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DIRECT DECOMPOSITIONS OF LATTICES, I

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This article contains the foundations of the algebraical theory of direct and subdirect decompositions of lattices and rings. Except for theorem 14 and most of theorems 2 and 3, all non-trivial results are new.

We shall, in general, use the notation of LT, with some exceptions. (LT means G. Birkhoff, Lattice Theory, 2nd. ed., New York, 1948.) In lattices, $\bar{a}$ is the set of $x \leqq a$, $\underline{a}$ the set of $x \geqq a$, $\langle a, b \rangle$ the set (interval) of $a \leqq x \leqq b$; x' is the complement of x even if not unique, x^* the pseudocomplement. In the general case, $\cup$, $\cap$ and $\subset$ are set-joins, meets and inclusions, reserving $\vee$, $\wedge$ and $\leqq$ for the lattice-operations; δ is the Kronecker delta:

$$\delta_a^b = \begin{cases} O \text{ if } a \neq b \\ I \text{ if } a = b \end{cases}, \qquad \text{and} \qquad \delta_x^A = \begin{cases} O \text{ if } x \text{ non} \in A \\ I \text{ if } x \in A \end{cases};$$

$\Rightarrow$ is implication; * in the text means end of proof. "Homomorphism" always means lattice-homomorphism.

For most of the elementary definitions use LT.

1. Preliminary notions

Definition. *If S_a $(a \in A)$ are abstract algebras of the same type with α-ary operations Σ, then $\mathsf{P}_a S_a$ is the abstract algebra (direct product) consisting of all maps*

$$[x_a]_a : A \rightarrow \mathbf{U}_a S_a \quad \text{with} \quad x_a \in S_a ,$$

with α-ary operations $\Sigma_b[x_a^b]_a = [\Sigma_b x_a^b]_a$. (For a finite A we shall, of course, use $S_1 \times S_2 \times \ldots \times S_n$, $[x_1, \ldots, x_n]$, etc.; also, often, $[x_a]$ instead of $[x_a]_a$.)

Definition. $S = \mathsf{P}_a S_a$, *read as "S is (decomposable into) the direct product of S_a's", means that there is an algebraic isomorphism between S and $\mathsf{P}_a S_a$.*

We do make an unlawful use of the equality sign here; but note that it is only in definitions that the equality sign between an algebra and a direct product is really justified — in such cases we shall use $\equiv$. Similarly we write $x = [x_a]$ for "x corresponds to $[x_a]$", usually adding "in $\mathsf{P}_a S_a$"; thus if we have two direct decompositions in which $x = [x_a]$ and $x = [y_a]$ respectively, then $x_a \neq y_a$ is usually true.

By definition, $S_1 \times S_2 \equiv S_2 \times S_1$; and $\mathsf{P}_{b \in B}(\mathsf{P}_{a \in A_b} S_{ab}) = \mathsf{P}_{c \in C} S_c$ with C consisting of pairs $[a, b]$ with $a \in A_b$. Thus direct products are commutative and associative.

Definition. *$S \leqq \mathsf{P}_a S_a$ read as "S is (decomposable into) the subdirect product of S_a's" means that* (a) *there is an algebraic isomorphism between S and a subalgebra of $\mathsf{P}_a S_a$, and* (b) *to every $b \in A$ and $x \in S_b$ there is an $[x_a] \in S$ with $x_b = x$.*

Note that again $S \leqq \mathsf{P}_{b \in B} S_b$, $S_b \leqq \mathsf{P}_{a \in A_b} S_{ab}$ imply $S \leqq \mathsf{P}_{b \in B} \mathsf{P}_{a \in A_b} S_{ab}$; and $S_b \leqq \mathsf{P}_{a \in A} S_{ab}$ implies $\mathsf{P}_{b \in B} S_b \leqq \mathsf{P}_{a \in A}(\mathsf{P}_{b \in B} S_{ab})$ (read carefully: subdirect product of direct products). But $S \leqq S_1 \times S_2 \times S_3$ does not imply $S \leqq S_1 \times (S_2 \times S_3)$; e. g., for lattices, $\mathbf{2} \leqq \mathbf{2} \times \mathbf{2} \times \mathbf{2}$*) (the isomorphism is $x \to [x, x, x]$, while $\mathbf{2}$ non $\leqq \mathbf{2} \times \mathbf{4}$. Note also that $S_1 \leqq S_2$ means $S_1 = S_2$, since $S_2 \equiv S_2 \times \mathbf{1}$, where $\mathbf{1}$ is the one-element algebra (if it exists).

Obviously, there is an intimate connection between lattices (and generally, algebras) and their direct decompositions. Thus if $L = \mathsf{P}_a L_a$, then L has I if and only if all L_a have I, L is distributive if and only if every L_a is such, $x = [x_a]$ is complemented if and only if every x_a is complemented, and then $x' = [x'_a]$, etc. Equally obviously, not all of this is true for subdirect decompositions. The following theorem is therefore of some interest.

Theorem 1. *Let $L \leqq \mathsf{P}_a L_a$ with lattices L, L_a. Then if L is modular, so are L_a; if L is distributive, so are L_a.*

Proof. Take $x, y, z \in L_a$; there exist $u, v, w \in L$ whose a-th coordinates are x, y, z respectively.

Now let L be modular and $x \leqq z$; set $u_1 = u \wedge w$, $v_1 = v$, $w_1 = u \vee w$; then $u_1, v_1, w_1 \in L$ and their a-th coordinates are x, y, z respectively again, and $u_1 \leqq w_1$; since L is modular, $(u_1 \vee v_1) \wedge w_1 = u_1 \vee (v_1 \wedge w_1) \Rightarrow (x \vee y) \wedge z = x \vee (y \wedge z)$ in L_a.

Finally let L be distributive; then $(u \vee v) \wedge w \leqq u \vee (v \wedge w)$, $\Rightarrow (x \vee y) \wedge z \leqq x \vee (y \wedge z)$, so that L_a is distributive (LT, IX, § 1, ex. 3).*

2. Central and neutral elements

In paragraphs 2 to 5 large letters denote lattices, excepting O, I, and A, which shall always mean a set of indices.

*) $\mathbf{n}$ is the chain of n elements.

