau_1) + \int_{\tau_1}^{\tau_2} \mathrm{D}f(x(\tau), \tau)\, t\,. \tag{2,2,5}$$

As $f(x(\tau), \tau)\, t$ depends continuously on t for fixed τ, from (2,2,5) and from Theorem (1,3,6) if follows, that $x(\tau)$ is continuous and according to Note (1,1,2) $\int_{\tau_1}^{\tau_2} \mathrm{D}f(x(\tau), \tau)\, t = \int_{\tau_1}^{\tau_2} f(x(\tau), \tau)\, \mathrm{d}\tau$. Consequently $x(\tau)$ is a solution of (2,2,1).

Finally let $x(\tau)$ be a solution of (2,2,3). As for fixed

$$\lim_{t \to \tau} \frac{f(x(\tau), \tau)t - F(x(\tau), \tau, t) - f(x(\tau), \tau)\,\tau + F(x(\tau), \tau, \tau)}{t - \tau} = 0\,,$$

according to Theorems 1,3,7 and 1,3,2 we get $\int_{\tau_1}^{\tau_2} \mathrm{D}f(x(\tau), \tau)\, t = \int_{\tau_1}^{\tau_2} \mathrm{D}F(x(\tau), \tau, t)$. It follows that $x(\tau)$ is a solution of (2,2,2) and (2,2,1). Theorem (2,3,1) is proved.

2,3. Dynamic systems. We shall use the following special definition of a dynamic system.

Definition 2,3,1. *Let $Q \subset E_n$. Let a homeomorphic mapping $T_t = T_t(x)$ of Q onto itself correspond to each real number t in such a way that*

1. $T_{t_1} T_{t_2} = T_{t_1 + t_2}$, where $(T_{t_1} T_{t_2})(x) = T_{t_1}(T_{t_2}(x))$ *for* $x \in Q$.
2. $T_t(x)$ *depends continuously on* (t, x).

The system of transformations T_t is called a dynamic system on Q.

Let G be an open subset of E_n and let $f(x)$ be defined and continuous on G, $f(x) \in E_n$. Let the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x) \tag{2,3,1}$$

satisfy the following condition: to every point $x_0 \in G$ there exists only one solution $x(\tau, x_0)$ fulfilling the condition $x(0, x_0) = x_0$ and this solution is defined for $-\infty < \tau < \infty$. Let us define the transformation T_t of G:

$$T_t(x_0) = x(t, x_0) .$$

The system of transformations T_t forms a dynamic system, as may be shown readily. We do not take into consideration that the dynamic system T_t is defined on an open set, but we emphasize the following special property: $T_t(x_0)$ has a derivative with respect to t for fixed x_0, which depends continuously on (t, x_0).

Definition 2,1,1 enables us to lessen the difference between the notion of a dynamic system defined by means of a differential equation and that of a dynamic system in the sense of Definition 2,3,1.

Let T_t be a group of transformations of Q onto itself satisfying 1) without any continuity condition. Specially T_t may be a dynamic system on Q in the sense of Definition 2,3,1 and let us consider the differential equation

$$\frac{dx}{d\tau} = DT_{t-\tau}(x) , \quad (x, \tau, t) \in Q \times E_1 \times E_1 . \tag{2,3,2}$$

Let us put $y(\tau) = T_\tau(x_0)$, $x_0 \in Q$ and let A be a subdivision of $\langle \tau_1, \tau_2 \rangle$ (subordinate to $S = E_2$). It is easy to verify that $B(A) = y(\tau_2) - y(\tau_1)$ and consequently $y(\tau)$ is a solution of (2,3,2).

2,4. Linear differential equations. If $h(\tau)$ is a continuous function, then all solutions of the equation

$$\frac{dx}{d\tau} = h(\tau)\, x \tag{2,4,1}$$

are expressed by the formula

$$x(\tau) = K \exp H(\tau) , \quad K \in E_1 , \tag{2,4,2}$$

where $H(\tau)$ is an arbitrary primitive function of $h(\tau)$. According to section 2,2 let us consider the equation

$$\frac{dx}{d\tau} = DxH(t) . \tag{2,4,3}$$

It follows from Theorem 2,2,1 that all solutions of (2,4,3) are given by the formula (2,4,2).

Let now the function $H(\tau)$ fulfil the condition $|H(\tau_1) - H(\tau_2)| \leqq \omega(|\tau_2 - \tau_1|)$, where $\eta^{-\frac{1}{2}}\omega(\eta) \to 0$ with $\eta \to 0+$. As

$$\begin{aligned}\exp\{H(\tau_2)\} - \exp\{H(\tau_1)\} &= \exp\{H(\tau_1)\}[\exp\{H(\tau_2) - H(\tau_1)\} - 1] = \\ &= \exp\{H(\tau_1)\} \,.\, (H(\tau_2) - H(\tau_1)) + O((H(\tau_2) - H(\tau_1))^2) ,\end{aligned}$$

it follows from simple considerations that $x(\tau) = K \exp H(\tau)$ is a solution of (2,4,3).

3. Existence of $\int_{\tau_*}^{\tau^*} \mathrm{D}U$

Let the real-valued function $U(\tau, t)$ be defined and continuous for $\tau_* \leqq \tau \leqq \leqq \tau^*$, $\tau - \sigma \leqq t \leqq \tau + \sigma$, $\tau_* \leqq t \leqq \tau^*$, where $\sigma > 0$. This set will be denoted by S. Let the function $\psi(\eta)$ be defined and continuous for $0 \leqq \eta \leqq \sigma$, $\eta^{-1}\psi(\eta)$ non-decreasing, $\psi(\eta) > 0$ for $\eta > 0$ and let $\sum_{n=1}^{\infty} 2^n \psi\left(\frac{\sigma}{2^n}\right) < \infty$. Let us put $\Psi(\eta) = \sum_{n=1}^{\infty} \psi\left(\frac{\eta}{2^n}\right) \frac{2^n}{\eta}$ for $0 < \eta \leqq \sigma$, $\Psi(0) = 0$. It is evident that the series converges for $0 < \eta \leqq \sigma$, that $\Psi(\eta)$ is continuous, non-decreasing and that $\Psi(\eta) > 0$ for $\eta > 0$.

Note 3,1. We may put for example $\psi(\eta) = \eta^{1+\varepsilon}$, $\eta\, |\log \eta|^{-1-\varepsilon}, \ldots, \varepsilon > 0$. The aim of this section is to prove the following

Theorem 3,1. *Let the function U fulfil the inequality*

$$|U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)| \leqq \psi(\eta)$$

if $0 < \eta < \sigma$ and if the points $(\tau + \eta, t + \eta)$, $(\tau + \eta, t)$, $(\tau, t + \eta)$, (τ, t) belong to S.

Then $U \in \mathfrak{P}\langle\tau_*, \tau^*\rangle$ and

$$\left|\int_{\tau_1}^{\tau_2} \mathrm{D}U - U(\tau_1, \tau_2) + U(\tau_1, \tau_2)\right| \leqq (\tau_2 - \tau_1)\, \Psi(\tau_2 - \tau_1) \tag{3,01}$$

for $\tau_* \leqq \tau_1 < \tau_2 \leqq \tau_1 + \sigma$, $\tau_2 \leqq \tau^*$.

Proof. Let us fix the numbers $\tau_1, \tau_2, \tau_* \leqq \tau_1 < \tau_2 \leqq \tau_1 + \sigma$, $\tau_2 \leqq \tau^*$, let us put $p(\lambda) = \tau_1 + \lambda(\tau_2 - \tau_1)$ and define the sequence $\{\varphi_n(\tau)\}$ in such a way that $\varphi_n(\tau)$ is defined for $\tau = p\left(\frac{k}{2^n}\right)$, $k = 0, 1, \ldots, 2^n$, $n = 0, 1, 2, \ldots$,

$$\varphi_0(p(0)) = 0\,, \quad \varphi_0(p(1)) = U(p(0), p(1)) - U(p(0), p(0))\,,$$
$$\varphi_1(p(0)) = 0\,, \quad \varphi_1(p(\tfrac{1}{2})) = U(p(0), p(\tfrac{1}{2})) - U(p(0), p(0))\,,$$
$$\varphi_1(p(1)) = U(p(\tfrac{1}{2}), p(1)) - U(p(\tfrac{1}{2}), p(\tfrac{1}{2})) + \varphi_1(p(\tfrac{1}{2}))\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\varphi_n(p(0)) = 0, \varphi_n\left(p\left(\frac{k}{2^n}\right)\right) = U\left(p\left(\frac{k-1}{2^n}\right), p\left(\frac{k}{2^n}\right)\right) -$$
$$- U\left(p\left(\frac{k-1}{2^n}\right), p\left(\frac{k-1}{2^n}\right)\right) + \varphi_n\left(p\frac{(k-1)}{2^n}\right), \quad k = 1, 2, \ldots, 2^n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Our aim is to prove that $\lim_{n\to\infty} \varphi_n(\tau) = \varphi(\tau)$ exists for $\tau = p\left(\frac{j}{2^m}\right)$, that $\varphi(\tau)$ may be extended to a continuous function on $\langle\tau_1, \tau_2\rangle$ and that $\varphi(\tau) = \int_{\tau_1}^{\tau} \mathrm{D}U$. The main difficulty is in establishing inequality (3,05). If $\tau = p\left(\frac{j}{2^m}\right)$, we have

$$|\varphi_{n+1}(\tau) - \varphi_n(\tau)| \leqq \sum_{k=1}^{2^n} \left| U\left(p\left(\frac{2k-1}{2^{n+1}}\right), p\left(\frac{2k}{2^{n+1}}\right)\right) - U\left(p\left(\frac{2k-1}{2^{n+1}}\right), p\left(\frac{2k-1}{2^{n+1}}\right)\right) - \right.$$

$$\left. - U\left(p\left(\frac{k-1}{2^n}\right), p\left(\frac{2k}{2^{n+1}}\right)\right) + U\left(p\left(\frac{k-1}{2^n}\right), p\left(\frac{2k-1}{2^{n+1}}\right)\right)\right| \leqq 2^n \psi\left(\frac{\tau_2 - \tau_1}{2^{n+1}}\right), \quad n \geqq m,$$

$$|\varphi_{n+k}(\tau) - \varphi_n(\tau)| \leqq \sum_{j=1}^{\infty} 2^{n-1+j} \psi\left(\frac{\tau_2 - \tau_1}{2^{n+j}}\right) = \frac{\tau_2 - \tau_1}{2} \sum_{j=1}^{\infty} \frac{2^{n+j}}{\tau_2 - \tau_1} \psi\left(\frac{\tau_2 - \tau_1}{2^{n+j}}\right) =$$

$$= \frac{\tau_2 - \tau_1}{2} \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right), \quad n > m. \tag{3,02}$$

