Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 7 (1957), No. 2, 248–253

Persistent URL: http://dml.cz/dmlcz/100245

Чехословацкий математический журнал, т. 7 (82) 1957, Прага

THE BAIRE AND BOREL MEASURE

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(Received September 14, 1956.)

This paper contains the main results of [5]. It is shown that in many important cases the Baire measure can be extended to a Borel measure.

1. Let $\mathfrak{A}$ be a non-empty family of sets. We say that $\mathfrak{A}$ is a *field* if the sum and the difference of each pair of elements $A, B \in \mathfrak{A}$ also belongs to $\mathfrak{A}$. If $\mathfrak{A}$ is a field and if $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$ (resp. $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$) whenever $A_n \in \mathfrak{A}$ ($n = 1, 2, \ldots$), then $\mathfrak{A}$ is called a *σ-field* (resp. *δ-field*). If $\mathfrak{A}$ is a σ-field and if $\bigcup \mathfrak{A} \in \mathfrak{A}$, we say that $\mathfrak{A}$ is a *σ-algebra*.

If $A_1, A_2, \ldots$ are sets and if $A_1 \subset A_2 \subset \ldots$, $\bigcup_{n=1}^{\infty} A_n = A$, we write $A_n \nearrow A$.

A non-negative σ-additive function μ (on a field $\mathfrak{A}$) such that $\mu(\emptyset) = 0$ is termed a *measure* (on $\mathfrak{A}$). If μ is a measure on a σ-algebra $\mathfrak{A}$, we put for each $M \subset \bigcup \mathfrak{A}$

$$\bar{\mu}(M) = \inf \mu(A)\,, \quad \text{where } A \in \mathfrak{A}\,, \quad A \supset M\,,$$
$$\underline{\mu}(M) = \sup \mu(A)\,, \quad \text{where } A \in \mathfrak{A}\,, \quad A \subset M\,.$$

Let P be a topological space.[1]) Let $\mathfrak{G}$ (resp. $\mathfrak{F}$) be the family of all open (resp. closed) subsets of P. Let $\mathfrak{G}^*$ (resp. $\mathfrak{F}^*$) be the family of all sets $E[x; f(x) > 0]$ (resp. $E[x; f(x) = 0]$, where f is a continuous function on P. Let $\mathfrak{B}$ (resp. $\mathfrak{B}^*$) be the smallest σ-algebra containing $\mathfrak{G}$ (resp. $\mathfrak{G}^*$). The elements of $\mathfrak{B}$ (resp. $\mathfrak{B}^*$) are called *Borel* (resp. *Baire*) *sets*; a measure on the system $\mathfrak{B}$ (resp. $\mathfrak{B}^*$) is termed a *Borel* (resp. *Baire*) *measure*.

Given a Baire measure μ, let $\mathfrak{P}$ be the family of all sets $A \subset P$ for which there exist $G_n \in \mathfrak{G}^*$ such that

$$A \subset \bigcup_{n=1}^{\infty} G_n\,, \quad \mu(G_n) < \infty \quad (n = 1, 2, \ldots)\,.$$

[1]) We suppose that the topology is defined by means of a system $\mathfrak{G}$ (whose elements are subsets of P) with the following properties: 1) $\emptyset, P \in \mathfrak{G}$; 2) $G_1, G_2 \in \mathfrak{G} \Rightarrow G_1 \cap G_2 \in \mathfrak{G}$; 3) $\mathfrak{G}_0 \subset \mathfrak{G} \Rightarrow \bigcup \mathfrak{G}_0 \in \mathfrak{G}$.

We say that μ has the property V_P if each set $A \in \mathfrak{B}^*$ such that $\mu(A) < \infty$ belongs to $\mathfrak{P}$.

2. We state now an elementary lemma which is important for further considerations:

Let P be an arbitrary set; let $\mathfrak{M}$, $\mathfrak{N}$ be systems of subsets of P and let $\emptyset \in \mathfrak{M} \cap \mathfrak{N}$. Let α (resp. β) be a finite non-negative function on $\mathfrak{M}$ (resp. on $\mathfrak{N}$). Suppose that the following conditions are fulfilled:

1) $M \in \mathfrak{M},\ N \in \mathfrak{N} \Rightarrow M - N \in \mathfrak{M},\ N - M \in \mathfrak{N}$;
2) $M_1, M_2 \in \mathfrak{M},\ M_1 \cap M_2 = \emptyset \Rightarrow M_1 \cup M_2 \in \mathfrak{M},\ \alpha(M_1) + \alpha(M_2) = = \alpha(M_1 \cup M_2)$;
3) $M \in \mathfrak{M},\ N \in \mathfrak{N},\ M \subset N \Rightarrow \beta(N - M) = \beta(N) - \alpha(M)$;
4) $N \in \mathfrak{N} \Rightarrow \beta(N) \leqq \sup \alpha(M)$, *where* $M \subset N,\ M \in \mathfrak{M}$;
5) $N_n \in \mathfrak{N}\ (n = 1, 2, \ldots),\ \sum_{n=1}^{\infty} \beta(N_n) < \infty \Rightarrow \bigcup_{n=1}^{\infty} N_n \in \mathfrak{N},\ \beta(\bigcup_{n=1}^{\infty} N_n) \leqq \sum_{n=1}^{\infty} \beta(N_n)$.