Theorem 2. *These properties of a (neutral) element $e \in L$ are equivalent:*

(a) *for any $x, y \in L$, the sublattice generated by (e, x, y) is distributive;*

(b) *for any $x, y \in L$, $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$ and dually, $x \wedge (e \vee y) = (x \wedge e) \vee (x \wedge y)$ and dually;*

(c) *for any $x, y \in L$, $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$ and dually, $e \wedge x = e \wedge y$, $e \vee x = e \vee y \Rightarrow x = y$;*

(d) $e = [\delta^A_a]_a$ *under some isomorphism between L and a sublattice of $\mathsf{P}_a L_a$;*

(e) $e = [I, O]$ *under some subdirect decomposition of L;*

(f) *there exist disjoint congruence relations Θ_1, Θ_2 such that $x \wedge e\Theta_1 x$ and $x \vee e\Theta_2 x$;*

(g) *any maximal distributive sublattice contains e.*

Proof. (See LT, II, § 10, including exer. 1a.) Obviously (e) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (b); (b) $\Rightarrow$ (c) is implicitly contained in the proof of th. 10, l. c.; for (c) $\Rightarrow$ (e), (g) $\Longleftrightarrow$ (*a*) see LT again (th. 11). For (e) $\Rightarrow$ (f) use LT, VI, th. 9; for (f) $\Rightarrow$ (b) note that $x \vee e\Theta_1 e$ $(x \vee e\Theta_1(x \vee e) \wedge e = e)$ and similarly $x \wedge e\Theta_2 e$, and then (b) is proved directly (e. g., for the second identity $x \wedge (e \vee y)\, \Theta_1 x \wedge e\Theta_1 x = x \vee (x \wedge y)\, \Theta_1(x \wedge e) \vee (x \wedge y)$ and $x \wedge (e \vee y)\, \Theta_2 x \wedge y\Theta_2 e \vee (x \wedge y)\, \Theta_2(x \wedge e) \vee (x \wedge y)$; as $\Theta_1 \wedge \Theta_2 = O$, this implies $x \wedge (e \vee y) = (x \wedge e) \vee (x \wedge y)$).*

It will be useful to note that the decomposition of (e) is $L \leqq \bar{e} \times \underline{e}$ with homomorphism $x \to [x \wedge e,\ x \vee e]$.

Corollary 1. *Let $d, e \in L$, d neutral. If e satisfies the identities of theorem* 2 (b) *whenever either $x, y \leqq d$ or $x, y \geqq d$, then e is neutral. Such is the case when either*

(a) $\bar{e} = \bar{d}$, $\underline{e} = \underline{d}$ *(isomorphisms), or, more generally,*

(b) *there exist homomorphisms f, g taking d into e such that $\bar{d}$, respectively $\underline{d}$ are the images of sublattices.*

Proof. If (b) holds and $x, y \leqq d$, then

$$e \wedge (x \vee y) = fd \wedge (fu \vee fv) = f(d \wedge (u \vee v)) = f(d \wedge u) \vee f(d \wedge v) = \\ = (e \wedge x) \vee (e \wedge y)$$

and dually, and similarly for the second identity; and for $x, y \geqq d$.

Now let $x, y \in L$ and use the decomposition of L corresponding to d by th. 2, (e). Then

$$\begin{aligned} e \wedge (x \vee y) &= [e \wedge d,\ e \vee d] \wedge ([x \wedge d,\ x \vee d] \vee [y \wedge d,\ y \vee d]) = \\ &= [d \wedge e \wedge ((x \wedge d) \vee (y \wedge d)),\quad (d \vee e) \wedge (x \vee d \vee y \vee d)] = \\ &= [d \wedge e \wedge ((x \wedge d) \vee (y \wedge d)),\quad (d \wedge (x \vee d \vee y \vee d)) \vee \\ &\quad \vee (e \wedge (x \vee d \vee y \vee d))] \end{aligned}$$

since d is neutral,

$$= [d \wedge ((e \wedge x \wedge d) \vee (e \wedge y \wedge d)),\quad d \vee (e \wedge (x \vee d)) \vee (e \wedge (y \vee d))]$$

since $x \wedge d, y \wedge d \leq d \leq x \vee d, y \vee d$,

$$= [d \wedge ((e \wedge x) \vee (e \wedge y)),\ d \vee (e \wedge x) \vee (e \wedge d) \vee (e \wedge y) \vee (e \wedge d)] =$$
$$= [d \wedge ((e \wedge x) \vee (e \wedge y))\ ,\quad d \vee (e \wedge x) \vee (e \wedge y)] = (e \wedge x) \vee (e \wedge y)$$

and dually; similarly $x \wedge (e \vee y) = (x \wedge e) \vee (x \wedge y)$ and dually.*

Corollary 2. *If $d, e \in L$ and both satisfy the first identity of th.* 2 (b) *and its dual, then $\langle e \wedge d, e\rangle = \langle d, e \vee d\rangle$.*

Proof. The mapping f, $fx = x \vee d$, takes $\langle e \wedge d, e\rangle$ into $\langle d, e \vee d\rangle$; with our assumptions, f is homomorphic. If we set $gx = x \wedge e$, then g takes $\langle d, e \vee d\rangle$ into $\langle e \wedge d, e\rangle$ and $gfx = (x \vee d) \wedge e = (x \wedge e) \vee (d \wedge e) = x$ whenever $e \wedge d \leq x \leq e$, so that f is $1 - 1$ onto. *

Theorem 3. *Let L have extremal elements. Then these properties of a* (*central*) *element $e \in L$ are equivalent*:

(a) *e is neutral and complemented*;

(b) *e is complemented and for all $x \in L$, $x = (x \wedge e) \vee (x \wedge e')$*;

(c) *e is pseudocomplemented and for all $x \in L$, $x = (x \wedge e) \vee (x \wedge e^*)$*;

(d) *$e = [I, O]$ under some direct decomposition of L*;

(e) *there exist disjoint congruence relations Θ_1, Θ_2 and an $d \in L$ such that $e\Theta_1 I$, $e\Theta_2 O$ and $d\Theta_1 O$, $d\Theta_2 I$*;

(f) *any maximal Boolean subalgebra containing the extremal elements of L contains e also.*

Proof. Most of the proof is in LT, II (corollary to th. 10, exer. a) in § 8). (b) $\Rightarrow$ (c) obviously; conversely, setting $x = I$ we obtain $e \vee e^* = I$, so that e^* is a complement — now (c) reduces to (b). (a) $\Leftrightarrow$ (e) obviously. The proof of equivalence of (f) follows the lines of th. 11 in LT, II.*

It will be useful to note that the decomposition of (d) is $L = \bar{e} \times \bar{e}'$ with isomorphism $x \to [x \wedge e, x \wedge e']$.