It follows that $\lim\limits_{n \to \infty} \varphi_n(\tau) = \varphi(\tau)$ exists for $\tau = p\left(\frac{j}{2^m}\right)$, $j = 0, 1, 2, \ldots, 2^m$, $m = 0, 1, 2, \ldots$

Let $\tau_3 = p\left(\frac{s}{2^q}\right)$, $\tau_4 = p\left(\frac{s}{2^q} + \frac{1}{2^l}\right)$, $0 \leqq l \leqq q$, $0 \leqq s < 2^q$, $\frac{s}{2^q} + \frac{1}{2^l} \leqq 1$, l, s, q integers. We shall prove that

$$|\varphi(\tau_4) - \varphi(\tau_3) - U(\tau_3, \tau_4) + U(\tau_3, \tau_3)| \leqq \frac{\tau_4 - \tau_3}{2} \Psi(\tau_4 - \tau_3). \tag{3,03}$$

For this purpose let us define the sequence of such functions $\{\tilde{\varphi}_n(\tau)\}$ that $\tilde{\varphi}_n(\tau)$ is defined for $\tau = p\left(\frac{s}{2^q} + \frac{j}{2^{l+n}}\right)$, $j = 0, 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$

$$\tilde{\varphi}_0\left(p\left(\frac{s}{2^q}\right)\right) = 0, \quad \tilde{\varphi}_0\left(p\left(\frac{s}{2^q} + \frac{1}{2^l}\right)\right) = U\left(p\left(\frac{s}{2^q}\right), p\left(\frac{s}{2^q} + \frac{1}{2^l}\right)\right) - U\left(p\left(\frac{s}{2^q}\right), p\left(\frac{s}{2^q}\right)\right),$$

$$\tilde{\varphi}_1\left(p\left(\frac{s}{2^q}\right)\right) = 0, \quad \tilde{\varphi}_1\left(p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right)\right) = U\left(p\left(\frac{s}{2^q}\right), p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right)\right) -$$

$$- U\left(p\left(\frac{s}{2^q}\right), p\left(\frac{s}{2^q}\right)\right),$$

$$\tilde{\varphi}_1\left(p\left(\frac{s}{2^q} + \frac{1}{2^l}\right)\right) = U\left(p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right), p\left(\frac{s}{2^q} + \frac{1}{2^l}\right)\right) -$$

$$- U\left(p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right), p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right)\right) + \tilde{\varphi}_1\left(p\left(\frac{s}{2^q} + \frac{1}{2^{l+1}}\right)\right),$$

. . . .

$$\tilde{\varphi}_n\left(p\left(\frac{s}{2^q}\right)\right) = 0,$$

$$\tilde{\varphi}_n\left(p\left(\frac{s}{2^q} + \frac{k}{2^{l+n}}\right)\right) = U\left(p\left(\frac{s}{2^q} + \frac{k-1}{2^{l+n}}\right), p\left(\frac{s}{2^q} + \frac{k}{2^{l+n}}\right)\right) -$$

$$- U\left(p\left(\frac{s}{2^q} + \frac{k-1}{2^{l+n}}\right), p\left(\frac{s}{2^q} + \frac{k-1}{2^{l+n}}\right)\right) +$$

$$+ \tilde{\varphi}_n\left(p\left(\frac{s}{2^q} + \frac{k-1}{2^{l+n}}\right)\right), \quad k = 1, 2, \ldots, 2^n,$$

. . . .

It is easy to verify that $\tilde{\varphi}_n(\tau) = \varphi_{l+n}(\tau) - \varphi_{l+n}(\tau_3)$ for $\tau = p\left(\frac{s}{2^q} + \frac{j}{2^{l+m}}\right)$, $j = 1, 2, \ldots, 2^m$, $n \geqq m$, $n \geqq q - l$. Hence

$$\tilde{\varphi}(\tau) = \lim_{n\to\infty} \tilde{\varphi}_n(\tau) = \varphi(\tau) - \varphi(\tau_3)\,. \tag{3,04}$$

As the sequence $\{\tilde{\varphi}_n(\tau)\}$ is defined in the same way as the sequence $\{\varphi_n(\tau)\}$, the inequality

$$|\tilde{\varphi}_{n+k}(\tau) - \tilde{\varphi}_n(\tau)| \leqq \frac{\tau_4 - \tau_3}{2}\,\Psi\left(\frac{\tau_4 - \tau_3}{2^n}\right), \tau = p\left(\frac{s}{2^q} + \frac{j}{2^{l+m}}\right), \; n \geqq m\,,$$

analogous to (3,02) holds. We put here $\tau = \tau_4$, $n = 0$, use the definition of $\tilde{\varphi}_0(\tau_4)$, pass to the limit for $k \to \infty$ and using (3,04) we finish the proof of (3,03).

By means of (3,03) we shall estimate that

$$|\varphi(\tau_4) - \varphi(\tau_3) - U(\tau_3, \tau_4) + U(\tau_3, \tau_3)| \leqq 3(\tau_4 - \tau_3)\,\Psi(2\,(\tau_4 - \tau_3)) \tag{3,05}$$

for $\tau_3 = p\left(\frac{k}{2^n}\right)$, $\tau_4 = p\left(\frac{k'}{2^n}\right)$, $k' > k$. Let us put

$$\tau_3 = p\left(\frac{k}{2^n}\right), \; \tau_4 = p\left(\frac{k}{2^n} + \frac{\delta_{l+1}}{2^{l+1}} + \frac{\delta_{l+2}}{2^{l+2}} + \ldots + \frac{\delta_n}{2^n}\right), \tag{3,06}$$

where $0 \leqq k < 2^n$, $l + 1 \leqq n$, $\delta_{l+1} = 1$, $\delta_j = 0$ or 1 for $j = l + 2, \ldots, n$, $\frac{k}{2^n} + \frac{\delta_{l+1}}{2^{l+1}} + \frac{\delta_{l+2}}{2^{l+2}} + \ldots + \frac{\delta_n}{2^n} \leqq 1$.

Let us put $\tilde{\tau}_0 = \tau_3 = p\left(\frac{k}{2^n}\right)$, $\tilde{\tau}_1 = p\left(\frac{k}{2^n} + \frac{\delta_{l+1}}{2^{l+1}}\right)$, $\tilde{\tau}_2 = p\left(\frac{k}{2^n} + \frac{\delta_{l+1}}{2^{l+1}} + \frac{\delta_{l+2}}{2^{l+2}}\right)$, $\ldots$, $\tilde{\tau}_{n-l} = \tau_4 = p\left(\frac{k}{2^n} + \frac{\delta_{l+1}}{2^{l+1}} + \ldots + \frac{\delta_n}{2^n}\right)$.

According to (3,03)

$$|\varphi(\tilde{\tau}_j) - \varphi(\tilde{\tau}_{j-1}) - U(\tilde{\tau}_{j-1}, \tilde{\tau}_j) + U(\tilde{\tau}_{j-1}, \tilde{\tau}_{j-1})| \leqq \frac{\tilde{\tau}_j - \tilde{\tau}_{j-1}}{2}\,\Psi\,(\tilde{\tau}_j - \tilde{\tau}_{j-1}) \leqq$$
$$\leqq \frac{\tau_2 - \tau_1}{2^{l+j+1}}\,\Psi\left(\frac{\tau_2 - \tau_1}{2^{l+j}}\right), \quad j = 1, 2, \ldots, n - l\,. \tag{3,07}$$

Further we have

$$\left|U\left(\tilde{\tau}_{j-1} - \frac{(i-1)(\tau_2 - \tau_1)}{2^{l+j}}, \tilde{\tau}_j\right) - U\left(\tilde{\tau}_{j-1} - \frac{(i-1)(\tau_2 - \tau_1)}{2^{l+j}}, \tilde{\tau}_{j-1}\right) - \right.$$
$$\left. - U\left(\tilde{\tau}_{j-1} - \frac{i(\tau_2 - \tau_1)}{2^{l+j}}, \tilde{\tau}_j\right) + U\left(\tilde{\tau}_{j-1} - \frac{i(\tau_2 - \tau_1)}{2^{l+j}}, \tilde{\tau}_{j-1}\right)\right| \leqq \psi\left(\frac{\tau_2 - \tau_1}{2^{l+j}}\right),$$
$$i = 1, 2, \ldots j, \; j = 1, 2, \ldots, n - l\,.$$

Hence

$$|U(\tilde{\tau}_{j-1}, \tilde{\tau}_j) - U(\tilde{\tau}_{j-1}, \tilde{\tau}_{j-1}) - U(\tilde{\tau}_0, \tilde{\tau}_j) + U(\tilde{\tau}_0, \tilde{\tau}_{j-1})| \leqq 2^j\psi\left(\frac{\tau_2 - \tau_1}{2^{l+j}}\right) \tag{3,08}$$

$j = 1, 2, \ldots, n-1$. From (3,07) and (3,08) we get

$$|\varphi(\tau_4) - \varphi(\tau_3) - U(\tau_3, \tau_4) + U(\tau_3, \tau_3)| \leqq \sum_{j=1}^{n-l} \{|\varphi(\tilde{\tau}_j) - \varphi(\tilde{\tau}_{j-1}) - U(\tilde{\tau}_{j-1}, \tilde{\tau}_j) +$$
$$+ U(\tilde{\tau}_{j-1}, \tilde{\tau}_{j-1})| + |U(\tilde{\tau}_{j-1}, \tilde{\tau}_j) - U(\tilde{\tau}_{j-1}, \tilde{\tau}_{j-1}) - U(\tilde{\tau}_0, \tilde{\tau}_j) + U(\tilde{\tau}_0, \tilde{\tau}_{j-1})|\} \leqq$$
$$\leqq \sum_{j=1}^{n-l} \left\{\frac{\tau_2 - \tau_1}{2^{l+j+1}} \Psi\left(\frac{\tau_2 - \tau_1}{2^{l+j}}\right) + 2^j \psi\left(\frac{\tau_2 - \tau_1}{2^{l+j}}\right)\right\}.$$

