Czechoslovak Mathematical Journal

Jan Mařík

The surface integral

Czechoslovak Mathematical Journal, Vol. 6 (1956), No. 4, 522–558

Persistent URL: http://dml.cz/dmlcz/100219

THE SURFACE INTEGRAL

JAN MAŘÍK, Praha.

(Received 10. X. 1955.)

In this paper fundamental properties of the $(m-1)$-dimensional integral in the m-dimensional space are studied.[1])

1. Some conventions and definitions. The symbol m denotes in this memoir a fixed integer > 1. If A is an arbitrary set and if $v_1, \ldots, v_m$ are finite real functions on A, we say that a *vector* $v = [v_1, \ldots, v_m]$ is defined on A and write $v(x) = [v_1(x), \ldots, v_m(x)]$ $(x \in A)$. The functions $v_1, \ldots, v_m$ are called *components* of the vector v. — Further, E_n (n natural) is the n-dimensional Euclidean space. (A vector is thus a mapping into E_m.) If $b \in E_m$, $b = [b_1, \ldots, b_m]$, then the number $\sqrt{\sum_{i=1}^{m} b_i^2}$ will be called the *norm* of b and will be denoted by $|b|$. If $v = [v_1, \ldots, v_m]$ is a vector on the set A, we put

$$||v||_A = \sup_{x \in A} |v(x)| \text{ or } ||v||_A = 0$$

according as $A \neq \emptyset$ or $A = \emptyset$.

If $f, f_1, f_2, \ldots$ are finite real functions on a set A and if $f_n(x) \to f(x)$ for each $x \in A$, we say that the sequence $f_1, f_2, \ldots$ is *convergent* and has the limit f. We then write $f_n \to f$ or $\lim_{n \to \infty} f_n = f$. — The meaning of the symbols $\max(f, g)$, $|f|$, $g \leqq 1$ and so on (f, g real functions on A) is obvious. Further we write

$$||f||_A = \sup_{x \in A} |f(x)| \quad \text{or} \quad ||f||_A = 0,$$

according as $A \neq \emptyset$ or $A = \emptyset$, for every real function f on A.

The sets belonging to the smallest σ-algebra $\mathfrak{B}$ that contains all closed sets of a metrical space A, are called *Borel sets* (Borel subsets of A). A (real) function, which is measurable with respect to $\mathfrak{B}$, is termed a *Borel function*. We say that continuous functions are of class 0. Given any countable ordinal number $\gamma > 0$, we say that a function f is of class γ, if there exist functions $f_1, f_2, \ldots$, the

[1]) The main ideas are explained also in [5].

class of each f_n being less than γ, such that $f_n \to f$ (definition by transfinite induction). It is known that f is a finite Borel function, if and only if there exists a countable ordinal number γ such that f is of class γ. — The meaning of the expressions "bounded vector", "Borel vector" and so on is obvious.

The words "measure", "measurable" and so on concern — whenever another sense has not been explicitly assigned to them — the usual Lebesgue measure in some space E_n; the meaning of n will always be clear from the context. The one-dimensional measure will be denoted by μ. If $A \subset E_n$, then $\overline{A}$ (resp. A^0) is the closure (resp. interior) of the set A.

Let n, q be natural numbers; let f be a function on an open set $G \subset E_n$. We say that f is *of class* C_q (on G), if all derivatives of the q-th order (and so all derivatives of the r-th order, where $r < q$) of f are continuous on G. We say that f is of class C_∞, if f is of class C_r for $r = 1, 2, \ldots$

2. Definition. Let A be a bounded measurable subset of E_m. Let $\mathfrak{V}_A$ be the family of all vectors v whose components are polynomials (in m variables) and which fulfil the relation $\|v\|_A \leqq 1$. Put

$$\|A\| = \sup \int_A \operatorname{div} v(x)\, dx\,, \quad \text{where} \quad v \in \mathfrak{V}_A\,.$$

(The integral $\int_A \operatorname{div} v(x)\, dx$ exists for every $v \in \mathfrak{V}_A$, because the set A is bounded and $\operatorname{div} v(x) = \sum_{i=1}^{m} \frac{\partial v_i(x)}{\partial x_i}$ is a polynomial again. Clearly $0 \leqq \|A\| \leqq \infty$. If the measure of A is zero, we have $\|A\| = 0$.)

3. Definition. Let A be a bounded measurable subset of E_m; let $\mathfrak{P}_A$ be the family of all polynomials f (in m variables) such that $\|f\|_A \leqq 1$. For $i = 1, \ldots, m$ we put

$$\|A\|_i = \sup \int_A \frac{\partial f(x)}{\partial x_i}\, dx\,, \quad \text{where} \quad f \in \mathfrak{P}_A\,.$$

Let $\mathfrak{A}_i$ be the system of all bounded measurable sets $A \subset E_m$, for which $\|A\|_i < < \infty$; finally, put $\mathfrak{A} = \bigcap_{i=1}^{m} \mathfrak{A}_i$.

4. Theorem. *If A is a bounded measurable subset of E_m, then*

$$\|A\|_m \leqq \|A\| \leqq \sum_{j=1}^{m} \|A\|_j\,.$$

Proof. If $f \in \mathfrak{P}_A$, put $v = [0, \ldots, 0, f]$. Then $v \in \mathfrak{V}_A$, $\operatorname{div} v = \frac{\partial f}{\partial x_m}$, whence $\int_A \frac{\partial f(x)}{\partial x_m}\, dx \leqq \|A\|$; it follows that $\|A\|_m \leqq \|A\|$. The second inequality is obvious.

Remark. By symmetry, $\|A\|_i \leqq \|A\|$ for $i = 1, \ldots, m$. We have therefore $A \in \mathfrak{A}$ if and only if $\|A\| < \infty$.

5. Lemma. *If K is compact, G open in E_m and if $K \subset G$, then there exists a function f with the following properties:*

1) *f is of class C_∞ on E_m;*
2) *$f(x) = 1$ for $x \in K$, $f(x) = 0$ for x non $\in G$;*
3) *$0 \leqq f(x) \leqq 1$ for $x \in E_m$.*

Proof. If δ is a sufficiently small positive number, the relations $x = [x_1, \ldots, x_m] \in K$, $y = [y_1, \ldots, y_m] \in E_m$, $\max_i |x_i - y_i| < \delta$ imply $y \in G$. For $t \in E_1$ put

$$\varphi_0(t) = 0 \,, \quad \text{if} \quad |t| \geqq \delta \,, \tag{1}$$

$$\varphi_0(t) = \exp\left(-\frac{1}{\delta^2 - t^2}\right) \,, \quad \text{if} \quad |t| < \delta \,. \tag{2}$$

It is known that the function φ_0 is of class C_∞ on E_1. There exists a (positive) constant α such that

$$\int_{-\infty}^{\infty} \alpha \, \varphi_0(t) \, \mathrm{d}t = 1 \,; \tag{3}$$

write

$$\varphi = \alpha\varphi_0 \,. \tag{4}$$

Then the function $\Phi(t) = \int_{-\infty}^{\delta(2t-1)} \varphi(\tau) \, \mathrm{d}\tau$ has all derivatives, is non-decreasing and fulfils the relations $\Phi(t) = 0$ for $t \leqq 0$, $\Phi(t) = 1$ for $t \geqq 1$, $0 \leqq \Phi(t) \leqq 1$ for $t \in E_1$. Put

$$\psi(x_1, \ldots, x_m) = \varphi(x_1) \ldots \varphi(x_m) \,. \tag{5}$$

Then ψ is of class C_∞ on E_m and we have

$$\max_i |x_i| < \delta \Rightarrow \psi(x_1, \ldots, x_m) > 0 \,,$$

$$\max_i |x_i| \geqq \delta \Rightarrow \psi(x_1, \ldots, x_m) = 0 \,,$$

$$\int_{E_m} \psi(x) \, \mathrm{d}x = 1 \,.$$

For $x = [x_1, \ldots, x_m] \in E_m$ put

$$L(x) = (x_1 - \delta, x_1 + \delta) \times \ldots \times (x_m - \delta, x_m + \delta) \,.$$

Since K is compact, there exists a finite set $\{b^1, \ldots, b^p\} \subset K$ [2]) such that $K \subset \subset \bigcup_{j=1}^{p} L(b^j)$. The function $g(x) = \sum_{j=1}^{p} \psi(x - b^j)$ is then of class C_∞, assumes only positive values on K and vanishes outside G. There exists consequently a finite positive constant β such that $\beta \, g(x) \geqq 1$ for all $x \in K$ and the function $f(x) = \Phi(\beta \, g(x))$ has all the required properties.

---

[2]) We write b^j, since b_j usually denotes the j-th coordinate of some point b.

6. Lemma. *If $A \in \mathfrak{A}_m$ and if the functions $f, \dfrac{\partial f}{\partial x_m}$ are continuous on E_m, then*

$$\left| \int_A \frac{\partial f(x)}{\partial x_m} \, \mathrm{d}x \right| \leqq \|f\|_A \cdot \|A\|_m . \tag{6}$$

Proof. Let the interval $I = \langle a_1, b_1 \rangle \times \ldots \times \langle a_m, b_m \rangle$ contain the set A. Let the sequence of polynomials $g_1, g_2, \ldots$ converge uniformly on I to the function $\dfrac{\partial f}{\partial x_m}$ [3]) and let $f_1, f_2, \ldots$ be polynomials in $m - 1$ variables such that the sequence $f_1, f_2, \ldots$ converges uniformly on the set $\langle a_1, b_1 \rangle \times \ldots \times \langle a_{m-1}, b_{m-1} \rangle$ to the function $f(x_1, \ldots, x_{m-1}, a_m)$. Since

$$f(x_1, \ldots, x_m) = f(x_1, \ldots, x_{m-1}, a_m) + \int_{a_m}^{x_m} \frac{\partial f(x_1, \ldots, x_{m-1}, t)}{\partial x_m} \, \mathrm{d}t ,$$

we see that the sequence of polynomials

$$G_n(x_1, \ldots, x_m) = f_n(x_1, \ldots, x_{m-1}) + \int_{a_m}^{x_m} g_n(x_1, \ldots, x_{m-1}, t) \, \mathrm{d}t$$

converges uniformly on the set I to the function f, whence

$$\|G_n\|_A \to \|f\|_A . \tag{7}$$

On the other hand,

$$\int_A \frac{\partial G_n(x)}{\partial x_m} \, \mathrm{d}x = \int_A g_n(x) \, \mathrm{d}x \to \int_A \frac{\partial f(x)}{\partial x_m} \, \mathrm{d}x . \tag{8}$$

Our assertion follows immediately from (7), (8) and from the obvious relation

$$\left| \int_A \frac{\partial G_n(x)}{\partial x_m} \, \mathrm{d}x \right| \leqq \|G_n\|_A \cdot \|A\|_m .$$

7. Lemma. *If $A \in \mathfrak{A}$ and if the functions $v_1, v_2, \ldots, v_m, \dfrac{\partial v_1}{\partial x_1}, \dfrac{\partial v_2}{\partial x_2}, \ldots, \dfrac{\partial v_m}{\partial x_m}$ are continuous on E_m, then*

$$\left| \int_A \operatorname{div} v(x) \, \mathrm{d}x \right| \leqq \|v\|_A \cdot \|A\| \tag{9}$$

(where $v = [v_1, \ldots, v_m]$).