For each $A \subset P$ put

$$\underline{\gamma}(A) = \sup \alpha(M), \quad \textit{where } M \in \mathfrak{M},\ M \subset A\,,$$
$$\overline{\gamma}(A) = \inf \beta(N), \quad \textit{where } N \in \mathfrak{N},\ N \supset A\,.$$ [2]

Let $\mathfrak{T}$ be the system of all sets $T \subset P$ for which $\underline{\gamma}(T) = \overline{\gamma}(T) < \infty$; let $\mathfrak{A}$ be the system of all $A \subset P$ such that $A \cap T \in \mathfrak{T}$ for each $T \in \mathfrak{T}$.

Then $\mathfrak{T}$ is a δ-field, $\mathfrak{A}$ is a σ-algebra and $\overline{\gamma}$ is a measure on $\mathfrak{A}$. Furthermore, $\mathfrak{N} \subset \mathfrak{T} \subset \mathfrak{A}$ and $\underline{\gamma}(N) = \overline{\gamma}(N) = \beta(N)$ for each $N \in \mathfrak{N}$.

(The proof is not difficult.)

3. Now let the measure μ have the property V_P. Let $\mathfrak{M}$ (resp. $\mathfrak{N}$) be the family of all $A \in \mathfrak{F}^*$ (resp. $\mathfrak{G}^*$) such that $\mu(A) < \infty$. If we put $\alpha(M) = \mu(M)$, $\beta(N) = \mu(N)$ for $M \in \mathfrak{M}$, $N \in \mathfrak{N}$, then all the conditions of the preceding lemma are satisfied and we easily obtain the following assertion:

If the measure μ has the property V_P, then

$$\overline{\mu}(A) = \inf \mu(G)\,, \quad \textit{where} \quad G \in \mathfrak{G}^*\,, \quad G \supset A\,, \tag{1}$$

for each $A \subset P$, and

$$\underline{\mu}(A) = \sup \mu(F)\,, \quad \textit{where} \quad F \in \mathfrak{F}^*\,, \quad F \subset A\,, \quad \mu(F) < \infty\,, \tag{2}$$

for each $A \in \mathfrak{P}$.

4. We say that the measure μ has the property W_P, if μ is a Baire measure and if there exists a Borel measure ν with the following properties:

1) $B \in \mathfrak{B}^* \Rightarrow \nu(B) = \mu(B)$;
2) $G \in \mathfrak{G} \cap \mathfrak{P} \Rightarrow \nu(G) = \underline{\mu}(G)$;

[2]) If there exist no $N \in \mathfrak{N}$ such that $N \supset A$, we have $\overline{\gamma}(A) = \inf \emptyset = \infty$.

3) $B \in \mathfrak{B} - \mathfrak{P} \Rightarrow \nu(B) = \infty$;

4) $B \in \mathfrak{B} \Rightarrow \nu(B) = \inf \nu(G)$, where $G \in \mathfrak{G}$, $G \supset B$.

Now we can state the following theorem:

Let the measure μ have the property V_P and let the implications

$$G_1, G_2 \in \mathfrak{G} \cap \mathfrak{P} \Rightarrow \underline{\mu}(G_1) + \underline{\mu}(G_2) \geqq \underline{\mu}(G_1 \cup G_2)\,, \tag{3}$$

$$G_n \in \mathfrak{G} \cap \mathfrak{P} \quad (n = 1, 2, \ldots), \quad G_n \nearrow G \Rightarrow \underline{\mu}(G_n) \to \underline{\mu}(G) \tag{4}$$

be valid. Then the measure μ has the property W_P.

The proof is based on the following ideas: Let $\mathfrak{M}$ (resp. $\mathfrak{N}$) be the family of all $A \in \mathfrak{F}$ (resp. $\mathfrak{G} \cap \mathfrak{P}$) such that $\bar{\mu}(A) < \infty$ (resp. $\underline{\mu}(A) < \infty$). If we put $\alpha(M) = \bar{\mu}(M)$, $\beta(N) = \underline{\mu}(N)$ for $M \in \mathfrak{M}$, $N \in \mathfrak{N}$, then it follows from (1), (2), (3), (4), that all the conditions of Section 2 are fulfilled. It is easy to see that the corresponding system $\mathfrak{A}$ contains each open subset of P and that the conditions 1)—4) are satisfied if we write $\nu(B) = \bar{\gamma}(B)$ for each $B \in \mathfrak{B}$.