As $\tau_4 - \tau_3 \geqq \frac{\tau_2 - \tau_1}{2^{l+1}}$ it follows that

$$|\varphi(\tau_4) - \varphi(\tau_3) - U(\tau_3, \tau_4) + U(\tau_3, \tau_3)| \leqq \sum_{j=1}^{n-l} \left\{\frac{\tau_4 - \tau_3}{2^j} \Psi\left(\frac{\tau_4 - \tau_3}{2^{j-1}}\right) +\right.$$
$$\left. + 2^j \psi\left(\frac{\tau_4 - \tau_3}{2^{j-1}}\right)\right\} \leqq (\tau_4 - \tau_3)\, \Psi(\tau_4 - \tau_3) + 2(\tau_4 - \tau_3) \sum_{j=1}^{\infty} \frac{2^j}{2(\tau_4 - \tau_3)} \cdot$$
$$\cdot \psi\left(\frac{2(\tau_4 - \tau_3)}{2^j}\right) = (\tau_4 - \tau_3)\, \Psi(\tau_4 - \tau_3) + 2(\tau_4 - \tau_3)\, \Psi(2(\tau_4 - \tau_3)) \leqq$$
$$\leqq 3(\tau_4 - \tau_3)\, \Psi(2(\tau_4 - \tau_3))$$

and (3,05) holds if τ_4, τ_3 are defined by means of (3,06). As the functions U, Ψ are continuous, φ may be extended in a unique way to a continuous function on $\langle\tau_1, \tau_2\rangle$ and (3,05) holds for all $\tau_3, \tau_4 \in \langle\tau_1, \tau_2\rangle$.

From the assumptions concerning U and from (3,05) it follows that

$$|\varphi(\tau_4) - \varphi(\tau_3) - U(\tau_3, \tau_4) + U(\tau_3, \tau_3)| \leqq 3|\tau_4 - \tau_3|\, \Psi(2|\tau_4 - \tau_3|) + \\ + \psi(|\tau_4 - \tau_3|) \tag{3,09}$$

for $\tau_1 \leqq \tau_3 < \tau_4 \leqq \tau_2$. As $\Psi(\eta) \to 0$, $\eta^{-1}\psi(\eta) \to 0$ with $\eta \to 0+$ and as (3,05) and (3,09) hold, it follows that

$$\varphi(\tau) + \varepsilon\tau \in \mathbf{M}(U)\,, \quad \varphi(\tau) - \varepsilon\tau \in \mathbf{m}(U) \quad \text{for every } \varepsilon > 0\,.$$

Consequently $U \in \mathfrak{P}\langle\tau_1, \tau_2\rangle$ and $\int_{\tau_1}^{\tau} \mathrm{D}U = \varphi(\tau)$. (3,01) is an obvious consequence of (3,03) for $\tau_3 = \tau_1$, $\tau_4 = \tau_2$. The proof of Theorem 3,1 is complete.

Let us turn to a theorem on the continuous dependence of $\int \mathrm{D}U$ on a parameter.

Theorem 3,2. *Let the functions $U_k(\tau, t)$, $k = 0, 1, 2, \ldots$ be defined and continuous on S and let the following inequality*

$$|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)| \leqq \psi(\eta)$$

hold if $0 \leqq \eta \leqq \sigma$ and if $(\tau + \eta, t + \eta)$, $(\tau, t + \eta)$, $(\tau + \eta, t)$, $(\tau, t) \in S$ ($\psi(\eta)$ and S are defined at the beginning of this section). Further let $U_k(\tau, t) \to U_0(\tau, t)$ uniformly for $k \to \infty$, $(\tau, t) \in S$.

Then

$$\int_{\tau_1}^{\tau_2} \mathrm{D}U_k \to \int_{\tau_1}^{\tau_2} \mathrm{D}U_0 \quad \textit{with} \quad k \to \infty, \textit{ uniformly for } \tau_* \leqq \tau_1 < \tau_2 \leqq \tau^* \tag{3,10}$$

Proof. $U_k \in \mathfrak{P}$ according to Theorem 3,2. Let us fix $\tau_1, \tau_2 \in \langle \tau_*, \tau^* \rangle$, $0 < \tau_2 - \tau_1 \leqq \sigma$. Let the sequence $\{\varphi_{kn}(\tau)\}_{n=0}^{\infty}$ correspond to the function $U_k(\tau, t)$ $k = 0, 1, 2, \ldots$ in the same way as the sequence $\{\varphi_n(\tau)\}$, was constructed to the function $U(\tau, t)$ at the beginning of the proof of Theorem 3,1. According to Theorem 3,1 we have

$$\left| \int\limits_{p\left(\frac{j-1}{2^n}\right)}^{p\left(\frac{j}{2^n}\right)} \mathrm{D}U_k - \varphi_{kn}\left(p\left(\frac{j}{2^n}\right)\right) + \varphi_{kn}\left(p\left(\frac{j-1}{2^n}\right)\right) \right| \leqq \frac{\tau_2 - \tau_1}{2^n} \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right)$$

for $k = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots, j = 1, 2, \ldots, 2^n$. Consequently

$$\left| \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_k - \varphi_{kn}(\tau_2) + \varphi_{kn}(\tau_1) \right| \leqq (\tau_2 - \tau_1) \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right).$$

Hence

$$\left| \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_k - \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_0 \right| \leqq 2(\tau_2 - \tau_1) \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right) + |\varphi_{k,n}(\tau_2) - \varphi_{0n}(\tau_2)| \quad (3,11)$$

as $\varphi_{kn}(\tau_1) = 0$, $k = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots$. Let us put $\zeta_k = \max\limits_{(\tau,t)\in S} |U_k(\tau, t) - U_0(\tau, t)|$, $k = 1, 2, 3, \ldots$ As $|\varphi_{k,n}(\tau_2) - \varphi_{0,n}(\tau_2)| \leqq 2^{n+1}\zeta_k$, from (3,11) we get

$$\left| \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_k - \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_0 \right| \leqq 2(\tau_2 - \tau_1) \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right) + 2^{n+1}\zeta_k . \quad (3,12)$$

To every integer n there obviously exists such a number $K(n)$ that

$$2^{n+1}\zeta_k \leqq 2(\tau_2 - \tau_1) \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right) \quad \text{for} \quad k \geqq K(n)$$

It follows from (3,12) that

$$\left| \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_k - \int\limits_{\tau_1}^{\tau_2} \mathrm{D}U_0 \right| \leqq 4(\tau_2 - \tau_1) \Psi\left(\frac{\tau_2 - \tau_1}{2^n}\right) \quad \text{for} \quad k \geqq K(n)$$

and the proof of Theorem 3,2 is finished.

4. The Existence of Solutions of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t)$ and the Continuous Dependence on a Parameter

4,1. The existence theorem. In section 2 we introduced the differential equation $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, \tau, t)$. As we showed this equation has the same solutions as the equation $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau)$ in the classical sense, if $\frac{\partial F}{\partial t} = f(x, t)$ is continuous. The aim of this section is to prove the existence theorem under more general conditions. It is useful to prove at first a theorem on the continuous dependence on a parameter.

Up to the end of this paper we shall assume that $\omega_1(\eta)$, $\omega_2(\eta)$, $0 \leqq \eta \leqq 1$ are continuous, increasing, $\omega_1(0) = \omega_2(0) = 0$, $\omega_1(\eta) \geqq c\eta$, $\omega_2(\eta) \geqq c\eta$, $c > 0$ and that the function $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ fulfils the conditions introduced at the beginning of section 3. Let G be an open subset of E_{n+1}, $0 < \sigma \leqq 1$. Let us denote by $\mathbf{F}(G, \omega_1, \omega_2, \sigma)$ the set of functions $F(x, t)$ which fulfil the following conditions:

$F(x, t)$ is defined and continuous for $(x, t) \in G$, $F(x, t) \in E_n$,

$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) \text{ for } (x, t_1), (x, t_2) \in G, |t_2 - t_1| \leqq \sigma\,, \quad (4,1,01)$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\,\omega_2(|t_2 - t_1|) \quad (4,1,02)$$

for $(x_2, t_2), (x_2, t_1), (x_1, t_2), (x_1, t_1) \in G$, $\|x_2 - x_1\| \leqq 2\omega_1(\sigma)$, $|t_2 - t_1| \leqq \sigma$.

Note 4,1,1. If we put $F(x, t) = \sum_{j=1}^{r} H_j(x)\,\varphi_j(t)$, where the functions $H_j(x)$ are bounded, satisfy a Lipschitz condition $\|H_j(x_2) - H_j(x_1)\| \leqq K\|x_2 - x_1\|$, $x_1, x_2 \in E_n$, $H_j(x) \in E_n$ and the functions $\varphi_j(t)$ satisfy the condition $|\varphi_j(t_2) - \varphi_j(t_1)| \leqq \omega(|t_2 - t_1|)$, $t_1, t_2 \in E_1$, $\varphi_j(t) \in E_1$, then $F(x, t) \in \mathbf{F}(E_n, k\omega, k\omega, 1)$ for k great enough.

Let $F_k(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$, $k = 0, 1, 2, \ldots$, $F_k \to F_0$ uniformly for $k \to \infty$.

Theorem 4,1,1. *Let for $k = 1, 2, 3, \ldots$ the functions $x_k(\tau)$ be defined for $\tau_* \leqq \tau \leqq \tau^*$, fulfil the differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t), \quad \|x_k(\tau_2) - x_k(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|)\,, \quad k = 1, 2, \ldots,$$
$$|\tau_1 - \tau_2| \leqq \sigma,\ \tau_2, \tau_1 \in \langle \tau_*, \tau^* \rangle$$

and $\lim_{k\to\infty} x_k(\tau) = x_0(\tau) \in G$ uniformly for $\tau \in \langle \tau_*, \tau^* \rangle$.