Note also that (1) a pseudocomplemented neutral element need not be central (e. g. in uncomplemented finite distributive lattices), and (2) the set-product of *all* the maximal Boolean subalgebras may be void — e. g. in $4 \oplus 1$, — while central elements exist — O, I.

The decompositions of th. 2 (e) and th. 3 (d) we shall name *corresponding* to the neutral or central element e.

Under lattice-homomorphisms onto, neutral elements go into neutral, central elements into central; this is implied by th. 2 (c), th.3 (a) respectively. More precise results are given in th. 9 and 10.

Let N_L be the set of all neutral elements of L, C_L the set of all central elements of L (= *center*). Then $N_{L \times M} = N_L \times N_M$, $C_{L \times M} = C_L \times C_M$. (This is implied by th. 2 (c), th. 3 (b) respectively.) If M is a sublattice of L, $M \cap N_L \subset$

$\subset N_M$; if even extremal elements in L, M coincide, $M \cap C_L \subset C_M$. (This is implied by th. 2 (a), th. 3 (a) respectively. See also th. 11.) If e is central in L and $a \leqq e \leqq b$, then e is central in $\langle a, b\rangle$. (e is neutral in $\langle a, b\rangle$, and has a relative complement there: $a \vee (b \wedge e')$.)

Every element of a distributive lattice is neutral; every element of a Boolean algebra is central; and conversely. One of the minor objects of this article is to examine which properties of distributive lattices and Boolean algebras can be "localised" to neutral, respectively central elements.

3. Factor isomorphism, the unicity theorem

Theorem 4. *Let $d, e \in L$ and let A_i $(1 \leqq i \leqq 4)$ be disjoint; let $L \leqq \mathsf{P}_{\substack{1 \leqq i \leqq 4 \\ a \in A_i}} L_{ia}$, under which $d = [\delta_a^{A_1 \cup A_2}]$, $e = [\delta_a^{A_2 \cup A_3}]$ $(a \in \bigcup_1^4 A_i)$; and also $L \leqq \mathsf{P}_1^4 M_i$, under which $d = [I, I, O, O]$, $e = [I, O, I, O]$. Then $M_i \leqq \mathsf{P}_{a \in A_i} L_{ia}$ for $1 \leqq i \leqq 4$.*

Proof. Write corresponding elements under the sets they are contained in:

$$\mathsf{P}_{A_1} L_{1a} \times \mathsf{P}_{A_2} L_{2a} \times \mathsf{P}_{A_3} L_{3a} \times \mathsf{P}_{A_4} L_{4a}\,, \qquad L\,, \qquad M_1 \times M_2 \times M_3 \times M_4\,.$$

Take any $x \in M_1$. There exists an $a \in L$ with

$$[[x_a^1], [x_a^2], [x_a^3], [x_a^4]]\,, \qquad a \qquad [x, x_2, x_3, x_4]\,.$$

Set $fx = [x_a^1]$; then f is single-valued, since if

$$[[z_a^1], [z_a^2], [z_a^3], [z_a^4]] \qquad b \qquad [x, z_2, z_3, z_4]\,,$$

then

$$[[x_a^1], [O], [O], [O]] \qquad a \wedge d \wedge e \qquad [x, O, O, O]\,,$$

and

$$[[z_a^1], [O], [O], [O]] \qquad b \wedge d \wedge e \qquad [x, O, O, O]\,,$$

so that (reading from right to left) $a \wedge d \wedge e = b \wedge d \wedge e$, $\Rightarrow [x_a^1] = [z_a^1]$. Obviously f is homomorphic. If $x \neq y$ in M_1, then for some $c \in L$

$$c \qquad [y, y_2, y_3, y_4]$$

with $a \wedge d \wedge e \neq c \wedge d \wedge e$, implying $fx \neq fy$. Obviously then $M_1 \leqq \mathsf{P}_{A_1} L_{1a}$. Similarly, for $i > 1$ $M_i \leqq \mathsf{P}_{A_i} L_{ia}$, paralleling the above proof and using in turn, instead of $a \wedge d \wedge e$, the elements $(a \wedge d) \vee e$, $(a \wedge e) \vee d$, $a \vee e \vee d$.*

Note that if the first decomposition is $L \leqq (\mathsf{P}_{A_1} L_{1a}) \times \mathsf{P}_{A_2} L_{2a} \times P_{A_3} L_{3a} \times \mathsf{P}_{A_4} L_{4a}$, then we can conclude $M_1 = \mathsf{P}_{A_1} L_{1a}$.

Theorem 4 is easily generalised to n neutral elements e_k, grouping the former direct decomposition into 2^n subgroups; but the conditions become too complex.

A special case ($e = d$, $L_{2a} = L_{3a} = 1$) of th. 4 is the following.

Theorem 5. (Unicity theorem.) *If* $L_1 \times L_2 \geqq L \leqq M_1 \times M_2$ *and* $e = [I, O]$ *in both, then* $L_1 = M_1$, $L_2 = M_2$.

Note that without some condition such as that on e no conclusion of the above type can be reached, not even $L \times M = L \times N \Rightarrow M = N$. E. g., if B_∞ is the Boolean algebra of (countably infinite) stationary sequences of O, I, then $\mathsf{B}_\infty = \mathsf{B}_\infty \times \mathsf{2} = \mathsf{B}_\infty \times \mathsf{4}$, etc.

Corollary. *If* $e \in L$ *is central, then* $\underline{e} = \bar{e}'$, $\bar{e} = \underline{e}'$.

Proof. Use the decompositions of th. 2 (e), th. 3 (d) (namely, $\bar{e} \times \underline{e}$ and $\bar{e} \times \bar{e}'$) and the unicity theorem; dualise.*

This generalises the theorem stating that Boolean algebras are self-dual. More precise results are given in the corollary to th. 15.

For lattices L, M we could define L/M thus: $M = \bar{e}$ for some neutral $e \in L$, and then $L/M = N$ if $L \leqq M \times N$. Then N is uniquely defined (by th. 2 (e) and the unicity theorem, $N = \underline{e}$), and if $L = L_1$, $M = M_1$, then $L/M = L_1/M_1$. In this manner we could develop the theory of factor lattices; many of the theorems on factor groups also hold for factor lattices.