Proof. By means of the method employed in the proof of the preceding lemma we can find vectors $v^1, v^2, \ldots$ whose components are polynomials, such that $\int_A \operatorname{div} v^n(x) \, \mathrm{d}x \to \int_A \operatorname{div} v(x) \, \mathrm{d}x$ and $\|v^n\|_A \to \|v\|_A$, whence the assertion easily follows.

---

[3]) See, for example, [1], p. 345, Weierstrass' theorem.

8. Lemma. *Let $\mathfrak{V}_1$ (resp. $\mathfrak{P}_1$) be the family of all vectors v (resp. functions f) which are of class C_∞ on E_m and fulfil the relation $\|v\|_{E_m} \leqq 1$ (resp. $\|f\|_{E_m} \leqq 1$). Then we have*

$$\|A\| = \sup \int_A \operatorname{div} v(x)\, dx\,, \quad \textit{where} \quad v \in \mathfrak{V}_1 \tag{10}$$

$$\left(\textit{resp.}\ \|A\|_m = \sup \int_A \frac{\partial f(x)}{\partial x_m}\, dx,\ \textit{where}\ f \in \mathfrak{P}_1\right) \tag{11}$$

for every bounded measurable subset A of E_m.

Proof. Let α, ε be real numbers, $\alpha < \|A\|$, $\varepsilon > 0$. There exists a vector $v^1 \in \mathfrak{V}_A$ (see definition 2) such that $\int_A \operatorname{div} v^1(x)\, dx > \alpha$. Lemma 5 shows that there exists a vector v^2 of class C_∞ on E_m which coincides with v^1 in some neighbourhood of A and fulfils the inequality $|v^2(x)| < 1 + \varepsilon$ for each $x \in E_m$. Put $v = v^2 \,.\, (1 + \varepsilon)^{-1}$. In virtue of the relation

$$\int_A \operatorname{div} v(x)\, dx = \frac{1}{1+\varepsilon} \int_A \operatorname{div} v^1(x)\, dx > \frac{\alpha}{1+\varepsilon}$$

we get

$$\|A\| \leqq \sup \int_A \operatorname{div} v(x)\, dx\,, \quad \text{where} \quad v \in \mathfrak{V}_1\,. \tag{12}$$

If $\|A\| = \infty$, (10) is an easy consequence of (12); if $\|A\| < \infty$, (10) follows from (12) and (9) (lemma 7). The proof of (11) is similar.

9. Theorem. *Let A_1, A_2 be bounded measurable subsets of E_m. Then the following assertions are true:*

a) *If the measure of $(A_1 - A_2) \cup (A_2 - A_1)$ is zero, then*

$$\|A_1\| = \|A_2\|\,, \quad \|A_1\|_i = \|A_2\|_i\,. \tag{13a}$$

b) *If the measure of $A_1 - A_2$ is zero, then*

$$\|A_2 - A_1\| \leqq \|A_1\| + \|A_2\|\,, \quad \|A_2 - A_1\|_i \leqq \|A_1\|_i + \|A_2\|_i\,. \tag{13b}$$

c) *If the measure of $A_1 \cap A_2$ is zero, then*

$$\|A_1 \cup A_2\| \leqq \|A_1\| + \|A_2\|\,, \quad \|A_1 \cup A_2\|_i \leqq \|A_1\|_i + \|A_2\|_i\,. \tag{13c}$$

d) *If $\bar{A}_1 \subset A_2^0$,[4]) then*

$$\|A_2 - A_1\| = \|A_1\| + \|A_2\|\,, \quad \|A_2 - A_1\|_i = \|A_1\|_i + \|A_2\|_i\,. \tag{13d}$$

e) *If $\bar{A}_1 \cap \bar{A}_2 = \emptyset$, then*

$$\|A_1 \cup A_2\| = \|A_1\| + \|A_2\|\,, \quad \|A_1 \cup A_2\|_i = \|A_1\|_i + \|A_2\|_i \tag{13e}$$

$$(i = 1, \ldots, m)\,.$$

Proof. The assertions a) — c) follow easily from (10) and (11), where we write, of course, i in place of m. Now suppose that $\bar{A}_1 \subset A_2^0$. There exist an

---

[4]) $\bar{A}$ is the closure, A^0 is the interior of A.

open set G such that $\overline{A}_1 \subset G$, $\overline{G} \subset A_2$, and (see lemma 5) a function g of class C_∞ on E_m which equals 1 in some neighbourhood of A_1, vanishes outside G and fulfils the relation $0 \leqq g \leqq 1$. Let α_j be real numbers, $\alpha_j < \|A_j\|_m$ ($j = 1, 2$). In virtue of lemma 8 we can determine functions $f_j \in \mathfrak{P}_1$ so as to have $\int\limits_{A_j} \frac{\partial f_j(x)}{\partial x_m}\, dx > \alpha_j$. Put $\varphi_j = f_j g$ ($j = 1, 2$), $\psi = f_2 - \varphi_2 = f_2(1 - g)$. Since $\overline{G} \subset A_2$ and since φ_1, φ_2 vanish outside G, we have

$$\int\limits_{A_2} \frac{\partial \varphi_j(x)}{\partial x_m}\, dx = \int\limits_{E_m} \frac{\partial \varphi_j(x)}{\partial x_m}\, dx = \int\limits_{E_{m-1}} \left(\int\limits_{-\infty}^{\infty} \frac{\partial \varphi_j(x_1, \ldots, x_m)}{\partial x_m}\, dx_m \right) dx_1 \ldots dx_{m-1} = 0\,. \tag{14}$$

Evidently

$$\int\limits_{A_1} \frac{\partial \psi(x)}{\partial x_m}\, dx = 0\,, \quad \int\limits_{A_1} \frac{\partial \varphi_1(x)}{\partial x_m}\, dx = \int\limits_{A_1} \frac{\partial f_1(x)}{\partial x_m}\, dx\,. \tag{15}$$

If we put $\varphi = f_2 - \varphi_1 - \varphi_2 = \psi - \varphi_1$, we get by (14), (15)

$$\int\limits_{A_2 - A_1} \frac{\partial \varphi(x)}{\partial x_m}\, dx = \int\limits_{A_2} \frac{\partial \varphi(x)}{\partial x_m}\, dx - \int\limits_{A_1} \frac{\partial \varphi(x)}{\partial x_m}\, dx =$$

$$= \int\limits_{A_2} \frac{\partial f_2(x)}{\partial x_m}\, dx + \int\limits_{A_1} \frac{\partial f_1(x)}{\partial x_m}\, dx > \alpha_2 + \alpha_1\,. \tag{16}$$

According to (6), the relation $|\varphi| \leqq |\psi| + |\varphi_1| = |f_2|(1 - g) + |f_1| g \leqq 1$ implies

$$\int\limits_{A_2 - A_1} \frac{\partial \varphi(x)}{\partial x_m}\, dx \leqq \|A_2 - A_1\|_m\,. \tag{17}$$

The equality $\|A_2 - A_1\|_m = \|A_1\|_m + \|A_2\|_m$ follows easily from (16), (17) and (13b). The rest of the proof may be left to the reader.

10. Lemma. *Let D be the boundary of the set $A \in \mathfrak{A}_m$. Then the relation*

$$\left| \int\limits_{A} \frac{\partial f(x)}{\partial x_m}\, dx \right| \leqq \|f\|_D \cdot \|A\|_m \tag{18}$$

holds good, whenever the functions f, $\frac{\partial f}{\partial x_m}$ are continuous on E_m.

Proof. Let ε be any positive number and let the functions f, $\frac{\partial f}{\partial x_m}$ be continuous on E_m. We put $\alpha = \|f\|_D$ and $K = \overline{A} \cap E[x; |f(x)| \geqq \alpha + \varepsilon]$. The set K is compact; since $|f(x)| \leqq \alpha$ for $x \in D$, we have $K \cap D = \emptyset$, whence $K \subset A^0$. Let G be open, $K \subset G$, $\overline{G} \subset A$. By lemma 5, there exists a function g_0 of class C_∞

on E_m which fulfils the relations $g_0(x) = 1$ for $x \in K$, $g_0(x) = 0$ for x non $\in G$ and $0 \leqq g_0(x) \leqq 1$ for $x \in E_m$; put $g_1 = 1 - g_0$, $f_0 = fg_0$, $f_1 = fg_1$. Since $\overline{G} \subset A$ and f_0 vanishes outside G, we have $\int\limits_A \frac{\partial f_0(x)}{\partial x_m} \, \mathrm{d}x = \int\limits_{E_m} \frac{\partial f_0(x)}{\partial x_m} \, \mathrm{d}x = 0$. (The function $\frac{\partial f_0}{\partial x_m}$ is obviously continuous.) As $|f_1(x)| = |f(x)| \, g_1(x) \leqq |f(x)| < \alpha + \varepsilon$ for $x \in A - K$, $f_1(x) = 0$ for $x \in K$ and $f(x) = f_0(x) + f_1(x)$ for all x, we have

$$\left| \int\limits_A \frac{\partial f(x)}{\partial x_m} \, \mathrm{d}x \right| = \left| \int\limits_A \frac{\partial f_1(x)}{\partial x_m} \, \mathrm{d}x \right| \leqq \|f_1\|_A \, . \, \|A\|_m \leqq (\alpha + \varepsilon) \, . \, \|A\|_m \, .$$

Making $\varepsilon \to 0$, we obtain our assertion.