Then the function $x_0(\tau)$ is a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t)$ on $\langle \tau_*, \tau^* \rangle$.

Proof. Let us put $F_k(x_k(\tau), t) = U_k(\tau, t)$, $k = 0, 1, 2, \ldots$ Let K be a compact subset of G containing all the points $(x_k(\tau), \tau)$, $k = 0, 1, 2, \ldots, \tau \in \langle \tau_*, \tau^* \rangle$. The functions $U_k(\tau, t)$ are defined and continuous on the set S: $\tau_* \leqq \tau \leqq \tau^*$, $|t - \tau| \leqq \sigma_1$, $\tau_* \leqq t \leqq \tau^*$, if σ_1 is small enough and we have

$$\|U_k(\tau, t) - U_0(\tau, t)\| = \|F_k(x_k(\tau), t) - F_0(x_0(\tau), t)\| \leqq$$
$$\leqq \|F_k(x_k(\tau), t) - F_0(x_k(\tau), t)\| + \|F_0(x_k(\tau), t) - F_0(x_0(\tau), t\|) \leqq$$
$$\leqq \sup_{(x,t)\in G} \|F_k(x, t) - F_0(x, t)\| + \|F_0(x_k(\tau), t) - F_0(x_0(\tau), t\|\,.$$

It follows that $U_k(\tau, t) \to U_0(\tau, t)$ uniformly with $k \to \infty$ as $F_k \to F_0$ uniformly on G, $x_k(\tau) - x_0(\tau)$ uniformly on $\langle \tau_*, \tau^* \rangle$ and F_0 is continuous on K. Obviously

$$\|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)\| =$$
$$= \|F_k(x_k(\tau + \eta), t + \eta) - F_k(x_k(\tau + \eta), t) - F_k(x_k(\tau), t + \eta) + F_k(x_k(\tau), t)\| .$$

If $(\tau, t) \in S$, $0 \leqq \eta \leqq \sigma_1$, then

$$\|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)\| \leqq$$
$$\leqq 2\omega_1(\eta) \,.\, \omega_2(\eta) = 2\psi(\eta) \,.$$

We apply Theorem 3,2 and get

$$\int_{t_1}^{t_2} \mathrm{D}F_k(x_k(\tau), t) \to \int_{t_1}^{t_2} \mathrm{D}F_0(x_0(\tau), t) \,, \quad t_1, t_2 \in \langle \tau_*, \tau^* \rangle$$

and consequently

$$x_0(t_2) = x_0(t_1) + \int_{t_1}^{t_2} \mathrm{D}F_0(x_0(\tau), t) \,.$$

$x_0(\tau)$ is a solution of $\dfrac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t)$. Theorem 4,1,1 is proved.

Let us pass to the existence theorem

Theorem 4,1,2. *Let $F(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$ and let K be a compact subset of G.*

There exists such a number $\sigma^* > 0$ that to every point $(x_0, t_0) \in K$ there exists a solution $x(\tau)$ of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t), \quad x(t_0) = x_0, \tag{4,1,03}$$

which is defined on $t_0 - \sigma^* \leqq \tau \leqq t_0 + \sigma^*$. This solution satisfies the condition $\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|)$ for $\tau_1, \tau_2 \in \langle t_0 - \sigma_*, t_0 + \sigma^* \rangle$.

Theorem 4,1,2 will be proved by means of two lemmas. Let K be a compact subset of G and let us choose such an open set G_1 that $\overline{G}_1$ is compact and that $K \subset G_1$, $\overline{G}_1 \subset G$. Let us denote by ϱ the distance from K to the complement of G_1. As $\omega_1(\eta) \geqq c\eta$, $\Psi(\eta) \to 0$ for $\eta \to 0+$, we have $2n\eta\Psi(\eta) < \omega_1(\eta)$ for η small enough. Obviously there exists such a number σ^* that $0 < \sigma^* \leqq \dfrac{\sigma}{2}$, $\sigma^* + \omega(\sigma^*) < \varrho$ and that $2n\eta\Psi(\eta) < \omega_1(\eta)$ for $0 < \eta < 2\sigma^*$.

Lemma 4,1,1. *Let the function $\omega_3(\eta)$ be continuous and increasing for $0 \leqq \eta \leqq \leqq 2\sigma^*$ and let the function $\psi_3(\eta) = \omega_3(\eta)\,\omega_2(\eta)$ fulfil the condition introduced at the beginning of section 3. Let $F(x, t) \in \mathbf{F}(G_1, \omega_1, \omega_2, 2\sigma^*)$ and let $x(\tau)$ be a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{4,1,04}$$

on $\langle t_0 - \lambda_1, t_0 + \lambda_2 \rangle$, $\lambda_1, \lambda_2 \in \langle 0, \sigma^* \rangle$, $(x(t_0), t_0) \in K$;

$$\|x(\tau_2) - x(\tau_1)\| \leqq \omega_3|\tau_2 - \tau_1|) \quad \textit{for} \quad \tau_1, \tau_2 \in \langle t_0 - \lambda_1, t_0 + \lambda_2 \rangle \,.$$

Then

$$\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|) \quad \textit{for} \quad \tau_1, \tau_2 \in \langle t_0 - \lambda_1, t_0 + \lambda_2 \rangle \,. \tag{4,1,05}$$

Proof. Let us choose a number δ, $0 < \delta \leqq 1$, $\omega_3(\delta.(\lambda_1 + \lambda_2)) \leqq \omega_1(\sigma^*)$. The function $F(x(\tau), t)$ is defined on the set S_1: $t_0 - \delta\lambda_1 \leqq \tau \leqq t_0 + \delta\lambda_2$, $\tau - \sigma^* \leqq \leqq t \leqq \tau + \sigma^*$ and

$$\|F(x(\tau_4), t_4) - F(x(\tau_4), t_3) - F(x(\tau_3), t_4) + F(x(\tau_3), t_3)\| \leqq$$
$$\leqq \|x(\tau_4) - x(\tau_3)\| \, \omega_2(|t_4 - t_3|) \leqq \omega_3(|\tau_4 - \tau_3|). \, \omega_2(|t_4 - t_3|)$$

for $(\tau_4, t_4), (\tau_4, t_3), (\tau_3, t_4), (\tau_3, t_3) \in S_1$. As $\omega_3(\eta)\, \omega_2(\eta) = \psi_3(\eta)$, we use (3,01) (for every component of $F(x(\tau), t)$ replacing $\psi(\eta)$, $\Psi(\eta)$ by $\psi_3(\eta)$, $\Psi_3(\eta)$) and get

$$\| \int_{\tau_1}^{\tau_2} \mathrm{D}F(x(\tau), t) - F(x(\tau_1), \tau_2) + F(x(\tau_1), \tau_1) \| \leqq n|\tau_2 - \tau_1| \, \Psi_3(|\tau_2 - \tau_1|)$$

for $\tau_1, \tau_2 \in \langle t_0 - \delta\lambda_1, t_0 + \delta\lambda_2 \rangle$. Because $x(\tau)$ is a solution of (4,1,03), according to (4,1,01) we get

$$\|x(\tau_2) - x(\tau_1)\| \leqq \omega_1(|\tau_2 - \tau_1|) + n|\tau_2 - \tau_1| \, \Psi_3(|\tau_2 - \tau_1|)$$

for $\tau_1, \tau_2 \in \langle t_0 - \delta\lambda_1, t_0 + \delta\lambda_2 \rangle$.

As $\Psi_3(\eta) \to 0$ for $\eta \to 0+$, $\omega_1(\eta) \geqq c\eta$, $c > 0$, there exists such a ϑ, $0 < \vartheta \leqq \leqq 1$ that

$$\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(\tau_2 - \tau_1) \quad \text{for} \quad \tau_1, \tau_2 \in \langle t_0 - \vartheta\lambda_1, t_0 + \vartheta\lambda_2 \rangle \,. \tag{4,1,06}$$

Let Θ be the upper bound of such numbers ϑ, $0 < \vartheta \leqq 1$ that (4,1,06) holds for $\tau_1, \tau_2 \in \langle t_0 - \vartheta\lambda_1, t_0 + \vartheta\lambda_2 \rangle$. We prove that $\Theta = 1$.

Let us suppose that $\Theta < 1$. We shall find such numbers $\tilde{\tau}_1 \neq \tilde{\tau}_2 \in \in \langle t_0 - \Theta\lambda_1, t_0 + \Theta\lambda_2 \rangle$ that

$$\|x(\tilde{\tau}_2) - x(\tilde{\tau}_1)\| = 2\omega_1(|\tilde{\tau}_2 - \tilde{\tau}_1|) \,. \tag{4,1,07}$$

In the same manner as above we find such a number $\chi > 0$ that

$$\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|) \text{ for } \tau_1, \tau_2 \in \langle t_0 + \Theta\lambda_2 - \chi, t_0 + \Theta\lambda_2 + \chi \rangle \,.$$

In every interval $\langle t_0 - \Theta\lambda_1, t_0 + \Theta\lambda_2 + \varepsilon \rangle$, $(\varepsilon > 0)$ there exist necessarily such $\tau', \tau'', \tau' < \tau''$ that $\|x(\tau'') - x(\tau')\| > 2\omega_1(\tau'' - \tau')$. It follows that $t_0 + + \Theta\lambda_2 < \tau'' \leqq t_0 + \Theta\lambda_2 + \varepsilon$, $t_0 - \Theta\lambda_1 \leqq \tau' \leqq t_0 + \Theta\lambda_2 - \chi$ if $\varepsilon < \chi$, $\varepsilon + \chi < < \Theta(\lambda_1 + \lambda_2)$. Passing to the limit for $\varepsilon \to 0+$ we obtain $\tilde{\tau}_1 \neq \tilde{\tau}_2$.