4. Complementation, homomorphisms, associativity conditions

Theorem 6. *If* $L \leqq M \times N$ *and* $e = [I, O] \in L$ *is complemented in* L, *then* $L = M \times N$ *under the same homomorphism.*

Proof. As e is complemented, L must contain extremal elements; then $M \times N$ has extremal elements also, and they must coincide with those of L; then, finally, e is complemented in $M \times N$ and $e' = [O, I] \in L$. If $[x_1, x_2] \in M \times N$, there exist $u, v \in L$ such that $u = [x_1, u_2]$, $v = [v_1, x_2]$, and then $[x_1, x_2] = [x_1, O] \vee [O, x_2] = (u \wedge e) \vee (v \wedge e') \in L$. *

A weaker form of theorem 6 follows from theorem 5: $L \leqq \bar{e} \times \underline{e} = \bar{e} \times \bar{e}' = L$, but the homomorphism is not the same.

We used the term *corresponding* decomposition, whether the element was central or only neutral. Theorem 6 justifies this.

Theorem 7. *The complement of a central element* e *is unique, and is an orthocomplement in the following sense: if* $x \in L$ *and an* x' *exists, then* $(x \wedge e)' = x' \vee e'$ *and dually,* $e'' = e$.

Proof. Use the direct decomposition corresponding to e. If $d = [u, v]$ and $d \wedge e = 0$, $d \vee e = I$, then $O = [u, O]$, $I = [I, v]$, so that $d = [O, I] = e'$; this is unique, and $e'' = e$.

Now assume that $x' = [u', v']$ is a complement of $x = [u, v]$; then $x' \vee e' = [u', I]$, $x \wedge e = [u, O]$, so that $[u', I] \wedge [u, O] = O$, $[u', I] \vee [u, O] = I$.

Conversely, if $[r, s]$ is a complement of $x \wedge e = [u, O]$, necessarily $r \wedge u = O$, $r \vee u = I$, $s = I$, so that $[r, v']$ is a complement of x and $([u, v] \wedge e)' = [r, v'] \vee e'$. Dualise. *

Theorem 8. *The complement of a central element e is a pseudocomplement, and is an orthopseudocomplement in the following sense: if $x \in L$ and x^* exists, then $(x \wedge e)^* = x^* \vee e'$ and dually.*

Proof. Use the direct decomposition corresponding to e. Let $x = [x_1, x_2]$, $y = [y_1, y_2]$. First, $x \wedge e = O \Longleftrightarrow x_1 = O \Longleftrightarrow x \leqq e'$, so that e' is a pseudocomplement. Note that the coordinates of a pseudocomplement are themselves pseudocomplements; i. e., $x^* = [x_1^*, x_2^*]$. Now

$$y \wedge x \wedge e = O \Leftrightarrow y_1 \wedge x_1 = O \Leftrightarrow y_1 \leqq x_1^* \Leftrightarrow y \leqq x^* \vee e' ,$$

$$y \wedge (x \vee e) = O \Leftrightarrow y \wedge e = O = y \wedge x \Leftrightarrow$$

$$\Leftrightarrow y_1 = O \quad \text{and} \quad y_2 \leqq x_2^* \Leftrightarrow y \leqq x^* \wedge e' .^*$$

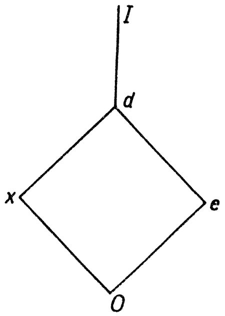

Fig. 1.

Theorems 7,8 generalise the theorem stating the complementation in Boolean algebras is orthocomplementation.

Note that if a neutral element e is pseudocomplemented, then for all pseudocomplemented $x \in L$, $(x \vee e)^* = x^* \wedge e^*$ (for $y \wedge (x \vee e) = O \Longleftrightarrow y \wedge e = O = y \wedge x \Longleftrightarrow x^* \geqq y \leqq e^* \Longleftrightarrow y \leqq x^* \wedge e^*$), but that $(x \wedge e)^* = x^* \vee e^*$, $e^{**} = e$ need not hold. E. g., in the lattice of fig. 1.
we have $(x \wedge e)^* = O^* = I$, $x^* \vee e^* = e \vee x = d$, and $d^{**} = O^* = I$.

Theorem 9. *If $L \leqq M \times N$ corresponding to the neutral element e, and h maps L homomorphously onto K, then there exist lattices R, S and homomorphisms f, g such that: f(g) maps M(N) homomorphously onto R(S), and $K \leqq R \times S$, $[f, g]\, e = [I, O]$, $[f, g]$ is an extension of h, if h is an isomorphism, so is $[f, g]$.*

Proof 1. For $x = [x_1, x_2] \in M \times N$ set $fx_1 = h(x \wedge e)$, $gx_2 = h(x \vee e)$, $R \equiv fM$, $S \equiv gN$. (Note that $x \wedge e \in L$, for there exists an $y = [x_1, v] \in L$, and then $x \wedge e = [x_1, O] = y \wedge e \in L$; similarly $x \vee e \in L$.) Obviously *f, g are homomorphisms.*

2. Prove that *K is isomorphous to a sublattice of $R \times S$*: for $\xi \in K$ choose $x \in L$ with $hx = \xi$, and set $\varphi\xi = [h(x \wedge e), h(x \vee e)] \in R \times S$ (obviously, if $hy = \xi = hx$, then $h(y \wedge e) = h(x \wedge e)$ and dually); so that φ is a homomorphism into $R \times S$. Now, if $[a, b] \in \varphi K$, take $x \in L$ with $a = h(x \wedge e)$, $b = h(x \vee e)$, and set $\psi[a, b] = hx$. To prove that ψ is univalent, note that $h(x \wedge e) = h(y \wedge e)$ and dually implies

$$\begin{aligned} hx &= h(x \wedge (x \vee e)) = hx \wedge h(x \vee e) = hx \wedge h(y \vee e) = h(x \wedge (y \vee e)) = \\ &= h(x \wedge y) \vee h(x \wedge e) = h(x \wedge y) \vee h(y \wedge e) = h(y \wedge (x \vee e)) = \\ &= hy \wedge h(x \vee e) = hy \wedge h(y \vee e) = h(y \wedge (y \vee e)) = hy . \end{aligned}$$

Now obviously $\psi\varphi\xi = \xi$, i. e., φ has an inverse, and is an $1-1$ homomorphism (into $R \times S$).

3. Prove $K \leqq R \times S$. Take $a \in R$, so that $a = h(x \wedge e)$ for some $x \in L$; setting $\xi = hx$, $b = h(x \vee e)$, we have $[a, b] = \varphi\xi$.