11. Lemma. *Let D be the boundary of the set $A \in \mathfrak{A}$ and let the functions v_1, $v_2, \ldots, v_m$, $\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \ldots, \frac{\partial v_m}{\partial x_m}$ be continuous on E_m. Then we have*

$$\left| \int\limits_A \operatorname{div} v(x) \, \mathrm{d}x \right| \leqq \|v\|_D \, . \, \|A\| \, . \tag{19}$$

Proof. Let ε be a positive number. We put $\beta = \|v\|_D$ and $K = \overline{A} \cap E[x; |v(x)| \geqq \beta + \varepsilon]$, determine the function g_0 as in the proof of the preceding lemma and write $g_1 = 1 - g_0$, $v^0 = vg_0$, $v^1 = vg_1$. It is easy to see that $\int\limits_A \operatorname{div} v^0(x) \, \mathrm{d}x = 0$ and that $|v^1(x)| < \beta + \varepsilon$ for every $x \in A$. Our assertion follows immediately from the relation

$$\left| \int\limits_A \operatorname{div} v(x) \, \mathrm{d}x \right| = \left| \int\limits_A \operatorname{div} v^1(x) \, \mathrm{d}x \right| \leqq (\beta + \varepsilon) \, . \, \|A\| \, .$$

12. Lemma. *Let D be the boundary of the set $A \in \mathfrak{A}_m$. Let the functions f_n, g_n, $\frac{\partial f_n}{\partial x_m}, \frac{\partial g_n}{\partial x_m}$ be continuous on E_m $(n = 1, 2, \ldots)$ and let the sequences $\{f_n\}$, $\{g_n\}$ converge uniformly on the set D to the same function. Then the limits $L = \lim\limits_{n \to \infty} \int\limits_A \frac{\partial f_n(x)}{\partial x_m} \, \mathrm{d}x$, $L' = \lim\limits_{n \to \infty} \int\limits_A \frac{\partial g_n(x)}{\partial x_m} \, \mathrm{d}x$ exist and have the same finite value.*

Proof. According to (18) (lemma 10) we have

$$\left| \int\limits_A \frac{\partial f_n(x)}{\partial x_m} \, \mathrm{d}x - \int\limits_A \frac{\partial f_p(x)}{\partial x_m} \, \mathrm{d}x \right| \leqq \|f_n - f_p\|_D \, . \, \|A\|_m$$

for arbitrary indices n, p, so that the limit L exists and is finite. From the relation

$$\left| \int\limits_A \frac{\partial f_n(x)}{\partial x_m} \, \mathrm{d}x - \int\limits_A \frac{\partial g_n(x)}{\partial x_m} \, \mathrm{d}x \right| \leqq \|f_n - g_n\|_D \, . \, \|A\|_m$$

we see that $L' = L$.

13. Definition. Let D be the boundary of the set $A \in \mathfrak{A}_m$; let f be a continuous function on D. We put

$$P_m(A, f) = \lim_{n\to\infty} \int_A \frac{\partial f_n(x)}{\partial x_m}\, dx\,, \tag{20a}$$

where the functions f_n, $\frac{\partial f_n}{\partial x_m}$ are continuous on E_m and $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on D. (Such functions f_n exist; because the set D is compact, there exist, in fact, such polynomials. The definition is correct according to lemma 12.)

If $A \in \mathfrak{A}_i$ for some i, we define similarly $P_i(A, f)$.

Remark. If the functions f, $\frac{\partial f}{\partial x_m}$ are continuous on E_m, then obviously

$$P_m(A, f) = \int_A \frac{\partial f(x)}{\partial x_m}\, dx\,. \tag{20b}$$

Further it is easy to see that $P_m(A, f) + P_m(A, g) = P_m(A, f + g)$, whenever f, g are continuous on D.

14. Theorem. *Let D be the boundary of the set $A \in \mathfrak{A}_m$; let $\mathfrak{F}$ be the system of all bounded Borel functions on D. Then there exists exactly one functional R on $\mathfrak{F}$ such that*

1) $R(f) = P_m(A, f)$, *if f is continuous on D;*

2) $R(f_n) \to R(f)$, *if $f_1, f_2, \ldots$ is a bounded sequence with limit f, $f_n \in \mathfrak{F}$ ($n = 1, 2, \ldots$).*

Furthermore, the functional R has the following properties:

3) $R(\alpha f + \beta g) = \alpha R(f) + \beta R(g)$, *if $\alpha, \beta \in E_1$, $f, g \in \mathfrak{F}$;*

4) $|R(f)| \leqq \|f\|_D \cdot \|A\|_m$, *if $f \in \mathfrak{F}$;*

5) $R(f) = P_m(A, f) = \int_A \frac{\partial f(x)}{\partial x_m}\, dx$, *if the functions f, $\frac{\partial f}{\partial x_m}$ are continuous in some neighbourhood of the set $\bar{A}$.*

Proof. We may obviously suppose that $D \neq \emptyset$. Let f be continuous on the set D; let the sequence of polynomials $f_1, f_2, \ldots$ converge uniformly to f on D. Lemma 10 yields

$$\left|\int_A \frac{\partial f_n(x)}{\partial x_m}\, dx\right| \leqq \|f_n\|_D \cdot \|A\|_m \quad (n = 1, 2, \ldots)\,; \tag{21}$$

evidently $\|f_n\|_D \to \|f\|_D$. Making $n \to \infty$ in (21), we obtain in virtue of (20a) the relation

$$|P_m(A, f)| \leqq \|f\|_D \cdot \|A\|_m .$$

Thus we see that the functional $P_m(A, f)$ is continuous on the normed linear space of all continuous functions f on D (with the norm $\|f\|_D$). It is easy to prove (see, for instance, [3], p. 18, § 46, b)) that the functional $P_m(A, f)$ can be expressed as a difference of two non-negative (linear) functionals. It is well-known (see e. g. [4], p. 478—479) that each non-negative functional which is defined on the space of all continuous functions f on a given metric space S can be written as $\int_S f \, \mathrm{d}\gamma$, where γ is a Borel measure. Hence there exists a finite σ-additive function λ on the system of all Borel subsets of D such that $P_m(A, f) = \int_D f \mathrm{d}\lambda$ for each continuous function f on D. This integral has of course a meaning for all $f \in \mathfrak{F}$; if we put

$$R(f) = \int_D f \, \mathrm{d}\lambda$$

for each $f \in \mathfrak{F}$, we see that the conditions 1), 2), 3) are satisfied.

Now suppose that a functional R' on $\mathfrak{F}$ also fulfils the conditions 1) and 2). The system $\mathfrak{F}_0$ of all $f \in \mathfrak{F}$, for which $R(f) = R'(f)$, includes all continuous functions on D. Since the limit of every bounded convergent sequence $f_1, f_2, \ldots$, where $f_n \in \mathfrak{F}_0$, also belongs to $\mathfrak{F}_0$, we have $\mathfrak{F}_0 = \mathfrak{F}$, $R' = R$.

Further let the functions f, $\frac{\partial f}{\partial x_m}$ be continuous on an open set $G \supset \bar{A}$. It is an easy consequence of lemma 5 that there exists a function f_1 such that $f_1(x) = f(x)$ in some neighbourhood of $\bar{A}$ and that the functions f_1, $\frac{\partial f_1}{\partial x_m}$ are continuous on E_m. Following (20b) we have

$$P_m(A, f) = \int_A \frac{\partial f_1(x)}{\partial x_m} \, \mathrm{d}x = \int_A \frac{\partial f(x)}{\partial x_m} \, \mathrm{d}x \, ,$$

which proves the relation 5).

Finally, let $\mathfrak{F}_1$ be the family of all functions $f \in \mathfrak{F}$, for which $|R(f)| \leqq \|A\|_m$. If f is a polynomial such that $\|f\|_D \leqq 1$, then $R(f) = P_m(A, f) = \int_A \frac{\partial f(x)}{\partial x_m} \, \mathrm{d}x$; according to (18) (lemma 10) we get

$$|R(f)| \leqq \|f\|_D \cdot \|A\|_m \leqq \|A\|_m \, ,$$

whence $f \in \mathfrak{F}_1$. If $f_1, f_2, \ldots$ is a bounded convergent sequence, where $f_n \in \mathfrak{F}_1$, then, in virtue of 2), the function $\lim_{n \to \infty} f_n$ also belongs to $\mathfrak{F}_1$. We thus see that every function $f \in \mathfrak{F}$ such that $\|f\|_D \leqq 1$ is an element of $\mathfrak{F}_1$, whence 4) follows at once.

Remark 1. Since the functional $R(f)$ is an extension of $P_m(A, f)$, we can write $P_m(A, f)$ instead of $R(f)$ again. If a misunderstanding is impossible, we write $P_m(A, f) = P_m(f)$. If $A \in \mathfrak{A}_i$ for some i, we define similarly the functional $P_i(A, f) = P_i(f)$.

Remark 2. Suppose that $A_1, A_2, A_3 \in \mathfrak{A}_m$ and that the measure of the sets $A_1 \cap A_2, A_2 - A_3$ is zero; let D_i be the boundary of A_i. It is easy to prove that the relations

$$P_m(A_1 \cup A_2, f) = P_m(A_1, f) + P_m(A_2, f)$$
$$(\text{resp. } P_m(A_3 - A_2, f) = P_m(A_3, f) - P_m(A_2, f))$$

hold for each bounded Borel function f on $D_1 \cup D_2$ (resp. $D_2 \cup D_3$).