Having proved the existence of $\tilde{\tau}_1, \tilde{\tau}_2$, let us define the set S_2 by meaans of the inequalities $t_0 - \Theta\lambda_1 \leqq \tau \leqq t_0 + \Theta\lambda_2$, $\tau - \sigma^* \leqq t \leqq \tau + \sigma^*$. As

$$\|F(x(\tau_4), t_4) - F(x(\tau_4), t_3) - F(x(\tau_3), t_4) + F(x(\tau_3), t_3)\| \leqq$$
$$\leqq \|x(\tau_4) - x(\tau_3)\| \, \omega_2(|t_4 - t_3|) \leqq 2\omega_1(|\tau_4 - \tau_3|) \, \omega_2(|t_4 - t_3|)$$

for $(\tau_4, t_4), (\tau_4, t_3), (\tau_3, t_4), (\tau_3, t_3) \in S_2$, we use (3,01) again and get similarly as in the previous case

$$\| \int_{\tau_3}^{\tau_4} \mathrm{D}F(x(\tau), t) - F(x(\tau_3), \tau_4) + F(x(\tau_3), \tau_3) \| \leqq 2n|\tau_4 - \tau_3| \, \Psi(|\tau_4 - \tau_3|)$$

and according to (4,1,01), we have

$$\|x(\tau_4) - x(\tau_3)\| \leqq \omega_1(|\tau_4 - \tau_3|) + 2n|\tau_4 - \tau_3| \, \Psi(|\tau_4 - \tau_3|) \,, \qquad (4,1,08)$$

as $x(\tau)$ is a solution of (4,1,03). If $\tau_4 = \tilde{\tau}_2$, $\tau_3 = \tilde{\tau}_1$, we get from (4,1,07) and (4,1,08) that

$$\omega_1(|\tilde{\tau}_2 - \tilde{\tau}_1|) \leqq 2n|\tilde{\tau}_2 - \tilde{\tau}_1| \, \Psi(|\tilde{\tau}_2 - \tilde{\tau}_1|) \,.$$

According to the choice of σ^* it follows that $|\tilde{\tau}_2 - \tilde{\tau}_1| \geqq 2\sigma^*$. At the same time $|\tilde{\tau}_2 - \tilde{\tau}_1| \leqq \Theta(\lambda_1 + \lambda_2) < 2\sigma^*$ and this contradiction proves Lemma 4,1,1.

We shall prove Theorem (4,1,2) from the classical existence theorem passing to a limit. For this purpose we shall need an approximation lemma.

Lemma 4,1,2. *There exist functions $F_k(x, t)$ for $(x, t) \in G_1$, $k = 1, 2, 3, \ldots$ having the following properties*:

1. *the functions $F_k(x, t)$ have continuous partial derivatives with respect to $x_1, \ldots, x_n, t$,*
2. *$F_k(x, t) \to F(x, t)$ uniformly for $(x, t) \in G_1$,*
3. *$F_k(x, t) \in \mathbf{F}(G_1, \omega_1, \omega_2, \sigma_1)$.*

Lemma 4,1,2 is proved by means of standard methods and we shall indicate the proof. Let us denote by ϱ^* the distance from G_1 to the complement of G. Let $\varrho^* > \varrho_1 > \varrho_2 > ,\ldots \lim\limits_{k\to\infty} \varrho_k = 0$ and let the functions $\zeta_k(y, \tau)$, $k = 1, 2, 3, \ldots$ be defined for $(y, \tau) \in E_{n+1}$ with continuous partial derivatives of the first order, $\zeta_k(y, \tau) = 0$ for $\|y\|^2 + \tau^2 \geqq \varrho_k^2$, $\zeta_k(y, \tau) \geqq 0$ for $(y, \tau) \in E_{n+1}$, $\int\limits_{E_{n+1}} \ldots \int \zeta_k(y, \tau) \, \mathrm{d}y_1, \ldots, \mathrm{d}y_n \, \mathrm{d}\tau = 1$. Let us define

$$F_k(x, t) = \int\limits_{E_{n+1}} \ldots \int F(y, \tau) \, \zeta_k(x - y, t - \tau) \, \mathrm{d}y_1 \ldots \mathrm{d}y_n \, \mathrm{d}\tau$$

(we put $F(y, \tau) = 0$ if $(y, \tau) \,\tilde{\in}\, G$). In a usual way we verify that $F_k(x, t)$ fulfil all conditions of Lemma 4,1,2.

Let us proceed to the proof of Theorem 4,1,2. Let $F_k(x, t)$ be a fixed sequence fulfilling the conditions of Lemma 4,1,2 and let us put $f_k(x, t) = \dfrac{\partial}{\partial t} F_k(x, t)$. As the functions $\int\limits_{\tau}^{t} f_k(x(\tau), \xi) \, \mathrm{d}\xi$ and $F_k(x(\tau), t)$ are almost identical for every function $x(\tau)$, $(x(\tau), \tau) \in G_1$, it follows from Lemma 1,3,1 and from Theorem 2,2,1 that every solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_k(x, \tau) \qquad (4,1,08)$$

is at the same time a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t) \qquad (4,1,09)$$

and conversely. Let us choose a point $(x_0, \tau_0) \in K$. If $x_k(\tau)$, $x_k(\tau_0) = x_0$ is a solution of (4,1,08), we may assume that $x_k(\tau)$ is defined on the maximal interval (α_k, β_k), $\alpha_k < \tau_0 < \beta_k)$. That means: if $y(\tau)$ is a solution of (4,1,08) on (α, β) and if $y(\tau) = x_k(\tau)$ for $\tau \in (\alpha, \beta) \cap (\alpha_k, \beta_k) \neq 0$, then $(\alpha, \beta) \subset (\alpha_k, \beta_k)$. We shall prove that

$$\alpha_k < \tau_0 - \sigma^* , \quad \tau_0 + \sigma^* < \beta_k , \quad k = 1, 2, 3, \ldots . \tag{4,1,10}$$

Let at least one of inequalities (4,1,10) be false for a fixed k. It is well known that the distance of the point $(x_k(\tau), \tau)$ from the complement of G_1 tends to zero for $\tau \to \beta_k -$ $(\tau \to \alpha_k +)$ if β_k (α_k) is finite. It follows that there exists such a number $\tilde{\tau}$ that

$$\lambda = |\tilde{\tau} - \tau_0| < \sigma^* , \quad \|x_k(\tilde{\tau}) - x_k(\tau_0)\| = 2\omega_1(\sigma^*) , \tag{4,1,11}$$

$\|x_k(\tau) - x_k(\tau_0)\| < 2\omega_1(\sigma^*)$ for $\tau \in \langle \tau_0 - \lambda, \tau_0 + \lambda \rangle$.

As $f_k(x_k(\tau), \tau)$ is continuous for $\tau \in \langle \tau_0 - \lambda, \tau_0 + \lambda \rangle$, there exists such a number L_k that

$$\|x_k(\tau_2) - x_k(\tau_1)\| \leqq L_k \omega_1(|\tau_2 - \tau_1|) , \quad \tau_1, \tau_2 \in \langle \tau_0 - \lambda, \tau_0 + \lambda \rangle . \tag{4,1,12}$$

Using Lemma 4,1,1 with $\omega_3(\eta) = L_k \omega_1(\eta)$ we get

$$\|x_k(\tau_2) - x_k(\tau_1)\| < 2\omega_1(|\tau_2 - \tau_1|) , \quad \tau_1, \tau_2 \in \langle \tau_0 - \lambda, \tau_0 + \lambda \rangle \tag{4,1,13}$$

and in particular

$$\|x_k(\tilde{\tau}) - x_k(\tau_0)\| \leqq 2\omega_1(\lambda) < 2\omega_1(\sigma^*) .$$

The contradiction with (4,1,11) proves (4,1,10).

As (4,1,12) holds for $\tau_1, \tau_2 \in \langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$ (with another constant L_k on the right side) using lemma 4,1,1 again we get that (4,1,13) holds for $\tau_1, \tau_2 \in \langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$.

$x_k(\tau)$ are defined and equicontinuous on $\langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$, $x_k(\tau_0) = x_0$. We extract a subsequence uniformly converging to $x(\tau)$. As (1,1,13) holds from Theorem 4,1,1 we get that $x(\tau)$ is the required solution of (4,1,03). Theorem 4,1,2 is proved.

4,2. Generalization of the theorem on the continuous dependence on a parameter. In this section we shall prove the theorem on the continuous dependence on a parameter under the assumption that the limit equation has a unique solution on $\langle \tau_*, \tau^* \rangle$. It is not necessary to assume that the solutions of all equations are defined on $\langle \tau_*, \tau^* \rangle$. The proof runs on similar lines as in the classical case and Theorems 4,1,1, 4,1,2 and Lemma 4,1,1 are used. Let $F(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$. Let us introduce the following definition:

Definition 4,2,1. *$x(\tau)$, $\tau \in \langle \tau_1, \tau_2 \rangle$ is a regular solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{4,2,1}$$

on $\langle \tau_1, \tau_2 \rangle$, if it is a solution of (4,2,1) according to Definition 2,2,1 and if to every $\tau \in \langle \tau_1, \tau_2 \rangle$ there exists such a $\sigma_2 = \sigma_2(\tau) > 0$, $\sigma_2 < \frac{1}{2}\sigma$ that $\|x(\tau_4) - x(\tau_3)\| \leqq 2\omega_1(|\tau_4 - \tau_3|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle \cap \langle \tau - \sigma_2, \tau + \sigma_2 \rangle$.