4. Finally, for $x \in L$ we obviously have $h[x_1x_2] = \varphi^{-1}[fx_1, gx_2]$; thus $[f, g]$ *is an extension of* h. Obviously $h[I, O] = \varphi^{-1}[I, O]$.*

Theorem 10. *If $L = M \times N$ corresponding to the central element e, and h maps L homomorphously onto K, then there exist lattices R, S and homomorphisms f, g such that $f(g)$ maps $M(N)$ homomorphously onto $R(S)$, and $K = R \times S$, $[f, g]\, e = [I, O]$, $[f, g]$ is an extension of h, if h is an isomorphism, so is $[f, g]$.* This follows from theorems 9 and 6.

Theorem 11. *If d is neutral (central) in L, and $e \leqq d$, then e is neutral (central) in $\overline{d}$ if and only if it is such in L.*

Proof. Obviously, e neutral or central in $L \Rightarrow e$ neutral, respectively central in $\overline{d}$. Conversely, we have $L \leqq \overline{d} \times \underline{d}$, $\overline{d} \leqq \overline{e} \times \langle e, d\rangle$, so that $L \leqq$ $\leqq \overline{e} \times \langle e, d\rangle \times \underline{d}$; in this decomposition, $e = [I, O, O]$; use theorem 2 (d). If e, d are central, the preceding decompositions are direct (theorem 6) and $L = \overline{e} \times (\langle e \times d\rangle \times \underline{d})$ with $e = [I, O]$.*

An alternate direct proof utilises only th. 2 (c) and th. 3 (a). Corollary 1 to theorem 2 is an interesting counterpart.

Theorem 12. *If for three different a's $L \leqq (\mathsf{P}_{b \in A,\, b \neq a} L_b) \times L_a$ under the same isomorphism, then $L = \mathsf{P}_A L_a$.*

Proof. Take $[x_a] \in \mathsf{P}_A L_a$. There exist $a_i \in A$, $y_{a_i} \in L_{a_i}$, $z_i \in L$ $(i = 1, 2, 3)$ such that $a_1 \neq a_2 \neq a_3 \neq a_1$, and

$$z_1 = [\ldots y_{a_1}, x_{a_2}, x_{a_3}, \ldots], \quad z_2 = [\ldots x_{a_1}, y_{a_2}, x_{a_3}, \ldots], \quad z_3 = [\ldots x_{a_1}, x_{a_2}, y_{a_3}, \ldots];$$

(this notation is perhaps obvious; only the a_i-th coordinates are written out; all other b-th coordinates are x_b). Now set

$$u_1 = z_1 \wedge (z_2 \vee z_3) = [\ldots y_{a_1} \wedge x_{a_1}, x_{a_2}, x_{a_3}, \ldots],$$

and u_2, u_3 cyclically. Then $u_i \in L$, and also

$$L \ni u_1 \vee (u_2 \wedge u_3) = [\ldots x_{a_1}, x_{a_2}, x_{a_3}, \ldots] = [x_a]\,.^*$$

Care must be taken to interpret "the same isomorphism" strictly; so as to exclude cases such as $4 \leqq (2 \times 2) \times 2$, $4 \leqq 2 \times (2 \times 2)$.

Slight generalisations of the above proof yield

Theorem 13. *If $L \leqq (\mathsf{P}_{a \in A_i} L_a) \times \mathsf{P}_{a \in A - A_i} L_a \times \mathsf{P}_{b \in B} M_b$ with $i = 1, 2, 3$ under the same isomorphism, and also $-A_i \subset A_j \cap A_k$ for $i \neq j \neq k \neq i$, $A_i \neq \emptyset$, then $L \leqq (\mathsf{P}_A L_a) \times \mathsf{P}_B M_b$.*

The conditions $-A_i \subset A_j \cap A_k$ are equivalent to $A_2 \supset -A_1$, $-A_3 \subset$ $\subset A_1 \cap A_2$.

5. The factor-theorems

Theorem 14. (Birkhoff factor-theorem.) *If L is a lattice with O, I and $\mathsf{P}_{A_1} M_a = L = \mathsf{P}_{A_2} N_b$, then there exist lattices L_{ab} for $[a, b] \in A_1 \times A_2$ such that $L = \mathsf{P}_{A_1 \times A_2} L_{ab}$, $M_a = \mathsf{P}_{b \in A_2} L_{ab}$, $N_b = \mathsf{P}_{a \in A_1} L_{ab}$.*

This is theorem 7 in LT, II.

The factor-theorem yields many elegant proofs. E. g., to prove that the center of a lattice is a sublattice (this is of course implied byth. 2 (f)), take central elements e, d and the corresponding direct decompositions $L = M_1 \times M_2$ $L = N_1 \times N_2$. Using the factor-theorem, there exists a direct decomposition $L = L_{11} \times L_{12} \times L_{21} \times L_{22}$ such that $M_1 = L_{11} \times L_{12}$, $M_2 = L_{21} \times L_{22}$, $N_1 = L_{11} \times L_{21}$, $N_2 = L_{12} \times L_{22}$. Now, as $d = [I, O]$ in $M_1 \times M_2$ and $e = [I, O]$ in $N_1 \times N_2$, we have $d = [I, I, O, O]$, $e = [I, O, I, O]$ in $L_{11} \times L_{12} \times L_{21} \times L_{22}$, so that $d \vee e = [I, I, I, O]$, $d \wedge e = [I, O, O, O]$. Thus, for example, $d \vee e = [I, O]$ in $(L_{11} \times L_{12} \times L_{21}) \times L_{22} = L$, and is therefore central.

We shall generalise the factor-theorem in two theorems; one is the following th. 15 and the other is in the second article of this series.

Theorem 15. *If $L \leqq M_1 \times M_2$ corresponding to d and $L \leqq N_1 \times N_2$ corresponding to e (d, e neutral), then there exists a subdirect decomposition $L \leqq L_{11} \times L_{12} \times L_{21} \times L_{22}$ such that*

$$M_1 \leqq L_{11} \times L_{12}\,, \quad M_2 \leqq L_{21} \times L_{22}, \quad N_1 \leqq L_{11} \times L_{21}\,, \quad N_2 \leqq L_{12} \times L_{22}\,,$$

and that $d = [I, I, O, O]$, $e = [I, O, I, O]$ in the third decomposition.