15. Definition. If $A \in \mathfrak{A}$ and if $v = [v_1, \ldots, v_m]$ is a bounded Borel vector on the boundary of the set A, we put

$$P(v) = P(A, v) = \sum_{i=1}^{m} P_i\,(A, v_i).$$

16. Theorem. *Let D be the boundary of the set $A \in \mathfrak{A}$. Let $\mathfrak{V}$ be the family of all bounded Borel vectors on D. Then we have*

a) $P(\alpha_1 v^1 + \alpha_2 v^2) = \alpha_1 P(v^1) + \alpha_2 P(v^2)$, *if* $\alpha_1, \alpha_2 \in E_1$, $v^1, v^2 \in \mathfrak{V}$;

b) $P(v^n) \to P(v)$, *if* $v^1, v^2, \ldots$ *is a bounded convergent sequence of elements of* $\mathfrak{V}$ *with limit* v;

c) $|P(v)| \leqq \|v\|_D \,.\, \|A\|$, *if* $v \in \mathfrak{V}$;

d) $P(v) = \int_A \operatorname{div} v(x)\, dx$, *if* $v = [v_1, \ldots, v_m]$, *where the functions* $v_1, v_2, \ldots, v_m$, $\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \ldots, \frac{\partial v_m}{\partial x_m}$ *are continuous in some neighbourhood of* $\bar{A}$.

Proof. The relations a), b), d) follow immediately from theorem 14. In order to prove c) we observe that (in virtue of d) and of (19) (lemma 11)) the relation $|P(v)| \leqq \|v\|_D \,.\, \|A\|$ holds good for each vector $v = [v_1, \ldots, v_m]$, where v_j are polynomials. We have therefore

$$|P(v)| \leqq \|A\| \tag{22}$$

for each vector v, which is continuous on D and fulfils the relation

$$\|v\|_D \leqq 1\,. \tag{23}$$

Let now $\gamma > 0$ be a countable ordinal number and suppose that (22) holds for each Borel vector v of the class $< \gamma$ on D such that $\|v\|_D \leqq 1$. Now let v be a Borel vector of the class γ and let (23) be valid. There exist Borel vectors v^n, the class of each v^n being $< \gamma$, such that $v^n \to v$. Put $f_n(x) = \max\,(1, |v^n(x)|)$, $w^n(x) = v^n(x)\,.\,(f_n(x))^{-1} (x \in D)$. Then the class of each vector w^n is $< \gamma$; moreover, $w^n \to v$ and $\|w^n\|_D \leqq 1$ $(n = 1, 2, \ldots)$. Since $P(w^n) \to P(v)$, $|P(w^n)| \leqq \|A\|$, we have $|P(v)| \leqq \|A\|$ again, which completes the proof of c).

17. Theorem. *Let $\mathfrak{B}$ be a σ-algebra on the set $D \neq \emptyset$.*[5] *Let $\mathfrak{V}$ be the family of all bounded $\mathfrak{B}$-measurable vectors (with m components). Finally, let P be a finite real functional on $\mathfrak{V}$ such that $P(u) + P(v) = P(u + v)$ for arbitrary elements $u, v \in \mathfrak{V}$ and that $P(v^n) \to P(v)$ for each bounded convergent sequence $v^1, v^2, \ldots$*

[5]) In this theorem, D is an arbitrary set; we do not suppose that $D \subset E_m$.

($v^n \in \mathfrak{V}$) *with limit v. Then there exists a $\mathfrak{B}$-measurable vector ν on the set D and a finite measure p on $\mathfrak{B}$ with the properties*

$$|\nu(x)| = 1 \quad \textit{for each} \quad x \in D , \tag{24}$$

$$P(v) = \int_D v \cdot \nu \, \mathrm{d}p \,^{6)} \quad \textit{for each} \quad v \in \mathfrak{V} . \tag{25}$$

If a $\mathfrak{B}$-measurable vector ν' and a measure p' on $\mathfrak{B}$ also satisfy the conditions (24) *and* (25), *then $p' = p$ (i. e. $p(B) = p'(B)$ for each $B \in \mathfrak{B}$) and $\nu(x) = \nu'(x)$ almost everywhere in D (with respect to p).*

Proof. Let $\mathfrak{F}$ be the family of all bounded $\mathfrak{B}$-measurable functions. For $f \in \mathfrak{F}$, $f \geqq 0$ put

$$Q(f) = \sup P(v) , \quad \text{where} \quad v \in \mathfrak{V} , \quad |v| \leqq f .^{7)}$$

Suppose for a moment that $Q(f) = \infty$ for some $f \in \mathfrak{F}$, $f \geqq 0$. Then there exist $v^n \in \mathfrak{V}$ such that $|v^n| \leqq f$, $P(v^n) > n$ $(n = 1, 2, \ldots)$, whence $\frac{1}{n} \cdot P(v^n) > 1$. But the relation $\left|\frac{v^n}{n}\right| \leqq \frac{f}{n}$ yields $\frac{1}{n} P(v^n) = P\left(\frac{v^n}{n}\right) \to 0$; we arrive at a contradiction which proves that $Q(f)$ is finite for all $f \in \mathfrak{F}$, $f \geqq 0$. As $Q(0) = 0$, we can put

$$Q(f) = Q(f_+) - Q(f_-)^{8)}$$

for an arbitrary function $f \in \mathfrak{F}$. We shall now study the properties of the functional Q. First, suppose that $f_1, f_2 \in \mathfrak{F}$, $f_1 \geqq 0$, $f_2 \geqq 0$; let v^1, v^2 be $\mathfrak{B}$-measurable vectors, $|v^i| \leqq f_i (i = 1, 2)$. Then $|v^1 + v^2| \leqq |v^1| + |v^2| \leqq f_1 + f_2$, whence $P(v^1) + P(v^2) = P(v^1 + v^2) \leqq Q(f_1 + f_2)$, consequently $Q(f_1) + Q(f_2) \leqq Q(f_1 + f_2)$. Suppose now that $f_1 + f_2 \geqq 1$, $f_i \geqq 0$, $f_i \in \mathfrak{F}$ $(i = 1, 2)$; let v be a $\mathfrak{B}$-measurable vector, $|v| \leqq f_1 + f_2$. If we put $v^i = v \frac{f_i}{f_1 + f_2}$, then $v = v^1 + v^2$, $|v^i| = |v| \cdot \frac{f_i}{f_1 + f_2} \leqq f_i$ $(i = 1, 2)$, whence $P(v) = P(v^1) + P(v^2) \leqq Q(f_1) + Q(f_2)$, consequently $Q(f_1 + f_2) \leqq Q(f_1) + Q(f_2)$. It follows that $Q(f_1 + f_2) = Q(f_1) + Q(f_2)$, whenever $f_i \in \mathfrak{F}$, $f_i \geqq 0$, $f_1 + f_2 \geqq 1$. For arbitrary non-negative functions $f_1, f_2 \in \mathfrak{F}$ we have, therefore, $Q(f_1) + Q(f_2) + Q(1) = Q(f_1) + Q(f_2 + 1) = Q(f_1 + f_2 + 1) = Q(f_1 + f_2) + Q(1)$, so that $Q(f_1) + Q(f_2) = Q(f_1 + f_2)$.

Let f_1, f_2 be non-negative functions of $\mathfrak{F}$ again; put $f = f_1 - f_2$. We get $f_+ \leqq f_1$, whence $f_1 = f_+ + g$, $f_2 = f_- + g$, where $g \geqq 0$. It follows that $Q(f) = Q(f_+) - Q(f_-) = Q(f_+) + Q(g) - Q(f_-) - Q(g) = Q(f_1) - Q(f_2)$. If f_1, f_2 are arbitrary functions of $\mathfrak{F}$, then $Q(f_1) + Q(f_2) = Q(f_{1+}) - Q(f_{1-}) + Q(f_{2+}) - Q(f_{2-}) = Q(f_{1+} + f_{2+}) - Q(f_{1-} + f_{2-}) = Q((f_{1+} + f_{2+}) - (f_{1-} + f_{2-})) =$

[6]) $v \cdot \nu$ is the scalar product.
[7]) $|v|$ denotes the function $|v(x)|$.
[8]) $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$.

$= Q(f_1 + f_2)$. Obviously $Q(f) \geqq 0$ for each $f \geqq 0$, whence $Q(f_1) \leqq Q(f_2)$ for $f_1 \leqq f_2$ $(f, f_1, f_2 \in \mathfrak{F})$. It follows that $Q(f) \leqq Q(|f|)$, $-Q(f) = Q(-f) \leqq Q(|f|)$, consequently $|Q(f)| \leqq Q(|f|)$ for each $f \in \mathfrak{F}$.

Let now $f_1, f_2, \ldots$ be a bounded convergent sequence of $\mathfrak{B}$-measurable functions with limit f; put $g_n = |f - f_n|$. Suppose that the relation $Q(g_n) \to 0$ does not hold. Then there exist a sequence of integers $k_1 < k_2 < \ldots$ and a positive number ε such that $Q(g_{k_n}) > \varepsilon$ for $n = 1, 2, \ldots$, and we can determine $\mathfrak{B}$-measurable vectors v^n so as to have $|v^n| \leqq g_{k_n}$, $P(v^n) > \varepsilon$ $(n = 1, 2, \ldots)$. But $v^1, v^2, \ldots$ is obviously a bounded sequence with limit 0, whence $P(v^n) \to \to P(0) = 0$. This contradiction shows that $Q(g_n) \to 0$. As $|Q(f) - Q(f_n)| = = |Q(f - f_n)| \leqq Q(g_n)$, we have $Q(f) - Q(f_n) \to 0$, i. e. $Q(f_n) \to Q(f)$.