Remark. If the space P is normal, we have clearly

$$\bar{\mu}(F) \leqq \underline{\mu}(G)\,, \tag{5}$$

whenever $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, $F \subset G$. It follows easily from (5) that (3) holds in each normal space.

5. Now we are able to prove the following assertion:

Let the measure μ have the property V_P and let some of the three following conditions be fulfilled:

1) *P is completely regular and for each $F \in \mathfrak{F}^*$, where $\mu(F) < \infty$, there exist compact sets K_n such that $\underline{\mu}(F - \bigcup_{n=1}^{\infty} K_n) = 0$.*

2) *P is normal and for each $F \in \mathfrak{F}^*$, where $\mu(F) < \infty$, there exist pseudocompact*[3]*) sets A_n such that $\underline{\mu}(F - \bigcup_{n=1}^{\infty} A_n) = 0$.*

3) *P is normal and countably paracompact.*[4]*)*

Then μ has the property W_P.

We have to prove that the conditions (3) and (4) are satisfied. If the space P has one of the properties 1) or 2), then the proof requires only elementary considerations. Now let P be normal and countably paracompact; let μ have the property V_P. If $G_n \in \mathfrak{G} \cap \mathfrak{P}$, $G_n \nearrow G$, we choose a set $F \in \mathfrak{F}$, $F \subset G$. Making use of the theorem which asserts that a normal space P is countably paracompact if and only if for each sequence $U_1, U_2, \ldots,$ where $U_n \in \mathfrak{G}$, $U_n \nearrow P$,

[3]) The space A is *pseudocompact* if each continuous function on A is bounded.

[4]) P is *countably paracompact* if for each countable open covering of P there exists a locally finite refinement.

there exist sets $D_n \in \mathfrak{F}$ such that $D_n \subset U_n$ $(n = 1, 2, \ldots)$ and $D_n \nearrow P$ (see [1] or [3]), we see that there exist $F_n \in \mathfrak{F}$ such that $F_n \subset G_n$, $F_n \nearrow F$. By (5) we get $\underline{\mu}(G_n) \geqq \overline{\mu}(F_n) \to \overline{\mu}(F)$, whence $\lim \underline{\mu}(G_n) \geqq \overline{\mu}(F)$, $\lim \underline{\mu}(G_n) \geqq \sup \overline{\mu}(F) = \underline{\mu}(G)$, which proves (4). Because the relation (3) holds in each normal space, we see that the proof is complete.

Remark. Combining this result with [4], p. 479, we obtain various theorems concerning the representation of a non-negative functional by means of an integral $\int_P f \, d\nu$, where ν is a Borel measure.

6. If J is a non-negative linear functional which is defined on the family of all continuous functions on a topological space P, then there exists (see [4], p. 479) a unique Baire measure μ such that

$$J(f) = \int_P f \, d\mu$$

for each continuous f. (The measure μ is obviously finite and has therefore the property V_P.) For each $G \in \mathfrak{G}$ put

$$\delta(G) = \sup J(f) ,$$

where f is continuous, $f(x) \leqq 1$ on G, $f(x) \leqq 0$ on $P - G$. It is easy to see that

$$\delta(G) \leqq \underline{\mu}(G) \quad (G \in \mathfrak{G}) ; \tag{6}$$

if P is normal, then

$$\delta(G) = \underline{\mu}(G) \quad (G \in \mathfrak{G}) . \tag{7}$$

If the space P fulfils the condition 1) (Section 5), then (7) holds again. If P is a completely regular Q-space, then there exists a compact set K such that $\underline{\mu}(P - K) = 0$ (see [2]) and (7) is fulfilled. We see at the same time that the measure μ has in this case the property W_P.

Now let P be an arbitrary topological space. If there exists a Borel measure ν such that

$$J(f) = \int_P f \, d\nu \quad (f \text{ continuous on } P) , \tag{8}$$

then obviously $\nu(B) = \mu(B)$ for each $B \in \mathfrak{B}^*$ and, consequently,

$$\underline{\mu}(B) \leqq \nu(B) \leqq \overline{\mu}(B) \tag{9}$$

for each $B \in \mathfrak{B}$. If, moreover,

$$\nu(G) = \delta(G) \quad (G \in \mathfrak{G}) , \tag{10}$$

then it follows from (6) and (9) that (7) holds again.