Proof. By th. 2 (g) $e \vee d$ and $e \wedge d$ are also neutral; by th. 11 e is neutral in $\overline{e \vee d}$, $e \wedge d$ is neutral in $\bar{e}$. Using th. 2 (e) thrice, we have $L \leqq \overline{e \vee d} \times \underline{e \vee d}$, $\underline{e \vee d} \leqq \bar{d} \times \langle d, e \vee d \rangle$, $\bar{d} \leqq \overline{e \wedge d} \times \langle e \wedge d, d \rangle$, so that $L \leqq \overline{e \wedge d} \times \langle e \vee d, d \rangle \times \langle d, e \vee d \rangle \times \underline{e \vee d}$ with the homomorphism $fx = [x \wedge e \wedge d, (x \vee e) \wedge d, (x \wedge e) \vee d, x \vee e \vee d]$. Thus $fd = [e \wedge d, d, d, e \vee d] = [I, I, O, O]$, $f(e) = [e \wedge d, e \wedge d, e \vee d, e \vee d] = [I, O, I, O]$. Using th. 2 (e) again

$$\begin{aligned}
M_1 &= \bar{d} \leqq \overline{e \wedge d} \times \langle e \wedge d, d \rangle && \text{under} \quad g_1 x = [x \wedge e \wedge d, x \vee (e \wedge d)]\,, \\
M_2 &= \underline{d} \leqq \langle d, e \vee d \rangle \times \underline{e \vee d} && \text{under} \quad g_2 x = [x \wedge (e \vee d), x \vee e \vee d]\,, \\
N_1 &= \bar{e} \leqq \overline{e \wedge d} \times \langle d, e \vee d \rangle && \text{under} \quad h_1 x = [x \wedge d, x \vee d]\,, \\
N_2 &= \underline{e} \leqq \langle e \wedge d, d \rangle \times \underline{e \vee d} && \text{under} \quad h_2 x = [x \wedge d, x \vee d].\,^*
\end{aligned}$$

Corollary 1. *If d, e are central in L, then*

$$\begin{aligned}
\langle O, d' \wedge e \rangle &= \langle d' \wedge e', d' \rangle = \langle d \wedge e', (d \vee e) \wedge (d' \vee e') \rangle = \langle d \wedge e, e \rangle = \\
&= \langle e', d' \vee e' \rangle = \langle (d \wedge e) \vee (d' \wedge e'), d' \vee e \rangle = \\
&= \langle d, d \vee e \rangle = \langle d' \vee e', I \rangle\,.
\end{aligned}$$

Taking $d \leqq e$, we have $\langle 0, d' \wedge e \rangle = \langle e', d' \rangle = \langle d, e \rangle = \langle d \vee e', I \rangle$.

Proof. In the decomposition constructed before th. 15 $d = [I, I, O, O]$, $e = [I, O, I, O]$, so that, e. g., $\langle O, d' \wedge e\rangle = \langle [O, O, O, O], [O, O, I, O]\rangle = L_{21}$, $\langle d' \wedge e', d'\rangle = \langle [O, O, O, I], [O, O, I, I]\rangle = L_{21}$, etc. If $d \leqq e$, then $L_{12} = \mathbf{1}$.*

Corollary 2. *If d is central and e neutral in L, then*

$$\langle O, d' \wedge e\rangle = \langle d, d \vee e\rangle \quad \text{and} \quad \langle e \wedge d, d\rangle = \langle d' \vee e, I\rangle .$$

Proof. Under $L \leqq \overline{d \wedge e} \times \langle e \wedge d, d\rangle \times \langle d, d \vee e\rangle \times \underline{d \vee e}$ we have $d = [I, I, O, O]$ and $e = [I, O, I, O]$. Since d is also central, $L \leqq \overline{d' \wedge e} \times \langle e \wedge d', d'\rangle \times \langle d', d' \vee e\rangle \times \underline{d' \vee e}$, under which $d' = [I, I, O, O]$, $e = [I, O, I, O]$ so that $d = [O, O, I, I]$. Rearranging the last decomposition, $L \leqq \langle d', d' \vee e\rangle \times \underline{d' \vee e} \times \overline{d' \wedge e} \times \langle e \wedge d', d'\rangle$, under which $d = [I, I, O, O]$, $e = [I, O, I, O]$. Using factor isomorphism (th. 4), we obtain

$$\langle O, d \wedge e\rangle = \langle d', d' \vee e\rangle , \quad \langle e \wedge d, d\rangle = \langle d' \vee e, I\rangle ,$$

$$\langle d, d \vee e\rangle = \langle O, d' \wedge e\rangle , \quad \langle d \vee e, I\rangle = \langle e \wedge d', d'\rangle .\text{*}$$

These two corollaries have some interesting consequences in the lattice-theory of projectivity. E. g., projective intervals with neutral end-points are isomorphic.

6. Application to ring theory

It has long been known that there is come connection between ring theory and lattice theory; this is suggested, for instance, by Stone's theorem on the correspondence between Boolean algebras and Boolean rings, and Newman's system of axioms common to Boolean rings and Boolean algebras.

Here we attempt to extend the connection to general lattices and rings. We shall not be concerned with the application to ring theory of facts, but only of the ideas and methods underlying the preceding sections. The treatement of the subject follows Stone's notions rather than Newman's. To a certain extent, we examine the consequences of the definition of the center of a ring (this is Boolean ring, but not a subring), paralleling the definition of the center of a lattice (this is a Boolean subalgebra).

In this paragraph, large letters always denote rings (not necessarily associative).

Definition. *Define a cross-ordering $\leqq$ in R thus: $x \leqq y$ whenever $xy = yx = x$.*

Elementary consequences. $\leqq$ is antisymmetric and transitive, and is reflexive only on idempotent elements. $0 \leqq x$, and $x \leqq 1$ if 1 exists. If x, y, z associate, $y \geqq x \leqq z \Rightarrow x \leqq yz$, $y \leqq x \geqq z \Rightarrow y + z \leqq x \geqq y + z - yz$. If x, z commute and x, y, z associate, $y \leqq x \Rightarrow yz \leqq x \geqq zy$. $y \leqq x \Rightarrow -y \leqq x$ and $ny \leqq x$ for integral n. If $R \leqq U \times V$ (subdirect decomposition), then $[x_1, x_2] \leqq [y_1, y_2] \Leftrightarrow x_1 \leqq y_1$ and $x_2 \leqq y_2$.

If R has 1 and ring operations $x + y$, xy, let dual R be the same set, with (ring-) operations $x \dot{+} y = x + y - 1$, $x \,.\, y = x + y - xy$. Then dual R is a ring, the zero and unit of dual R are 1, 0 respectively, the inverse element is $\dot{-}\, x = 2 - x$, substraction $x \dot{-} y = x - y + 1$. As can be shown directly, $x \overset{:}{+} y = x + y$, $x : y = xy$, so that dual dual $R \equiv R$. Also dual $R = R$, with isomorphism $x \to 1 - x$ ("dual automorphism", "involution"). x idempotent in $R \Longleftrightarrow x$ idempotent in dual R; x, y commute in $R \Longleftrightarrow x, y$ commute in dual R; x, y, z associate in $R \Longleftrightarrow x, y, z$ associate in dual R. $x \leqq y$ in $R \Longleftrightarrow$ $\Longleftrightarrow y \overset{\cdot}{\leqq} x$ in dual R. x central in $R \Longleftrightarrow x$ central in dual R; centers of R and dual R are dual. Thus the definition of dual rings is natural and (dual $R = R$) uninteresting — as in Boolean algebras.