Further we put for each $B \in \mathfrak{B}$

$$p(B) = Q(c_B).\ ^{9})$$

Then p is a finite measure and $Q(f) = \int_D f \, dp$ for each $f \in \mathfrak{F}$. We define the functionals $P_1, \ldots, P_m$ on the family $\mathfrak{F}$ by means of the formulae $P_1(f) = P([f, 0, \ldots, 0]), \ldots, P_m(f) = P([0, \ldots, 0, f])$ and put $p_i(B) = P_i(c_B)$ for each $B \in \mathfrak{B}$ $(i = 1, \ldots, m)$. If $f \in \mathfrak{F}$ and $v = [0, \ldots, 0, f, 0, \ldots, 0]$, then $|v| \leqq |f|$, therefore $P_i(f) = P(v) \leqq Q(|f|)$. In particular, $\pm p_i(B) = P_i(\pm c_B) \leqq Q(c_B) = p(B)$, whence $|p_i(B)| \leqq p(B)$ for each set $B \in \mathfrak{B}$ and each index i. It is easy to see that the functions p_i are σ-additive; consequently, there exist (see e. g. [7], p. 36, theorem of Radon-Nikodym) $\mathfrak{B}$-measurable functions ν_i such that $p_i(B) = = \int_B \nu_i \, dp$ $(B \in \mathfrak{B}, i = 1, \ldots, m)$. Obviously $P_i(f) = \int_D f \nu_i \, dp$ for each $f \in \mathfrak{F}$. If we put

$$\nu = [\nu_1, \ldots, \nu_m]\,,$$

we have

$$P(v) = \sum_{i=1}^{m} P_i(v_i) = \sum_{i=1}^{m} \int_D v_i \,.\, \nu_i \, dp = \int_D v \,.\, \nu \, dp$$

for each $v \in \mathfrak{V}$.

We shall prove that $|\nu(x)| = 1$ almost everywhere in D.[10] For this purpose put $B = E[x; |\nu(x)| > 1]$. We define a vector v by means of the relations

$$v(x) = \nu(x) \,.\, |\nu(x)|^{-1} \,(x \in B)\,, \quad v(x) = 0 \,(x \in D - B)\,.$$

Obviously $|v| = c_B$, whence $\int_B 1 \,.\, dp = Q(c_B) \geqq P(v) = \int_B v \,.\, \nu \, dp = \int_B |\nu| \, dp$, consequently $\int_B (|\nu| - 1) \, dp \leqq 0$. As $|\nu(x)| > 1$ on B, it follows that $p(B) = 0$. We can therefore suppose that $|\nu(x)| \leqq 1$ for all $x \in D$.

For an arbitrary $\mathfrak{B}$-measurable vector v such that $|v| \leqq 1$ we have $|v \,.\, \nu| \leqq \leqq |v| \,.\, |\nu| \leqq |\nu|$, whence $P(v) = \int_D v \,.\, \nu \, dp \leqq \int_D |\nu| \, dp$, so that $\int_D 1 \,.\, dp = Q(1) =$

---

[9]) c_B is the characteristic function of the set B.
[10]) With respect to the measure p.

$= \sup_{|v|\leq 1} P(v) \leq \int_D |\nu| \, dp$; it follows that $\int_D (1 - |\nu|) \, dp \leq 0$. Since $|\nu| \leq 1$, we have $|\nu(x)| = 1$ for almost all $x \in D$.[10]) This being so, we may suppose that $|\nu(x)| = 1$ for all $x \in D$; the relations (24), (25) are then satisfied.

Finally, let a $\mathfrak{B}$-measurable vector ν' and a measure p' on the system $\mathfrak{B}$ also fulfil the conditions (24) and (25). If $B \in \mathfrak{B}$, $v \in \mathfrak{V}$, $|v| \leq c_B$, then $P(v) = \int_D v \cdot \nu' \, dp' \leq \int_D |v| \cdot |\nu'| \cdot dp' \leq \int_D c_B \, dp' = p'(B)$; moreover, $P(c_B \cdot \nu') = \int_D c_B \cdot \nu' \cdot \nu' \, dp' = \int_D c_B \, dp' = p'(B)$, $|c_B \cdot \nu'| \leq c_B$. We see that $p'(B) = \max P(v)$, where $v \in \mathfrak{V}$, $|v| \leq c_B$; therefore $p'(B) = Q(c_B) = p(B)$. Further $\int_D (\nu - \nu') \cdot \nu \, dp = \int_D (\nu - \nu') \cdot \nu' \, dp$, whence $0 = \int_D (\nu - \nu') \cdot (\nu - \nu') \, dp = \int_D |\nu - \nu'|^2 \, dp$, so that $\nu(x) = \nu'(x)$ for almost all $x \in D$.[10])

18. Theorem. *Let A be an arbitrary set of $\mathfrak{A}$. Then there exist on the boundary D of A a Borel measure p and a Borel vector ν such that $|\nu| = 1$ and that*

$$\int_A \operatorname{div} v(x) \, dx = \int_D v \cdot \nu \, dp \tag{26}$$

for each $v \in \mathfrak{V}_A$.[11])

If a Borel measure p' and a Borel vector ν' on the set D fulfil the relations $|\nu'(x)| = 1$ for each $x \in D$ and $\int_A \operatorname{div} v(x) \, dx = \int_D v \cdot \nu' \, dp'$ for each $v \in \mathfrak{V}_A$, then $p' = p$ and $\nu'(x) = \nu(x)$ almost everywhere.[10]) *Further, we have*

$$p(D) = \|A\| \tag{27}$$

and

$$P(v)^{12)} = \int_D v \cdot \nu \, dp \tag{28}$$

for each bounded Borel vector v on the set D.

Proof. According to theorems 16 and 17 there exist a Borel vector ν and a Borel measure p on the set D such that $|\nu| = 1$ and that (28) is valid for each bounded Borel vector v on the set D. Following 16, d) we see that (26) holds for each $v \in \mathfrak{V}_A$.

Now let p' be a Borel measure and ν' a Borel vector on D such that $|\nu'| = 1$ and that $\int_A \operatorname{div} v(x) \, dx = \int_D v \cdot \nu' \, dp'$ for each $v \in \mathfrak{V}_A$. If i is an integer, $1 \leq i \leq m$, we have

$$\int_D f \cdot \nu_i \cdot dp = \int_A \frac{\partial f(x)}{\partial x_i} \, dx = \int_D f \cdot \nu'_i \cdot dp' \tag{29}$$

for each polynomial f. For $f = 1$ we get $0 = \int_D \nu'_i \cdot dp'$, whence $\int_D |\nu'_i| \, dp' < \infty$.

[11]) See definition 2.
[12]) See definition 15.

The relation $1 = \sqrt{\sum_{i=1}^{m} \nu_i'^2} \leqq \sum_{i=1}^{m} |\nu_i'|$ yields $\int_D 1 \, . \, dp' \leqq \int_D \sum_{i=1}^{m} |\nu_i'| \, dp < \infty$; we see that the measure p' is finite. From (29) it follows that the equality $\int_D f \, . \, \nu_i \, dp = \int_D f \, .$ $. \, \nu_i' \, dp'$ holds for each bounded Borel function f on the set D. We have therefore $P(v) = \int_D v \, . \, \nu \, dp = \int_D v \, . \, \nu' \, dp'$ for an arbitrary bounded Borel vector on D. By theorem 17, $p = p'$ and $\nu(x) = \nu'(x)$ for almost all $x \in D$.[10])

According to 16., c) and (28) we have

$$p(D) = \int_D \nu \, . \, \nu \, dp = P(\nu) \leqq \|A\| \, ; \tag{30}$$

since $\int_A \operatorname{div} v(x) \, dx = \int_D v \, . \, \nu \, dp \leqq \int_D |v| \, . \, |\nu| \, dp \leqq p(D)$ for each $v \in \mathfrak{V}_A$, we get $\|A\| = \sup \int_A \operatorname{div} v(x) \, dx \leqq p(D)$, which together with (30) proves the relation (27) and completes the proof.

Remark 1. It follows from (27) and (30) that $\|A\| = P(\nu)$.

Remark 2. According to the definition of the vector ν we have $P_m(f) = \int_D f \, . \, \nu_m \, dp$. From 14., 5) we thus see that the relation

$$\int_A \frac{\partial f(x)}{\partial x_m} \, dx = P_m(f) \leqq \int_D |\nu_m| \, dp$$

holds for each $f \in \mathfrak{P}_A$ (see definition 3), whence $\|A\|_m \leqq \int_D |\nu_m| \, dp$. But, if we put $f = \operatorname{sgn} \nu_m$, we get by 14., 4) $\int_D |\nu_m| \, dp = \int_D \operatorname{sgn} \nu_m \, . \, \nu_m \, dp = P_m(\operatorname{sgn} \nu_m) \leqq \|A\|_m$, so that $\|A\|_m = \int_D |\nu_m| \, dp = P_m(\operatorname{sgn} \nu_m)$.

Remark 3. The measure p and the vector ν will be called the *surface measure* and the *normal vector* of A respectively.

19. Notation. If $A \subset E_m$, $x = [x_1, \ldots, x_{m-1}] \in E_{m-1}$ and if i is an integer, $1 \leqq i \leqq m$, let A_x^i be the set of all $t \in E_1$ such that $[x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{m-1}] \in A$.

20. Theorem. *Let A be a bounded measurable subset of E_m. Let φ be a non-negative function on E_{m-1} such that $\int_{E_{m-1}} \varphi(x) \, dx < \infty$. Suppose that for almost each point $x \in E_{m-1}$ there exist a non-negative integer $r \leqq \varphi(x)$ and real numbers $a_1 < b_1 < \ldots < a_r < b_r$ such that the set A_x^m is equivalent*[13]) *to $\bigcup_{j=1}^{r} (a_j, b_j)$. Further, if f is a finite function on the boundary D of A, we define almost everywhere on E_{m-1} the function*

$$^{m}f(x) = \sum_{j=1}^{r} (f(x, b_j) - f(x, a_j)) \, .^{14)} \tag{31}$$

13) We say that the sets A_1, A_2 are equivalent, if the measure of $(A_1 - A_2) \cup (A_2 - A_1)$ is zero.

14) For $z = [z_1, \ldots, z_m] \in E_m$ we sometimes write $z = [x, y]$, where $x = [x_1, \ldots, x_{m-1}]$, $y = z_m$. — The points $[x, a_j]$, $[x, b_j]$ lie obviously in D.

Then we have $\|A\|_m \leqq 2\int_{E_{m-1}} \varphi(x)\,\mathrm{d}x$, *consequently* $A \in \mathfrak{A}_m$, *and the relation*

$$P_m(f) = \int_{E_{m-1}} {}^m f(x)\,\mathrm{d}x \tag{32}$$

is valid for each bounded Borel function f *on* D.