Lemma 4,2,1. *Let $x(\tau)$ be a regular solution of (4,2,1) on $\langle \tau_1, \tau_2 \rangle$. Let us choose a $\tau_0 \in \langle \tau_1, \tau_2 \rangle$ and let us find the number σ^* according to Theorem 4,1,2 for the set $K = ((x(\tau_0), \tau_0))$. Then $\|x(\tau_3) - x(\tau_4)\| \leqq 2\omega_1(|\tau_3 - \tau_4|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle \cap \cap \langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$.*

Proof. We shall prove Lemma 4,2,1 by means of Lemmas 4,1,1 and 1,1,1. Let us put $\langle \tau_1, \tau_2 \rangle \cap \langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle = \langle \tau_0 - \lambda_1, \tau_0 + \lambda_2 \rangle$, $S = \underset{(\tau, t)}{\mathrm{E}} [\tau \in \langle \tau_0 - \lambda_1, \tau_0 + \lambda_2 \rangle, |t - \tau| \leqq \sigma_2(\tau)]$. According to Lemma 1,1,1 there exists a subdivision $A = (\alpha_0, \tilde{\tau}_1, \alpha_1, \ldots, \tilde{\tau}_k, \alpha_k)$ of $\langle \tau_0 - \lambda_1, \tau_0 + \lambda_2 \rangle$ subordinate to S. Let $\alpha_{r-1} \leqq \tau_3 \leqq \alpha_r$, $\alpha_{s-1} \leqq \tau_4 \leqq \alpha_s$. If $r = s$, then obviously $\|x(\tau_4) - x(\tau_3)\| \leqq 2\omega_1(|\tau_4 - \tau_3|)$, if $r < s$ then

$$\|x(\tau_4) - x(\tau_3)\| \leqq \|x(\tau_4) - x(\alpha_{s-1})\| + \| x(\alpha_{s-1}) - x(\alpha_{s-2})\| + \ldots$$
$$\ldots + \|x(\alpha_n) - x(\tau_3)\| \leqq 2(s - r + 1)\, \omega_1(|\tau_4 - \tau_3|)\,.$$

In any case we have $\|x(\tau_4) - x(\tau_3)\| \leqq k\omega_1(|\tau_4 - \tau_3|)$. Lemma 4,2,1 follows from Lemma 4,1,1.

Let $F_0(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$. Let $x_0(\tau)$ be a regular solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t) \tag{4,2,2}$$

on $\langle \tau_1, \tau_2 \rangle$. Let us suppose further that if $z(\tau)$ is a solution of (4,2,2) regular on $\langle \tau_1, \tau_7 \rangle \subset \langle \tau_1, \tau_2 \rangle$, $z(\tau_1) = x_0(\tau_1)$ then $z(\tau) = x(\tau)$ for $\tau \in \langle \tau_1, \tau_7 \rangle$.

Theorem 4,2,1. *To every number $\varepsilon > 0$ there exists such a $\delta > 0$ that the following assertion is true:*

If $F_1(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$, $\|F_1(x, t) - F_0(x, t)\| \leqq \delta$ for $(x, t) \in G$ and if $y(\tau)$ is a regular solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_1(x, t) \tag{4,2,3}$$

on $\langle \tau_1, \tau_3 \rangle \subset \langle \tau_1, \tau_2 \rangle$, $\|y(\tau_1) - x_0(\tau_1)\| \leqq \delta$, then there exists such a regular solution $x_1(\tau)$ of (4,2,3) on $\langle \tau_1, \tau_2 \rangle$ that $x_1(\tau) = y(\tau)$ for $\tau \in \langle \tau_1, \tau_3 \rangle$, $\|x_1(\tau) - x_0(\tau)\| < \varepsilon$ for $\tau \in \langle \tau_1, \tau_2 \rangle$.

Proof. Let us choose an open set G_1, $\overline{G}_1 \subset G$, $\overline{G}_1$ compact, $(x_0(\tau), \tau) \in G_1$ for $\tau \in \langle \tau_1, \tau_2 \rangle$. Let us find a number σ^* to the set $\overline{G}_1$ according to Theorem 4,1,2 and let $\bar{\varepsilon}$ be such a number that $(y, \tau) \in G_1$ for $\tau \in \langle \tau_1, \tau_2 \rangle$, $\|y - x_0(\tau)\| \leqq \bar{\varepsilon}$. Let us further choose τ_4, $\tau_1 \leqq \tau_4 < \tau_2$. Let us replace the function $x_0(\tau)$ which is defined on $\langle\tau_1, \tau_2\rangle$ by its partial function on $\langle\tau_1, \tau_4\rangle$. If $\tau_4 = \tau_1$ Theorem 4,2,1 obviously holds on the degenerate interval $\langle\tau_1, \tau_4\rangle$. Theorem 4,2,1 will be proved fully if we prove the following lemma.

Lemma 4,2,2. *If Theorem 4,2,1 holds on $\langle\tau_1, \tau_4\rangle$, then it holds on $\langle\tau_1, \tau_4 + \eta\rangle$ where $0 < \eta \leqq \min(\tau_2 - \tau_4, \sigma^*)$, $4\omega_1(\eta) < \bar{\varepsilon}$.*

Proof. Let us assume that Lemma 4,2,2 is false. It follows that there exist a number ε, $0 < \varepsilon \leqq \bar{\varepsilon}$ and sequences of functions $F_k(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$, $F_k \to F_0$ uniformly on G and $y_k(\tau)$, $k = 2, 3, 4, \ldots$, in such a way that $y_k(\tau)$ are regular solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t)\,, \quad y_k(\tau_1) \to x_0(\tau_1) \tag{4,2,4}$$

on $\langle\tau_1, \tilde{\tau}_k\rangle$ $\tau_1 \leqq \tilde{\tau}_k \leqq \tau_4 + \eta$ and that, for every $k \geqq 2$, it is impossible to find a regular solution $x_k(\tau)$ of (4,2,4) in $\langle\tau_1, \tau_4 + \eta\rangle$, $x_k(\tau) = y_k(\tau)$ for $\tau \in \langle\tau_1, \tilde{\tau}_k\rangle$, $\|x_k(\tau) - x_0(\tau)\| < \varepsilon$ for $\tau \in \langle\tau_1, \tau_4 + \eta\rangle$.

As Theorem 4,2,1 holds on $\langle\tau_1, \tau_4\rangle$ according to the assumptions of Lemma 4,2,2, we may suppose without a loss on generality that $\tilde{\tau}_k \geqq \tau_4$ and that $y_k(\tau) \to x_0(\tau)$ uniformly for $k \to \infty$ on $\langle\tau_0, \tau_4\rangle$. As $\tau_k - \tau_4 \leqq \eta$ we have (according to Lemma 4,2,1)

$$\begin{aligned}\|y_k(\tilde{\tau}_k) - x_0(\tilde{\tau}_k)\| &\leqq \|y_k(\tilde{\tau}_k) - y_k(\tau_k)\| + \|y_k(\tau_4) - x_0(\tilde{\tau}_4)\| + \\ &+ \|x_0(\tau_4) - x_0(\tau_k)\| \leqq 4\omega_1(\eta) + \|y_k(\tau_4) - x_0(\tau_4)\|\end{aligned}$$

and $(y_k(\tilde{\tau}_k), \tilde{\tau}_k) \in G_1$ for $k > K$. According to Theorem 4,1,2 there exists such a solution $u_k(\tau)$ of (4,2,4) on $\langle\tilde{\tau}_k, \tilde{\tau}_k + \sigma^*\rangle$, that $\|u_k(\tau_6) - u_k(\tau_5)\| \leqq$ $\leqq 2\omega_1(|\tau_6 - \tau_5|)$ for $\tau_5, \tau_6 \in \langle\tilde{\tau}_k, \tilde{\tau}_k + \sigma^*\rangle$.

Let us put $x_k(\tau) = y_k(\tau)$ for $\tau \in \langle\tau_1, \tilde{\tau}_k\rangle$, $x_k(\tau) = u_k(\tau)$ for $\tau \in \langle\tilde{\tau}_k, \tau_4 + \eta\rangle$, $k > K$. $x_k(\tau)$ is a solution of (4,2,4) regular on $\langle\tau_1, \tau_4 + \eta\rangle$. (See Lemma 4,2,1.) As $(x_k(\tau), \tau) \in G_1$ for $\tau \in \langle\tau_1, \tau_4\rangle$ according to Lemma 4,2,1 we have

$$\begin{aligned}\|x_k(\tau_6) - x_k(\tau_5)\| \leqq 2\omega_1(|\tau_6 - \tau_5|) \quad \text{for} \quad &\tau_5, \tau_6 \in \langle\tau_0 - \sigma_1^*, \tau_0 + \sigma^*\rangle \cap \\ &\cap \langle\tau_1, \tau_4 + \eta\rangle\,, \; \tau_0 \in \langle\tau_1, \tau_4\rangle\end{aligned} \tag{4,2,5}$$

and as $\eta \leqq \sigma^*$, (4,2,5) holds for $\tau_5, \tau_6 \in \langle\tau_1, \tau_4 + \eta\rangle$, $|\tau_6 - \tau_5| \leqq 2\sigma^*$.

From the sequence $\{x_k(\tau)\}$ let us extract a uniformly converging subsequence, $\{x_{k_j}(\tau)\}$, $\lim x_{k_j}(\tau) = z(\tau)$. According to Theorem (4,1,1) $z(\tau)$ is a solution of (4,2,2) and

$$\|z(\tau_6) - z(\tau_5)\| \leqq 2\omega_1(|\tau_6 - \tau_5|) \text{ for } \tau_5, \tau_6 \in \langle\tau_1, \tau_4 + \eta\rangle,\ |\tau_6 - \tau_5| \leqq 2\sigma^*$$

$z(\tau) = x_0(\tau)$ for $\tau \in \langle\tau_1, \tau_4\rangle$ as $x_k(\tau) = y_k(\tau) \to x_0(\tau)$ uniformly an $\langle\tau_1, \tau_4\rangle$. Consequently $z(\tau)$ is a regular solution of (4,2,2) on $\langle\tau_1, \tau_4 + \eta\rangle$ $z(\tau_1) = x_0(\tau_1)$ and according to the unicity assumption of Theorem 4,2,1, it follows that $z(\tau) =$ $= x_0(\tau)$ for $\tau \in \langle\tau_1, \tau_4 + \eta\rangle$. For j great enough we have $\|x_{k_j}(\tau) - x_0(\tau)\| < \varepsilon$ for $\tau \in \langle\tau_1, \tau_4 + \eta\rangle$ and this contradiction proves Lemma 4,2,2. At the same time the proof of Theorem 4,2,1 is finished.