Theorem 16. *These properties of a (central) element e of a ring R are equivalent:*

(a) *e is idempotent, for all $x, y \in R$: x, e commute, x, y, e associate,*

(b) *$e = [\delta_a^A]_a$ under some subdirect decomposition of R,*

(c) *$e = [1, 0]$ under some direct decomposition of R,*

(d) *there exist disjoint congruence relations Θ_1, Θ_2 such that for all $x \in R$: $ex\Theta_1 x\Theta_1 xe$, $e\Theta_2 0$.*

In commutative associative rings of characteristic 2, properties (a)—(e) are equivalent, where

(e) every maximal Boolean subring containing 1 contains e also.

Proof. (c) $\Rightarrow$ (b) obviously, (b) $\Rightarrow$ (*a*) directly. Prove (a) $\Rightarrow$ (c): For $x \in R$ set $fx = xe$, $gx = x - xe$. Then f, g are homomorphisms onto subrings U, V; if $xe = ye$, $x - xe = y - ye$, then $x = y$, so that $x \to [fx, gx]$ is a $1 - 1$ homomorphism into U, V. For any $[x, y] \in U \times V$ take $z = xe + y - ye \in R$; then $ze = xe$, $z - ze = y - ye$, so that the homomorphism is onto $U \times V$. Finally, $fxfe = xeee = xe = fx$, $ge = O$, so that $e \to [1, 0]$. Note that U is the subset of $x \leqq e$ and V the subset of all x with $xe = O$, and both are ideals. For (c) $\Rightarrow$ $\Rightarrow$ (d) set $[x_1, x_2]\, \Theta_i [y_1, y_2]$ if and only if $x_i \Theta_i y_i$ $(i = 1, 2)$; (d) $\Rightarrow$ (a) is proved directly. To prove (a) $\Rightarrow$ (e) note that if B is a Boolean subring with 1, then the subring generated by e, B consists of $ex + y$ with $x, y \in B$; this is Boolean, so that $e \in B$ if B is maximal. Conversely, in the special case of (e) there exist maximal Boolean subrings with 1; then e is idempotent, so that (a) holds.*

If R has 1, condition (d) is much simpler: $e\Theta_1 1$, $e\Theta_2 0$.

Note that if R has 1, then $1 - e$ is an associated divisor of zero (the least, in the cross-ordering), and $1 - e = [0, 1]$ in the decomposition of (c). Thus decomposable rings with 1 have zero-divisors. But there exist indecomposable rings with 1 and non-trivial zero-divisors, see 2. in the following paragraph.

Theorem 17. *If $U \times V \geqq R \leqq W \times Z$ and $e = [1, 0]$ in both, then $U = W$, $V = Z$.*

The proof follows that of th. 5., using ae, $a + e - ae$ instead of $a \wedge e$, $a \vee e$ respectively.

Theorem 18. *If $R \leqq U \times V$ and $e = [1, 0] \epsilon R$, then $R = U \times V$ under the same homomorphism.*

Proof. For $[x, y] \epsilon U \times V$ there exist $[u, v] \epsilon U \times V$ such that $[x, v] \epsilon R$ and $[u, y] \epsilon R$. Then also $R \ni [u, y] + ([x, v] - [u, y]) [1, 0] = [x, y]$. *

Theorem 19. *If $R = U \times V$ with $e = [1, 0]$ and if h maps R homomorphously onto S, then there exist rings W, Z and homomorphisms f, g such that $f(g)$ maps $U(V)$ homomorphously onto $W(Z)$ and $S = W \times Z$, $[f, g] e = [1, 0]$, $[f, g]$ is an extension of h, if h is an isomorphism, then so is $[f, g]$.*

The proof follows the lines of that of th. 9, using $fx = h(xe)$, $gx = h(x - xe)$.

Theorem 20. *If d is central in R and $e \leqq d$, then e is central in $\bar{d}$ if and only if it is such in R.*

This follows immediately from th. 16 (c).

Theorem 21. *If R is a ring with 1, and if $\mathsf{P}_A U_a = R = \mathsf{P}_B V_b$, then there exist rings R_{ab} ($[a, b] \epsilon A_1 \times A_2$) such that*

$$R = \mathsf{P}_{A_1 \times A_2} R_{ab} , \quad U_a = \mathsf{P}_{b \epsilon A_2} R_{ab} , \quad V_b = \mathsf{P}_{a \epsilon A_1} R_{ab} .$$

Proof. Set $e_a = [\delta^a_i]_{i \epsilon A_1}$ in $\mathsf{P}_{A_1} U_a$, $d_b = [\delta^b_i]_{i \epsilon A_2}$ in $\mathsf{P}_{A_2} V_b$, $f_{ab} = e_a d_b$, $R_{ab} = \bar{f}_{ab}$ (in R). Then $x \to [x f_{ab}]_{a,b}$ is a homomorphism into $\mathsf{P}_{A_1 \times A_2} R_{ab}$. It is $1 - 1$, since if $x f_{ab} = y f_{ab}$ for all a, b, then $x e_a = y e_a$ for all a (since $[(x e_a) d_b]_b = [(y e_a) d_b]_b$ in $R = \mathsf{P}_{A_2} V_b$), and then $x = y$ (since $[x e_a]_a = [y e_a]_a$ in $R = \mathsf{P}_{A_1} U_a$). The rest is obvious.*

Note that the center (= set of central elements) of a ring is generally not a subring; if R has 1, a necessary and sufficient condition is $2 = 0$. But it is a Boolean ring under these ring operations: sum $x + y - 2xy$, product xy.

Associative rings are a special case of commutative groups with endomorphisms. This suggest another generalisation of central elements.

Let A be a set of endomorphisms of an additive abelian group G. If A, G are subdirectly factorisable into $A_1 \times A_2$, $G_1 \times G_2$ in such a manner that $[\alpha_1, \alpha_2][x_1, x_2] = [\alpha_1 x_1, \alpha_2 x_2]$, and if $\omega \epsilon A$ is such that $\omega = [1, 0]$, then ω is idempotent and commutes with all $\alpha \epsilon A$.