Proof. If f is a polynomial, we have

$$\int_{A_x^m} \frac{\partial f(x, y)}{\partial y}\,\mathrm{d}y = \sum_{j=1}^{r} (f(x, b_j) - f(x, a_j)) = {}^m f(x)$$

for almost all $x \in E_{m-1}$, so that

$$\int_A \frac{\partial f(z)}{\partial z_m}\,\mathrm{d}z = \int_{E_{m-1}} \left(\int_{A_x^m} \frac{\partial f(x, y)}{\partial y}\,\mathrm{d}y \right) \mathrm{d}x = \int_{E_{m-1}} {}^m f(x)\,\mathrm{d}x\,. \tag{33}$$

If $\|f\|_D \leqq 1$, then $|{}^m f(x)| \leqq 2\varphi(x)$ almost everywhere on E_{m-1}, whence $\|A\|_m \leqq$ $\leqq 2 \int_{E_{m-1}} \varphi(x)\,\mathrm{d}x < \infty$. Further, let $\mathfrak{F}$ be the family of all bounded Borel functions on D, for which (32) is valid. If $f_1, f_2, \ldots \in \mathfrak{F}$ is a bounded ($|f_n(x)| \leqq C$) convergent sequence with limit f, we have obviously ${}^m(f_n)(x) \to {}^m f(x)$, $|{}^m(f_n)(x)| \leqq$ $\leqq 2C\,\varphi(x)$ for almost all $x \in E_{m-1}$. As the function φ is summable, we get $\int_{E_{m-1}} {}^m(f_n)(x)\,\mathrm{d}x \to \int_{E_{m-1}} {}^m f(x)\,\mathrm{d}x$. From the relation $\int_{E_{m-1}} {}^m(f_n)(x)\,\mathrm{d}x = P_m(f_n) \to P_m(f)$ it follows that $P_m(f) = \int_{E_{m-1}} {}^m f(x)\,\mathrm{d}x$, whence $f \in \mathfrak{F}$. Since each polynomial f satisfies (33) (and thus belongs to $\mathfrak{F}$), we see that $\mathfrak{F}$ is the family of all bounded Borel functions on D.

Remark. In an analogous manner, we can define the symbol ${}^i f$ and prove a similar theorem for $i = 1, \ldots, m - 1$. Thus we can compute $P(A, v)$ with the help of the $(m - 1)$-dimensional Lebesgue integral.

From theorem 20 it follows that, for instance, every bounded convex set belongs to $\mathfrak{A}$. If, in particular, A is a cube with edge ε, i. e. $A = \langle a_1, a_1 + \varepsilon\rangle \times$ $\times \ldots \times \langle a_m, a_m + \varepsilon\rangle$, then $P_m(A, f) = \int_{A_0} (f(x, a_m + \varepsilon) - f(x, a_m))\,\mathrm{d}x$, where $A_0 = \langle a_1, a_1 + \varepsilon\rangle \times \ldots \times \langle a_{m-1}, a_{m-1} + \varepsilon\rangle$. Obviously $\|A\| \leqq 2m\,.\,\varepsilon^{m-1}$. Computing $P(A, \nu)$, where ν is the normal vector of A, we see that the sign of equality holds here.

21. Notation. If n is a natural number, we can determine a function ψ as in the proof of lemma 5 (see (1)—(5)), choosing $\delta = \dfrac{1}{n}$. Then we write $\psi = \psi_n$.

22. Lemma. *Let the function* g *be summable (in the sense of Lebesgue) on* E_m. *For* $n = 1, 2, \ldots$ *put* $g_n(x) = \int_{E_m} g(t)\,\psi_n(x - t)\,\mathrm{d}t$. *Then each* g_n *is of class* C_∞ *on* E_m *and*

$$\int_A g_n(x)\,\mathrm{d}x \to \int_A g(x)\,\mathrm{d}x \tag{34}$$

for every measurable set $A \subset E_m$.

Proof. From the relations

$$\frac{\partial g_n(x)}{\partial x_i} = \int\limits_{E_m} g(t) \; \frac{\partial \psi_n(x-t)}{\partial x_i} \, \mathrm{d}t \quad \text{etc.}$$

(see, for example, [2], p. 281) we see that each function g_n is of class C_∞.

Choose any measurable set $A \subset E_m$. If the function g is continuous and if there exists a compact set K such that g vanishes outside K, we can find a compact set K_0 such that all the functions g_n vanish on $E_m - K_0$ and the sequence $g_1, g_2, \ldots$ converges uniformly to g on E_m, so that the relation (34) holds.

Now let g be an arbitrary summable function and let ε be any positive number. There exist a compact set K and a function γ, which is continuous on E_m, vanishes on $E_m - K$ and fulfils the relation $\int\limits_{E_m} |g(x) - \gamma(x)| \, \mathrm{d}x < \varepsilon$. Put $\gamma_n(x) = \int\limits_{E_m} \gamma(t) \, \psi_n(x-t) \, \mathrm{d}t$. By what has just been proved, there exists an index n_0 such that $|\int\limits_A \gamma(x) \, \mathrm{d}x - \int\limits_A \gamma_n(x) \, \mathrm{d}x| < \varepsilon$ for each $n > n_0$. Further,

$$|\int\limits_A \gamma_n(x) \, \mathrm{d}x - \int\limits_A g_n(x) \, \mathrm{d}x| = |\int\limits_A (\int\limits_{E_m} (\gamma(t) - g(t)) \, \psi_n(x-t) \, \mathrm{d}t) \, \mathrm{d}x| \leqq$$
$$\leqq \int\limits_{E_{2m}} |\gamma(t) - g(t)| \, \psi_n(x-t) \, \mathrm{d}x \, \mathrm{d}t =$$
$$= \int\limits_{E_m} |\gamma(t) - g(t)| \, (\int\limits_{E_m} \psi_n(x-t) \, \mathrm{d}x) \, \mathrm{d}t = \int\limits_{E_m} |\gamma(t) - g(t)| \, \mathrm{d}t < \varepsilon ;$$

clearly also $|\int\limits_A g(x) \, \mathrm{d}x - \int\limits_A \gamma(x) \, \mathrm{d}x| < \varepsilon$. For $n > n_0$ we have, therefore, $|\int\limits_A g(x) \, \mathrm{d}x - \int\limits_A g_n(x) \, \mathrm{d}x| < 3\varepsilon$ and the proof is complete.

23. Definition. We say that the vector v and the function f are *associated* on the set G, if G is open in E_m, v is continuous on G and if the equality

$$P(K, v) = \int\limits_K f(x) \, \mathrm{d}x$$

(with the Lebesgue integral on the right) holds for each cube[15]) $K \subset G$.

Remark. The function f is then summable on each compact set $M \subset G$.

24. Theorem. *Let the vector v and the function f be associated on the set G. Then the relation*

$$P(A, v) = \int\limits_A f(z) \, \mathrm{d}z \tag{35}$$

holds for each set $A \in \mathfrak{A}$, where $\overline{A} \subset G$.

Proof. If $\emptyset \neq A \in \mathfrak{A}$, $\overline{A} \subset G$, we can determine a positive number ε such that the relations $z \in \overline{A}$, $|t - z| \leqq 2\varepsilon$ imply $t \in G$. Let H (resp. L) be the set of all t whose distance from A is less than ε (resp. is not greater than 2ε). Then $\overline{A} \subset H$, $\overline{H} \subset L \subset G$, H is open, L compact. There exists a vector w which is continuous on E_m, vanishes outside a bounded set $G_0 \subset G$ and which coincides with v on L. If

[15]) I. e. a Cartesian product of m closed one-dimensional intervals of equal finite and positive length.

$t, z \in E_m$, put $\varphi_t(z) = w_m(z - t)$ (where w_m is the m-th component of w). Choose an arbitrary natural number n and take the function ψ_n (see notation 21). If K is any cube, $K = \langle a_1, b_1 \rangle \times \ldots \times \langle a_m, b_m \rangle$ and if we put $K_0 = \langle a_1, b_1 \rangle \times \times \ldots \times \langle a_{m-1}, b_{m-1} \rangle$, we have

$$\int_{E_m} \psi_n(t)\, P_m(K, \varphi_t)\, dt = \int_{E_m} \psi_n(t) \left(\int_{K_0} (w_m([x, b_m] - t) - w_m([x, a_m] - t))\, dx \right) dt =$$
$$= \int_{K_0} \left(\int_{E_m} \psi_n(t)\, (w_m([x, b_m] - t) - w_m([x, a_m] - t))\, dt \right) dx = P_m(K, v_m^n)\,,$$

where v_m^n is the m-th component of the vector $v^n(z) = \int_{E_m} \psi_n(t)\, w(z - t)\, dt$. Since similar relations hold for $1, \ldots, m - 1$, we see that

$$\int_{E_m} \psi_n(t)\, P(K, w^t)\, dt = P(K, v^n)\,, \tag{36}$$

where $w^t(z) = w(z - t)$. Evidently

$$P(K, w^t) = P(K_t, w)\,, \tag{37}$$

where $K_t = E[z;\, z = \zeta - t,\, \zeta \in K]$.