5. Applications

5,1. Linear equations. In this section $x = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ \dots. \\ B_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ .. \\ b_n \end{pmatrix}$ denote column vectors, $A = (A_{ij})$, $a = (a_{ij})$ denote square matrices. Let us suppose that the functions $A(\tau)$, $B(\tau)$ are defined for all τ and that

$$\|A(\tau_1) - A(\tau_2)\| \leqq K_1|\tau_2 - \tau_1|^\alpha \quad \text{for} \quad |\tau_2 - \tau_1| \leqq 1\,, \tag{5,01}$$

$$\|B(\tau_2) - B(\tau_1)\| \leqq K_2|\tau_2 - \tau_1|^\beta \quad \text{for} \quad |\tau_2 - \tau_1| \leqq 1\,, \tag{5,02}$$

where $\alpha > \frac{1}{2}$, $\alpha + \beta > 1$. Let us consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)\, x + B(t)]\,. \tag{5,03}$$

Obviously $A(t)\, x + B(t) \in \mathbf{F}(G, K_3\eta^\gamma, K_1\eta^\alpha, 1)$, where $\gamma = \min(\alpha, \beta)$, G is an arbitrary open bounded subset of E_{n+1} and K_3 is great enough. In the same way as in Lemma 4,1,2 we find that there exist sequences $\{A_k(\tau)\}$, $\{B_k(\tau)\}$ satisfying the following conditions

$$\left.\begin{array}{l} A_k(\tau) \to A(\tau) \\ B_k(\tau) \to B(\tau) \end{array}\right\} \text{ uniformly for } k \to \infty \text{ on every bounded interval,}$$

there exist continuous derivatives $\dfrac{\mathrm{d}}{\mathrm{d}\tau} A_k(\tau) = a_k(\tau)$, $\dfrac{\mathrm{d}}{\mathrm{d}\tau} B_k(\tau) = b_k(\tau)$, $A_k(\tau)$ fulfil (5,01), $B_k(\tau)$ fulfil (5,02).

It follows from Theorem 2,2,1 that every solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A_k(t)\, x + B_k(t)]\,, \tag{5,04}$$

is at the same time a solution of the classical equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = a_k(\tau)\, x + b_k(\tau) \tag{5,05}$$

and conversely. Let us look for a solution $x(\tau)$ of (5,03), $x(0) = x_0$. The only solution of (5,05) $x_k(\tau)$, $x_k(0) = x_0$ is given by the formula

$$x_k(\tau_1) = x_0 \exp\{A_k(\tau_1) - A_k(0)\} + \exp\{A_k(\tau_1)\} \int_0^{\tau_1} \exp\{- A_k(\tau)\}\, \mathrm{d}B_k(\tau)$$

and according to Note 1,1,2 we may write

$$x_k(\tau_1) = x_0 \exp\{A_k(\tau_1) - A_k(0)\} + \exp\{A_k(\tau_1)\} \int_0^{\tau_1} \mathrm{D} \exp\{- A_k(\tau)\}\, B_k(t)\,. \tag{5,06}$$

Theorem 3,2 implies that

$$\int_0^{\tau_1} \mathrm{D} \exp\{- A_k(\tau)\}\, B_k(t) \to \int_0^{\tau_1} \mathrm{D} \exp\{- A(\tau)\}\, B(t)\,. \tag{5,07}$$

As $x_k(\tau)$ is the only solution of (5,04) with $x_k(0) = x_0$, according to Theorem 4,1,2 it follows that

$$\|x_k(\tau_2) - x_k(\tau_1)\| \leqq 2K_3|\tau_2 - \tau_1|^\gamma \quad \text{for} \quad |\tau_2 - \tau_1| \leqq \sigma^* \,. \tag{5,08}$$

Hence, from (5,06), (5,07) and (5,08) it follows that $x_k(\tau_1)$ tend almost uniformly to

$$x(\tau_1) = x_0 \exp\{A(\tau_1) - A(0)\} + \exp\{A(\tau_1)\} \int_0^{\tau_1} \mathrm{D} \exp\{- A(\tau)\} B(t)$$

and $x(\tau_1)$ is a solution of (5,03).

We shall prove that $x(\tau)$ is the only regular solution of (5,03),

Lemma 5,1. *Let $y(\tau)$ be a regular solution of* (5,03) *on* $\langle\tau_*, \tau^*\rangle$ $(\tau_* < 0 < \tau^*$, *we admit* $\tau_* = -\infty$ *or* $\tau^* = \infty)$, $y(0) = x(0)$. *Then* $y(\tau) = x(\tau)$ *for* $\tau \in \langle\tau_*, \tau^*\rangle$.

Proof. If Lemma 5,1 is false, then $u(\tau) = y(\tau) - x(\tau)$ is a solution of

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \mathrm{D}A(t)\, u \tag{5,09}$$

and $u(0) = 0$, $u(\tilde{\tau}) \neq 0$ for a certain $\tilde{\tau} \in \langle\tau_*, \tau^*\rangle$.

Let τ_3 be such a number from $\langle\tau_*, \tau^*\rangle$ that $u(\tau_3) = 0$ but in every neighbourhood of τ_3 there exist such τ that $u(\tau) \neq 0$. Obviously $\|u(\tau_2) - u(\tau_1)\| \leqq$ $\leqq 4K_3|\tau_2 - \tau_1|^\gamma$ for $\tau_2, \tau_1 \in \langle\tau_3 - \eta, \tau_3 + \eta\rangle$ where $\eta > 0$.

For every real ξ the function $\xi u(\tau)$ is a solution of (5,09) and

$$\xi u(\tau_3) = 0\,, \quad \|\xi u(\tau_2) - \xi u(\tau_1)\| \leqq 4\xi K_3|\tau_2 - \tau_1|^\gamma$$
$$\text{for} \quad \tau_2, \tau_1 \in \langle\tau_3 - \eta, \tau_3 + \eta\rangle \,.$$

According to Lemma 4,1,1 we have

$$\|\xi u(\tau_2) - \xi u(\tau_1)\| \leqq 2K_3|\tau_2 - \tau_1|^\gamma \quad \text{for} \quad \tau_1, \tau_2 \in (\tau_3 - \eta_1, \tau_3 + \eta_1)\,,$$

where $0 < \eta_1 \leqq \eta$ and η_1 does not depend on ξ.

Consequently $u(\tau) = 0$ for $\tau \in \langle\tau_3 - \eta_1, \tau_3 + \eta_1\rangle$ and we arrive at a contradiction. Lemma 5,1,1 is proved.

5,2. The convergence of solutions. Let the functions $\Phi_i(x)$, $i = 1, 2 \ldots, k$, $\Phi_i(x) \in E_n$ fulfil a Lipschitz condition $\|\Phi_i(x_2) - \Phi_i(x_1)\| \leqq K\|x_2 - x_1\|$ and let $\alpha_i > 0$, $\beta > 0$, $\zeta_i \neq 0$, $i = 1, 2, \ldots, k$, $\zeta_{k+1} \neq 0$, $\Lambda \in E_n$. It is known that the solution $x(\tau)$ of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \sum_{i=1}^{k} \Phi_i(x)\, j^{1-\alpha_i} \sin(j\zeta_i\tau + \eta_i) + \Lambda j^{1-\beta} \sin(j\zeta_{k+1}\tau + \eta_{k+1}) \tag{5,2,1}$$

is uniquely defined on $(-\infty, \infty)$. According to Theorem 2,2,1 (5,2,1) has the same solution as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_j(x, \tau)\,, \; x(0) = 0\,, \tag{5,2,2}$$

where

$$F_j(x, t) = -\sum_{i=1}^{k} \zeta_i^{-1} \Phi_i(x)\, j^{-\alpha_i} \cos(j\zeta_i t + \eta_i) - \zeta_{k+1}^{-1} A j^{-\beta} \cos(j\zeta_{k+1} t + \eta_{k+1}) .$$

If $\alpha + \beta > 1$, $\alpha = \min_{i=1,\dots,k} \alpha_i > \frac{1}{2}$, then $F_j(x, t) \in \mathbf{F}(G, K_1\eta^\gamma, K_2\eta^\alpha, 1)$, where $\gamma = \min(\alpha, \beta)$, G is an open bounded subset of E_n and according to Theorem 4,2,1 $x_j(\tau) \to 0$ almost uniformly.

LITERATURE

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Резюме

ОБОБЩЕННОЕ ОБЫКНОВЕННОЕ ДИФФЕРЕНЦИАЛЬНОЕ УРАВНЕНИЕ И НЕПРЕРЫВНАЯ ЗАВИСИМОСТЬ ОТ ПАРАМЕТРА

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Цель работы — объяснение явления сходимости, с которым мы встречаемся, напр., у последовательности дифференциальных уравнений

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = x_k k^{1-\alpha} \cos kt + k^{1-\beta} \sin kt\,, \quad x_k(0) = 0,\ k = 1, 2, 3, \dots,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0\,, \quad x(0) = 0\,. \tag{1}$$

Нетрудно видеть, что решения x_k сходятся равномерно к функции $x(t) \equiv 0$ на всяком замкнутом интервале, если $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$ и что они не сходятся к $x(t) \equiv 0$, если $\alpha + \beta = 1$ ($\alpha > 0$, $\beta > 0$).[1])

Прежде всего мы вводим *обобщение интеграла Перрона*: Пусть действительная функция $U(\tau, t)$ определена для $\tau_* \leqq \tau \leqq \tau^*$, $\tau - \delta(\tau) \leqq t \leqq \tau + \delta(\tau)$, где $\delta(\tau)$ — положительная функция. Функцию $M(\tau)$ мы назовем

[1]) Общая теорема, которую можно применить в случае $\alpha = 1$, $\beta = 1$, была доказана *М. А. Красносельским* и *С. Г. Крейном* в работе [1]. При более общих предположениях эта теорема доказана в работе [2] автора и З. Ворла и ее можно использовать в случае $\alpha = 1$, $1 \geqq \beta > 0$.