Conversely, let $\omega \epsilon A$ commute with all $\alpha \epsilon A$ and be idempotent. Let R consist of 1 and all endomorphisms $\Sigma_1^n \varepsilon_i \alpha_i$, where $\alpha_i \epsilon A$ and $\varepsilon_i = \pm 1$ $((\Sigma_1^n \varepsilon_i \alpha_i)(x)$ is defined as $\Sigma_1^n \varepsilon_i (\alpha_i x)$; since G is Abelian, this is an endomorphism). Then R is an associative ring containing A, and ω commutes with all elements of R, so that (th. 16 (c)) R is directly decomposable into $R_1 \times R_2$ in such a manner that $\omega = [1, 0]$; in this decomposition α goes into $[\omega\alpha, (1 - \omega) \alpha]$. G is also decomposable, $x \longleftrightarrow [\omega x, (1 - \omega) x]$.

If $\alpha = [\alpha_1, \alpha_2] \epsilon A$, $x = [x_1, x_2] \epsilon G$, then

$[\alpha_1, \alpha_2][x_1, x_2] = [\omega\alpha, (1 - \omega)\,\alpha][\omega x, (1 - \omega)\,x] = [\omega\alpha x, (1 - \omega)\,\alpha x] = \alpha x$.
The decomposition of G is direct: if $[u, v] \in G_1 \times G_2$, then $u = \omega x$, $v = y - \omega y$ for some $x, y \in G$; then taking $z = y - \omega y + \omega x \in G$, we have $\omega z = u$, $z - \omega z = v$, $\Rightarrow [u, v] \in G$. The factorisation of A is of course only subdirect; it is direct if A is a ring, or if $\omega = 1$ or $\omega = 0$.

7. Examples

1. Let D_n be the (commutative and associative) ring of n-vectors with numerical coordinates. Then

$x \in D_n$ is central if and only if each coordinate is 0 ot 1.

Thus there are exactly $2n$ direct two-factor decompositions, of which exactly $\left[\frac{n}{2}\right]$ are non-trivial nonisomorphic, i. e., $D_n = D_k \times D_{n-k}$.

2. Let R_n be the (associative) ring of $n \times n$ matrices with numerical coefficients. Then R_n is directly indecomposable, since the only commutative matrix is λI, and this is idempotent only for $\lambda = 0, 1$. (Let $\|d_{ik}\|$ be a commutative matrix, and set $x_{ik} = \delta_k^v$; then $\Sigma_\lambda d_{i\lambda} \,.\, x_{\lambda k} = \Sigma_\lambda d_{\lambda k} x_{i\lambda}$ is $0 = d_{vk}$ for $k \neq v$, and $d_{ii} = d_{vv}$ for $k = v$.)

3. Let T be a topological T_1-space. Let L_T be the lattice of its closed sets (cf. LT, IV, § 2). Let R_T be the ring of continuous real functions on T (cf. LT, v XI, § 4). Then L_T is complete and distributive, so that using th. 3 (a),

$X \in L_T$ is central if and only if it is closed and open.

R_T is commutative and associative, so that central elements are the idempotent elements (th. 16 (a)); as the $x \in R_T$ are continuous,

$x \in R_T$ is central if and only if it is the characteristic function of some closed and open set.

Thus central elements of L_T, of R_T and the closed-open sets of T correspond. Finite direct multiplication (of L_T or R_T) corresponds to the seldom used operation of topological addition. The factor-theorems have an interesting, though trivial, interpretation in T. Note also that conversely, if L is a complete distributive lattice with O, I, then with M-closure, L forms a topological space T and $L = L_T$.

4. Let X be a linear space and L the set of its endomorphisms (linear maps into itself). Then

$x \in L$ is central if and only if x is a commutative projection.

It is interesting that for normed linear spaces and closed (but not necessarily commutative) projections, some of the results of 6. are well-known. (E. g., that $x + y - xy$ is a projection whenever x, y are projections and commute.)

5. Let R be a set of real functions on $(-\infty, +\infty)$, with $x + y, xy \in R$ whenever $x, y \in R$ $((x + y)\, t = xt + yt,\ (xy)\, t = x(y(t)))$. Then the idempotent elements of R are constructed thus. Take a set X, and for $t \in X$ set $et = t$, for t non $\in X$ choose et arbitrarily in X. Obviously e is idempotent, and all idempotent elements of R are of this form. Now, if R contains a constant, c, and if an idempotent e is central, it must commute with c; this implies $c \in X$ $(xc = (xc)\, t = (cx)\, t = c)$. This will help us to construct an R with non-trivial central elements.

Define et: for rational t, $et = t$, and $et = O$ otherwise. Let R consist of all functions of the form $mI + ne$ with m, n rational. Then R is obviously closed under addition; and the superposition of $x, y \in R$ is

$$(mI + ne)(pI + qe) = mpI + mqe + n(p + q)\, e$$

since $e(pt + qe(t)) = (p + q)\, e(t)$ for rational p, q; and this implies that R is commutative. The central elements of R are I, e, $I - e$, O.

Резюме

ПРЯМЫЕ РАЗЛОЖЕНИЯ В СТРУКТУРАХ

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В статье изучаются алгебраические свойства центров и нейтральных элементов структуры и соответствующих прямых, соотв., полупрямых разложений структуры. На втором плане стоит задача исследовать такие свойства булевых алгебр, которые могут быть ,,локализированы“ на отдельные элементы структуры.

В § 2 подробно разбирается определение центра и нейтрального элемента и выводятся некоторые простые следствия этих определений. В § 3 доказывается теорема об изоморфизме множителей двух полупрямых разложений структуры (теорема единственности).

В § 4 исследуется дополнительность центров и нейтральных элементов (орто- и псевдо-дополнительность); изучается прямое и полупрямое разложение при гомоморфизме; наконец, выводится одно интересное условие для того, чтобы полупрямое разложение было прямым.

В § 5 приводится (несколько более слабая) варианта важной теоремы Биркгофа об общем ,,уплотнении“ двух прямых разложений структуры, которая может быть приложена и к полупрямым разложениям — вслед-

ствие получается предложение о проективности интервалов с нейтральными концами.

В § 6 приводятся положения, аналогичные изложенной теории, имеющие место в кольцах; в некотором смысле дело касается разложений Пеирце. В кольцах нельзя отличить центры от нейтральных элементов; с таким положением не встречаемся в случае абелевых групп с операторами, на которые обобщено изложение.