Put $g(z) = f(z)$ for $z \in L$, $g(z) = 0$ elsewhere. Then g is summable on E_m. If K is a cube, $K \subset H$, and if $|t| < \varepsilon$, we obviously have $K_t \subset L$, whence

$$P(K_t, w) = P(K_t, v) = \int_{K_t} f(z)\, dz = \int_{K_t} g(z)\, dz = \int_K g(z - t)\, dz\,. \tag{38}$$

Let n be an index greater than $\dfrac{\sqrt{m}}{\varepsilon}$. For each $t \in E_m$, where $|t| \geqq \varepsilon$, we then have $\psi_n(t) = 0$. It follows from (36)—(38), that $P(K, v^n) = \int_{E_m} \psi_n(t)\, P(K, w^t)\, dt = \int_{|t|<\varepsilon} \psi_n(t)\, P(K_t, w)\, dt = \int_{|t|<\varepsilon} \psi_n(t) \left(\int_K g(z - t)\, dz \right) dt = \int_K \left(\int_{E_m} \psi_n(t)\, g(z - t)\, dt \right) dz = \int_K g_n(z)\, dz$, where

$$g_n(z) = \int_{E_m} g(z - t)\, \psi_n(t)\, dt = \int_{E_m} g(t)\, \psi_n(z - t)\, dt\,.$$

Thus we see that the vector v^n and the function g_n are associated on H. But, since the vector v^n is of the class C_1 (in fact, of the class C_∞) on E_m, we have evidently $P(K, v^n) = \int_K \operatorname{div} v^n(z)\, dz$ for each cube K. This shows that $\int_K \operatorname{div} v^n(z)\, dz = \int_K g_n(z)\, dz$ for each cube $K \subset H$. The functions $\operatorname{div} v^n$, g_n being continuous, it follows that $g_n(z) = \operatorname{div} v^n(z)$ for each $z \in H$, so that by 16., d)

$$P(A, v^n) = \int_A g_n(z)\, dz\,. \tag{39}$$

This relation holds for each $n > \dfrac{\sqrt{m}}{\varepsilon}$. Since w is uniformly continuous, we have $v^n \to w$ uniformly on E_m and so $v^n \to v$ uniformly on $L \supset \overline{A}$, which yields

$$P(A, v^n) \to P(A, v)\,. \tag{40}$$

Moreover, on account of lemma 22 we have

$$\int_A g_n(z)\,dz \to \int_A g(z)\,dz = \int_A f(z)\,dz\,. \tag{41}$$

The equality (35) follows at once from (39)—(41).

Remark 1. The reader may compare this theorem with the remark to theorem 43.

Remark 2. Theorem 20 enables us to give examples of sets $A \in \mathfrak{A}_m$. We shall prove (see theorem 33) that conversely each set of $\mathfrak{A}_m$ fulfils the conditions of theorem 20. The proof is complicated and depends upon several lemmas.

Remark 3. The relation "to be associated" between a vector and a function is "invariant" (see theorem 53). Let us still mention that the paragraphs 44—53 do not depend upon the paragraphs 25—43.

25. Definition. Let G be a bounded open subset of E_1; let $\mathfrak{M}$ be the system of all components of G. We order the set $\mathfrak{M}$ as follows: If $I, J \in \mathfrak{M}$, then $I < J$ denotes that either $\mu(I) > \mu(J)$ [16]) or $\mu(I) = \mu(J)$ and $x < y$ for all $x \in I$, $y \in J$. (More intuitively: We order the intervals of $\mathfrak{M}$ according to their length and, if the length is equal, from the left to the right. It is easy to see that this is indeed an ordering.) If $I < J$, we thus have $\mu(I) \geqq \mu(J)$; as the set G is bounded, there exist for each $J \in \mathfrak{M}$ at most a finite number of I's such that $I < J$. Now we define an infinite sequence $I_1, I_2, \ldots$ in the following way: If the set $\mathfrak{M}$ has at least n elements, let I_n be the n-th element of $\mathfrak{M}$ in the ordering just defined; if $\mathfrak{M}$ has less than n elements, put $I_n = \emptyset$. It is easy to see that $\bigcup_{n=1}^{\infty} I_n = G$. We say that $I_1, I_2, \ldots$ is the *canonical sequence* of G.

26. Lemma. *Let Z be an arbitrary non-empty set; suppose that an open bounded set $G_x \subset E_1$ is given for each $x \in Z$. If $-\infty < a < b < \infty$, put*

$$M_{a,b} = E[x; \langle a, b\rangle \subset G_x]\,. \tag{42}$$

Let $I_1^x, I_2^x, \ldots$ be the canonical sequence of G_x; put

$$f_n(x) = \inf I_n^x\,, \quad g_n(x) = \sup I_n^x \quad (x \in Z,\ n = 1, 2, \ldots)\,.^{17)} \tag{43}$$

Further, let $\mathfrak{B}$ be a σ-algebra on Z such that each set $M_{a,b}$ belongs to $\mathfrak{B}$. Then the functions f_n, g_n are $\mathfrak{B}$-measurable ($n = 1, 2, \ldots$).

Proof. First, we prove that the function g_1 is $\mathfrak{B}$-measurable. Let c be a fixed real number. For each $d > 0$ put

$$C_d = \bigcup M_{b-d,b} \ (\text{resp. } D_d = \bigcup M_{b-d,b})\,,$$

[16]) μ is the one-dimensional Lebesgue measure (length).

[17]) We thus have $I_n^x = (f_n(x), g_n(x))$, if $I_n^x \neq \emptyset$, but $f_n(x) = \infty$, $g_n(x) = -\infty$ for $I_n^x = \emptyset$.

where b runs over all the rational numbers greater (resp. smaller) than c. We shall prove that

$$E[x; g_1(x) > c] = \bigcup (C_d - D_d) \quad (d > 0 \text{ rational}) . \tag{44}$$

Indeed, if $g_1(x) > c$, then $\mu(I_1^x) > 0$. There exists an index q such that

$$\mu(I_1^x) = \ldots = \mu(I_q^x) > \mu(I_{q+1}^x) . \tag{45}$$

Since $c - f_1(x) < g_1(x) - f_1(x) = \mu(I_q^x)$, we can determine a rational number d so as to have

$$\max (c - f_1(x), \mu(I_{q+1}^x)) < d < \mu(I_q^x) . \tag{46}$$

As $f_1(x) + d < f_1(x) + \mu(I_q^x) = g_1(x)$, there exists a rational number b such that

$$f_1(x) + d < b < g_1(x) . \tag{47}$$

According to (47), we have $\langle b - d, b\rangle \subset (f_1(x), g_1(x)) \subset G_x$; the relation (46) yields

$$d > c - f_1(x) , \tag{48}$$

whence $b > f_1(x) + d > c$, so that $x \in C_d$.

Assume for a moment that $x \in D_d$. Then there exists a number $b_1 < c$ such that $\langle b_1 - d, b_1\rangle \subset G_x$. Let $\langle b_1 - d, b_1\rangle$ be contained in the component I_n^x of G_x. From the relation $\mu(I_n^x) > d$ it follows that $n \leqq q$ (see (45), (46)). Because $t > f_1(x)$ for each $t \in \bigcup_{j=1}^{q} I_j$, we get

$$b_1 - d > f_1(x) . \tag{49}$$

But the relations (48), (49) yield the inequality $b_1 > c$; we arrive at a contradiction which proves that x non $\in D_d$, so that $x \in C_d - D_d$.

If, conversely, $x \in C_d - D_d$ for some $d > 0$, then obviously $\mu(I_1) > d$, and, further, $g_1(x) > c$. For if not, we could take a rational b such that $f_1(x) + {} + d < b < g_1(x)$ and we would obtain $\langle b - d, b\rangle \subset G_x$, where $b < g_1(x) \leqq c$, which is impossible, since x non $\in D_d$. This completes the proof of the formula (44); the function g_1 is therefore $\mathfrak{B}$-measurable.

Now we shall consider the function f_1. Obviously

$$E[x; f_1(x) < \infty] = \bigcup M_{a,b} \quad (a, b \text{ rational}, \ a < b) .$$

It follows that

$$E[x; f_1(x) < \infty] \in \mathfrak{B} . \tag{50}$$

Let c be a fixed real number again. For each $d > 0$ put

$$A_d = \bigcup M_{b,b+d} \text{ (resp. } B_d = \bigcup M_{b,b+d}),$$

where b runs over all the rational numbers greater (resp. smaller) than c. We shall prove the relation

$$E[x; c \leqq f_1(x) < \infty] = \bigcup (A_d - B_d) \quad (d > 0 \text{ rational}) . \tag{51}$$

Indeed, suppose that $c \leqq f_1(x) < \infty$ and

$$\mu(I_1^x) = \ldots = \mu(I_q^x) > \mu(I_{q+1}^x) . \tag{52}$$

There exists a rational d such that

$$\mu(I_{q+1}^x) < d < \mu(I_q^x) . \tag{53}$$

If we select a rational b from $(f_1(x), g_1(x) - d)$, then $\langle b, b + d\rangle \subset (f_1(x), g_1(x)) \subset \subset G_x$, whence (as $b > f_1(x) \geqq c$) follows $x \in A_d$. Now suppose that the number b_1 has the property that $\langle b_1, b_1 + d\rangle \subset G_x$. Let $\langle b_1, b_1 + d\rangle$ be contained in the component I_n^x of G_x. Since $\mu(I_n^x) > d$, we have $n \leqq q$, whence $b_1 \geqq f_1(x)$ and so $b_1 \geqq c$. This proves that x non $\in B_d$, so that $x \in A_d - B_d$.

Conversely, let $x \in A_d - B_d$ for some $d > 0$. Admit that $f_1(x) < c$. Since obviously $\mu(I_1) > d$, there exists a rational number b_2 such that $f_1(x) < b_2 < < \min(c, g_1(x) - d)$. We get $b_2 < c$, $f_1(x) < b_2$, $b_2 + d < g_1(x)$, whence $x \in B_d$ — contradiction. It follows that $f_1(x) \geqq c$; obviously $f_1(x) < \infty$. This proves the relation (51); according to (50) and (51), f_1 is $\mathfrak{B}$-measurable.

Finally, let $N_{a,b}$ $(a < b)$ be the set of all x such that $\langle a, b\rangle \subset I_1^x$. Evidently $N_{a,b} = E[x; f_1(x) < a] \cap E[x; g_1(x) > b] \in \mathfrak{B}$. For each $x \in Z$ put $G_x^1 = \bigcup_{n=2}^{\infty} I_n^x$ $(= G_x - I_1^x)$. If $M_{a,b}^1 = E[x; \langle a, b\rangle \subset G_x^1]$, we have $M_{a,b}^1 = M_{a,b} - N_{a,b}$ for all a, b $(a < b)$. If we apply our results to the system of sets G_x^1 $(x \in Z)$, we see that the functions f_2, g_2 are $\mathfrak{B}$-measurable too. In an analogous way we can prove that the functions $f_3, g_3, \ldots$ are $\mathfrak{B}$-measurable.