верхней функцией, если существует положительная функция $\delta'(\tau) \leqq \delta(\tau)$ такая, что

$$(\tau - \tau_0)(M(\tau) - M(\tau)_0) \geqq (\tau - \tau_0)\,(U(\tau, \tau_0) - U(\tau_0, \tau_0))\,,$$

если только $\tau_* \leqq \tau_0 \leqq \tau^*$, $\tau_0 - \delta'(\tau_0) \leqq \tau \leqq \tau_0 + \delta'(\tau_0)$. (Если справедливо обратное неравенство, то функцию мы называем нижней функцией.) Если существует и верхняя функция и нижняя функция и если $\inf\limits_{M(t)} (M(\tau^*) - M(\tau_*)) = \sup\limits_{m(\tau)}(m(\tau^*) - m(\tau_*))$, где $M(\tau)$ пробегает все верхние функции, а $m(\tau)$ пробегает все нижние функции, то это число мы называем интегралом Перрона от $\mathrm{D}U$ от τ_* до τ^* и обозначаем символом $\int\limits_{\tau_*}^{\tau^*} \mathrm{D}U$.

Интеграл $\int\limits_{\tau_*}^{\tau^*} \mathrm{D}U$ можно определить также при помощи известных обобщенных сумм Римана. Если положить $U(\tau, t) = f(\tau)\,\varphi(t)$, где $\varphi(t)$ есть функция с ограниченным изменением, то $\int\limits_{\tau_*}^{\tau^*} \mathrm{D}U$ существует в точности тогда, когда существует $\int\limits_{\tau_*}^{\tau^*} f(\tau)\,\mathrm{d}\varphi(\tau)$ в смысле Перрона и эти два интеграла равны между собой.

Во втором параграфе мы приступаем к обобщенным дифференциальным уравнениям. Пусть дана функция $F(x, \tau, t)$. Функцию $x(\tau)$ мы называем *решением обобщенного дифференциального* уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, \tau, t)\,, \tag{2}$$

если $x(\tau_2) - x(\tau_1) = \int\limits_{\tau_1}^{\tau_2} \mathrm{D}F(x(\tau), \tau, t)$ для любых τ_1, τ_2. (Мы всюду предполагаем, что x и F — векторы евклидова пространства E_n, $F = (F_1, \ldots, F_n)$, и естественно полагаем $\int\limits_{\tau_1}^{\tau_2} \mathrm{D}F = (\int\limits_{\tau_1}^{\tau_2} \mathrm{D}F_1, \ldots, \int\limits_{\tau_1}^{\tau_2} \mathrm{D}F_n)$.) Если $f(x, \tau)$ — непрерывная функция и если положить $F(x, \tau, t) = f(x, \tau)t$ или $F(x, \tau, t) = \int\limits_{\tau}^{t} f(x, \sigma)\,\mathrm{d}\sigma$, то каждое решение уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau) \tag{3}$$

будет в то же время решением уравнения (2) и наоборот.

В третьем параграфе мы доказываем существование интеграла $\int\limits_{\tau_*}^{\tau^*} \mathrm{D}U$ в предположении, что функция $U(\tau, t)$ непрерывна и что разность $U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)$ приближается достаточно быстро к нулю для $\eta \to 0$. Далее мы доказываем теорему о сходимости $\int\limits_{\tau_*}^{\tau^*} \mathrm{D}U_k \to \int\limits_{\tau_*}^{\tau^*} \mathrm{D}U$.

Четвертый параграф содержит основы теории уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t)\,. \tag{4}$$

Пусть $G \subset E_{n+1}$ — открытое множество, $\sigma > 0$, пусть функции $\omega_i(\eta)$, $i = 1, 2$ определены, непрерывны и не убывают для $0 \leqq \eta \leqq \sigma$, $\omega_i(\eta) \geqq c\eta$, $\omega_i(0) = 0$ $(c > 0)$ для $0 \leqq \eta \leqq \sigma$. Пусть $\mathbf{F} = \mathbf{F}(G, \omega_1, \omega_2, \sigma)$ обозначает множество функций $F(x, t)$, определенных и непрерывных в G и выполняющих неравенства

$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) \quad \text{для} \quad (x, t_1), (x, t_2) \in G,\ |t_2 - t_1| \leqq \sigma,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\, \omega_2(|t_2 - t_1|)$$
$$\text{для } (x_2, t_2), (x_2, t_1), (x_1, t_2), (x_1, t_1) \in G,\ \|x_2 - x_1\| \leqq 2\omega_1(\sigma),\ |t_2 - t_1| \leqq \sigma\,.$$

Положим $\psi(\eta) = \omega_1(\eta)\, \omega_2(\eta)$ и предположим, что функция $\eta^{-1}\psi(\eta)$ не убывает и что $\sum_{i=1}^{\infty} 2^i \psi\left(\frac{\sigma}{2^i}\right) < \infty$ [2].

Если $F(x, t) \in \mathbf{F}$, то для уравнения (4) имеет место *теорема о существовании*, подобная классической теореме о существовании: Если $(x_0, t_0) \in G$, то существует *регулярное решение* $x(\tau)$ уравнения (4), определенное на некотором интервале $\langle t_0 - \sigma_*, t_0 + \sigma^* \rangle$. (Решение $x(\tau)$ уравнения (4), определенное на интервале $\langle \tau_1, \tau_2 \rangle$ называется *регулярным*, если для любого $\tau \in \langle \tau_1, \tau_2 \rangle$ существует $\lambda > 0$ так, что $\|x(\tau_4) - x(\tau_3)\| \leqq 2\omega_1(|\tau_4 - \tau_3|)$, если $\tau_3, \tau_4 \in \langle \tau - \lambda, \tau + \lambda \rangle \cap \langle \tau_1, \tau_2 \rangle$).

Точно так же и теорема *о непрерывной зависимости от параметра* подобна соответствующей хорошо известной классической теореме: Пусть $F_k(x, t), F(x, t) \in \mathbf{F}$, $k = 1, 2, 3, \ldots$ $F_k(x, t) \to F(x, t)$ равномерно на G. Пусть $(x_0, t_0) \in G$ и пусть регулярное решение $y(\tau)$ уравнения (4) определено для $t_0 \leqq \tau \leqq t_1$, $(t_0 < t_1)$, $y(t_0) = x_0$, и пусть выполняется следующее условие однозначности: если $z(\tau)$ — регулярное решение уравнения (4), определенное для $t_0 \leqq \tau \leqq t_2$, где $t_0 < t_2 \leqq t_1$, $z(t_0) = x_0$, то $z(\tau) = y(\tau)$ для $t_0 \leqq \tau \leqq t_2$. Пусть $x_k \in E_n$, $x_k \to x_0$. Тогда для всех достаточно больших k существует регулярное решение $y_k(\tau)$ уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t)\,, \tag{5}$$

определенное на интервале $\langle t_0, t_1 \rangle$, $y_k(t_0) = x_k$. Решение $y_k(\tau)$ не должно быть однозначно определенным, но всегда имеет место $y_k(\tau) \to y(\tau)$ равномерно на $\langle t_0, t_1 \rangle$.

Применим этот результат к уравнению (1). Положим $F_k(x, t) = xk^{-\alpha} \sin kt + k^{-\beta}(1 - \cos kt)$, $F(x, t) = 0$ и легко обнаружим, что $F_k(x, t), F(x, t) \in \mathbf{F}(G, K\eta^\gamma, K\eta^\alpha, 1)$, где G — любое открытое ограниченное

[2]) Напр., можно положить $\psi(\eta) = \eta^{1+\varepsilon}$, $\varepsilon > 0$.

подмножество пространства E_{n+1}, K — достаточно большая постоянная и $\gamma = \min(\alpha, \beta)$. Итак, $x_k(t) \to 0$ равномерно на всяком замкнутом интервале, если $\gamma + \alpha > 1$, т. е. если $\alpha > \frac{1}{2}$, $\alpha + \beta > 1$. Явление сходимости для уравнения (1) полностью не объяснено, так как мы получаем добавочное условие $\alpha > \frac{1}{2}$.

В пятом параграфе подробно исследуется случай линейных уравнений, доказывается теорема об однозначности для обобщенных линейных уравнений и указано применение теоремы о непрерывной зависимости от параметра к классическим дифференциальным уравнениям.

Заметим, наконец, что полученные результаты можно истолковать при помощи т. наз. *обобщенных функций* (*распределений*). Если $F(x, t) \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$ $\Bigg($положим снова $\psi(\eta) = \omega_1(\eta)\, \omega_2(\eta)$ и предположим, что $\eta^{-1}\psi(\eta)$ не убывает и что $\sum\limits_{i=1}^{\infty} 2^i\, \psi\left(\dfrac{\sigma}{2^i}\right) < \infty\Bigg)$, положим $f(x, t) = \dfrac{\partial F}{\partial t}$, где производную мы берем в смысле теории обобщенных функций. Пусть функция $y(\tau)$ определена на интервале $\langle \tau_1, \tau_2 \rangle$ и пусть выполняются следующие условия:

1. $y(\tau) \in G$ для $\tau \in \langle \tau_1, \tau_2 \rangle$.

2. Существует функция $\omega_3(\eta)$, определенная, непрерывная и неубывающая для $0 \leqq \eta \leqq \sigma_1$, $(0 < \sigma_1 \leqq \sigma)$, $\omega_3(\eta) > c\eta$, $c > 0$, $\omega_3(0) = 0$. Если положить $\psi_3(\eta) = \omega_3(\eta)\, \omega_2(\eta)$, то $\eta^{-1}\psi_3(\eta)$ не убывает, $\sum\limits_{i=1}^{\infty} 2^i \psi_3\left(\dfrac{\sigma_1}{2^i}\right) < \infty$ и имеет место

$$\|y(\tau_4) - y(\tau_3)\| \leqq \omega_3(|\tau_4 - \tau_3|) \text{ для } \tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle, |\tau_4 - \tau_3| \leqq \sigma_1 .$$

В таком случае существует функция $g(\xi) = \int\limits_{\tau_0}^{\xi} \mathrm{D}F(y(\tau), t)$ для $\tau_0, \xi \in \langle \tau_1, \tau_2 \rangle$; эта функция непрерывна. Можно определить

$$f(y(t), t) = \frac{\mathrm{d}}{\mathrm{d}t}\, g(t) .$$

Функцию $y(t)$ мы назовем решением уравнения

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t) \qquad \textbf{(6)}$$

если выполнены условия 1., 2. и если обобщенные функции в обеих частях уравнения (6) равны, если подставить $y = y(t)$. Можно доказать, что $y(t)$ будет решением уравнения (6) в точности тогда, когда $y(\tau)$ — регулярное решение уравнения (4).