In [2], p. 170, Hewitt raised the following question: Let J be a non-negative linear functional which is defined on the system of all continuous functions on a normal space P. Does there exist a Borel measure ν such that the relations

(8) and (10) are true? It follows from Section 5 that the answer is affirmative if P is countably paracompact; but we do not yet know a normal space which has not this property.

If the space P is not normal, it may happen that $\delta(G) < \underline{\mu}(G)$ for some open set G; then there exists no Borel measure ν such that the relations (8) and (10) hold good. Such example (where P is completely regular) is constructed in [2], pp. 169—170 (Remark 1); but the corresponding Baire measure has the property W_P again.

Now let Ω be the smallest non-countable ordinal number; let T be the space of all the ordinal numbers $\xi \leqq \Omega$ and put $P = T \times T - \{[\Omega, \Omega]\}$. It is easy to see that we can put

$$J(f) = \lim_{[\xi,\eta] \to [\Omega,\Omega]} f(\xi, \eta)$$

for each continuous f. Then the corresponding Baire measure has not the property W_P, but it is possible to extend it to a Borel measure.

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Резюме

МЕРЫ БЭРА И БОРЕЛЯ

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(Поступило в редакцию 14/IX 1956 г.)

Пусть P — топологическое пространство. Пусть $\mathfrak{F}$ (соотв. $\mathfrak{G}$) — система всех замкнутых (соотв. открытых) подмножеств пространства P; пусть $\mathfrak{F}^*$ (соотв. $\mathfrak{G}^*$) — система всех множеств вида $E[x; f(x) = 0]$ (соотв. $E[x; f(x) > 0]$), где f — непрерывная функция на пространстве P. Далее пусть

$\mathfrak{B}$ (соотв. $\mathfrak{B}^*$) — наименьшая σ-алгебра, содержащая систему $\mathfrak{F}$ (соотв. $\mathfrak{F}^*$). Неотрицательную σ-аддитивную функцию на системе $\mathfrak{B}$ (соотв. $\mathfrak{B}^*$) назовем мерой Бореля (соотв. Бэра).

Пусть μ — мера Бэра: пусть $\mathfrak{P}$ — система всех множеств $A \subset P$, для которых существуют $G_n \in \mathfrak{G}^*$ так, что $\mu(G_n) < \infty$ $(n = 1, 2, \ldots)$, $A \subset \bigcup_{n=1}^{\infty} G_n$. Предположим, что мера μ обладает следующим свойством: Если $B \in \mathfrak{B}^*$, $\mu(B) < \infty$, то $B \in \mathfrak{P}$. Далее положим для каждого $A \subset P$

$$\underline{\mu}(A) = \sup \mu(B)\,, \quad \text{где} \quad B \subset A\,, \quad B \in \mathfrak{B}^*\,,$$
$$\overline{\mu}(A) = \inf \mu(B)\,, \quad \text{где} \quad B \supset A\,, \quad B \in \mathfrak{B}^*\,.$$

В работе намечены главные идеи доказательства следующих двух теорем:

Теорема 1. *Для любого $A \subset P$ имеет место*

$$\overline{\mu}(A) = \inf \mu(G)\,, \quad \text{где} \quad G \in \mathfrak{G}^*\,, \quad G \supset A\,;$$

если $\underline{\mu}(A) < \infty$, будет также

$$\underline{\mu}(A) = \sup \mu(F)\,, \quad \text{где} \quad F \subset A\,, \quad F \in \mathfrak{F}^*\,.$$

Теорема 2. *Пусть выполняется какое-либо из следующих трех условий:*

а) *Пространство P вполне регулярно и для любого $F \in \mathfrak{F}^*$, где $\mu(F) < \infty$, существуют компактные множества K_n так, что $\underline{\mu}(F - \bigcup_{n=1}^{\infty} K_n) = 0$.*

б) *Пространство P нормально и для любого $F \in \mathfrak{F}^*$, где $\mu(F) < \infty$, существуют псевдокомпактные множества A_n так, что $\underline{\mu}(F - \bigcup_{n=1}^{\infty} A_n) = 0$.*

в) *Пространство P нормально и счетно-паракомпактно.*

Тогда существует мера Бореля ν, обладающая следующими свойствами:

α) $B \in \mathfrak{B}^* \Rightarrow \nu(B) = \mu(B)$;

β) $G \in \mathfrak{G} \cap \mathfrak{P} \Rightarrow \nu(G) = \underline{\mu}(G)$;

γ) $B \in \mathfrak{B} - \mathfrak{P} \Rightarrow \nu(B) = \infty$;

δ) $B \in \mathfrak{B} \Rightarrow \nu(B) = \inf \nu(G)$, где $G \in \mathfrak{G}$, $G \supset B$.

Подробное доказательство этих теорем приведено в [5].