27. Lemma. *Let Z be an arbitrary non-empty set. For each $x \in Z$ let G_x be a bounded open subset of E_1 which has only a finite number of components. If $G_x = (a_1, b_1) \cup \ldots \cup (a_r, b_r)$ $(a_1 < b_1 \leqq a_2 < b_2 \leqq \ldots \leqq a_r < b_r$, r integer $\geqq 0)$, put $f_i(x) = a_i$, $g_i(x) = b_i$ for $i \leqq r$, $f_i(x) = \infty$, $g_i(x) = -\infty$ for $i > r$. Let $\mathfrak{B}$ be a σ-algebra on Z which contains all sets $M_{a,b}$ (see* (42)*). Then all functions f_n, g_n are $\mathfrak{B}$-measurable.*

Proof. For each $c \in E_1$ we have $E[x; f_1(x) < c] = \bigcup M_{a,b}$, where a, b are rational, $a < b < c$, and $E[x; -\infty < g_1(x) < c] = \bigcup (M_{\alpha,\beta} - M_{\beta,\gamma})$, where α, β, γ are rational, $\alpha < \beta < \gamma < c$; obviously $E[x; g_1(x) = -\infty] = E[x; f_1(x) = \infty]$. We see that the functions f_1, g_1 are $\mathfrak{B}$-measurable. Considering the sets $G_x^1 = G_x - (a_1, b_1)$ we prove that the functions f_2, g_2 are $\mathfrak{B}$-measurable, and so on.

28. Lemma. *Let A be a bounded Borel subset of E_m. If $x \in E_{m-1}$, let G_x*[18]*) be a subset of E_1 which is defined as follows: The number t belongs to G_x if and only if there exists a neighbourhood U of t such that $\mu(U - A_x^m) = 0$ (see notation* 19*). Then every set G_x is open, $M_{a,b} = E[x; \langle a, b\rangle \subset G_x]$ is a Borel subset of E_{m-1} whenever $a < b$, and the functions $f_1, g_1, f_2, g_2, \ldots$, which are defined by* (43)*, are Borel functions on E_{m-1}.*

---

[18]) If necessary, we write $G_x = G_x^A$.

[19]) G_x is, of course, the greatest open set with this property.

Proof. The set G_x is obviously open for each $x \in E_{m-1}$. From the separability of G_x it follows that $\mu(G_x - A_x^m) = 0$.[19] Let $a, b \in E_1$, $a < b$. We have $(a, b) \subset \subset G_x$, if and only if $\mu(A_x^m \cap (a, b)) = b - a$. If we put $B = A \cap E[[x_1, \dots, x_m]; a < x_m < b]$, we obtain $A_x^m \cap (a, b) = B_x^m$. As $\mu(B_x^m)$ is a Borel function of x,

$$m_{a,b} = E[x; (a, b) \subset G_x] = E[x; \mu(B_x^m) = b - a] \tag{54}$$

and, consequently, also $M_{a,b} = \bigcup_{n=1}^{\infty} m_{a-\frac{1}{n}, b+\frac{1}{n}}$ $(a < b)$ is a Borel subset of E_{m-1}. Our assertion follows immediately from lemma 26.

29. Lemma. *Let f be a bounded non-negative Borel function on E_m and let B be a Borel subset of E_{m-1}. Let ε be a positive number. If the relation $\int_0^\infty f(x, t)\,\mathrm{d}t > \varepsilon$ holds for each $x \in B$, then there exists a positive Borel function ψ on the set B such that $\int_0^{\psi(x)} f(x, t)\,\mathrm{d}t = \varepsilon$ for each $x \in B$.*

Proof. Let S be the set of all $[x, y]$, where $x \in E_{m-1}$, $y > 0$, $\int_0^y f(x, t)\,\mathrm{d}t < \varepsilon$. For each $x \in B$ there exists a finite positive number b such that $S_x^m = (0, b)$; put $b = \psi(x)$. Obviously $\int_0^{\psi(x)} f(x, t)\,\mathrm{d}t = \varepsilon$. We have now to prove that ψ is a Borel function. For each $y \in E_1$ let ${}_yS$ be the set of all $x \in E_{m-1}$ such that $[x, y] \in S$. As $F(x, y) = \int_0^y f(x, t)\,\mathrm{d}t$ is a Borel function, S is a Borel set; ${}_yS$ is therefore a Borel subset of E_{m-1}. We prove that for an arbitrary $c \in E_1$

$$E[x; \psi(x) > c] = B \cap (\bigcup {}_yS)\,, \tag{55}$$

where y runs over all rational numbers $> c$. Indeed, if $\psi(x) > c$, we can select a positive rational $y \in (c, \psi(x))$ and have $\int_0^y f(x, t)\,\mathrm{d}t < \varepsilon$, whence $[x, y] \in S$, $x \in {}_yS$. Conversely, if $x \in B \cap {}_yS$, where $y > c$, then $[x, y] \in S$, whence $\psi(x) > > y > c$. From (55) we see that ψ is a Borel function.

30. Lemma. *Suppose that $A \in \mathfrak{A}_m$ and that f is a bounded Borel function on E_m. For $x \in E_{m-1}$, $y \in E_1$ put $F(x, y) = \int_0^y f(x, t)\,\mathrm{d}t$. Then*

$$P_m(F) = \int_A f(z)\,\mathrm{d}z\,. \tag{56}$$

Proof. If f is continuous, (56) follows easily from (20b). Let $\mathfrak{F}$ be the family of all bounded Borel functions f on E_m for which (56) holds good. If $f_1, f_2, \dots$ $(f_n \in \mathfrak{F})$ is a bounded convergent sequence with limit f, then the function f evidently also belongs to $\mathfrak{F}$, so that $\mathfrak{F}$ is the family of all bounded Borel functions on E_m.

31. Lemma. *Let A (resp. Z) be a Borel subset of E_m (resp. E_{m-1}) and let f, g be finite Borel functions on Z. Suppose that $f(x) < g(x)$, $\mu(A_x^m \cap (f(x), g(x))) > 0$ for each $x \in Z$ and that $\mu((g(x), g(x) + \varepsilon) - A_x^m) > 0$ for each $x \in Z$ and each $\varepsilon > 0$. We define the function Φ on E_m by means of the relations*

$$\Phi(x, y) = \frac{\mu(A_x^m \cap (f(x), y))}{\mu(A_x^m \cap (f(x), g(x)))}, \quad \text{if } x \in Z, f(x) < y \leqq g(x),$$

$$\Phi(x, y) = 0 \text{ elsewhere.}$$

Then $P_m(\Phi)$ equals the $(m-1)$-dimensional measure of Z.

Proof. There exists a number c such that the relation $[x, y] \in A$ ($x \in E_{m-1}$, $y \in E_1$) implies $y > c$. Considering, if necessary, the set $E[[x, y]; [x, y + c] \in A]$ instead of A, we can suppose that $y > 0$ for each point $[x, y] \in A$. Since $\mu(A_x^m \cap (f(x), g(x))) > 0$, we then have $g(x) > 0$ for all $x \in Z$. Now we define the function h on E_m, writing

$$h(x, y) = 1, \quad \text{if} \quad x \in Z, \quad y > g(x), \quad [x, y] \text{ non} \in A,$$

and

$$h(x, y) = 0 \quad \text{elsewhere}.$$

Let n be a natural number. By lemma 29 there exists a Borel function ψ_n on Z such that $\int_0^{\psi_n(x)} h(x, t)\, dt = \frac{1}{n}$. As $h(x, t) = 0$ for $t \leqq g(x)$, we have $\psi_n(x) > g(x)$,

$$\mu((g(x), \psi_n(x)) - A_x^m) = \int_{g(x)}^{\psi_n(x)} h(x, t)\, dt = \frac{1}{n}.$$

Further, put

$$\gamma(x) = \mu(A_x^m \cap (f(x), g(x))) \quad (x \in Z).$$

We define a sequence $f_1, f_2, \dots$ of functions on E_m in the following way: If $x \in Z$, $[x, y] \in A$, $f(x) < y < g(x)$, put

$$f_n(x, y) = \min\left(n, \frac{1}{\gamma(x)}\right);$$

if $x \in Z$, $[x, y]$ non $\in A$, $g(x) < y < \psi_n(x)$, put

$$f_n(x, y) = -\min(n^2 \gamma(x), n);$$

in the remaining cases put $f_n(x, y) = 0$. Further write

$$F_n(x, y) = \int_0^y f_n(x, t)\, dt. \tag{57}$$

Then, for each $x \in Z$,

$$\int_{f(x)}^{g(x)} f_n(x, t)\, dt = \gamma(x) \,.\, \min\left(n, \frac{1}{\gamma(x)}\right) = \min(n\, \gamma(x), 1),$$

$$-\int_{g(x)}^{\psi_n(x)} f_n(x, t)\, dt = \frac{1}{n} \min(n^2 \gamma(x), n) = \min(n\, \gamma(x), 1),$$

whence $\int_{f(x)}^{\psi_n(x)} f_n(x, t)\, dt = 0$. Let x be a fixed element of Z. For $y \leqq f(x)$ and $y \geqq \psi_n(x)$ we have $F_n(x, y) = 0$; in the interval $\langle f(x), g(x)\rangle$ the function $F_n(x, y)$ of the variable y is non-decreasing and $F_n(x, g(x)) = \min(n\,\gamma(x), 1)$; in the interval $\langle g(x), \psi_n(x)\rangle$ $F_n(x, y)$ is non-increasing. Obviously

$$|f_n(z)| \leqq n,\ 0 \leqq F_n(z) \leqq 1\ (n = 1, 2, \ldots;\ z \in E_m)\,.$$

Finally, we define a function f_0 on E_m by means of the relations

$$f_0(x, y) = \frac{1}{\gamma(x)}\,,\ \text{if}\ x \in Z,\ f(x) < y < g(x),\ [x, y] \in A,$$

$$f_0(x, y) = 0\ \text{elsewhere.}$$

For $z \in A$ we evidently have $0 \leqq f_1(z) \leqq f_2(z) \leqq \ldots,\ f_n(z) \to f_0(z)$, whence

$$\int_A f_n(z)\, dz \to \int_A f_0(z)\, dz = \int_Z \frac{1}{\gamma(x)} \cdot \gamma(x)\, dx = \text{measure of}\ Z\,. \tag{58}$$

By (56) (lemma 30) and (57)

$$P_m(F_n) = \int_A f_n(z)\, dz \quad (n = 1, 2, \ldots)\,. \tag{59}$$

Similar reasonings show that for $x \in Z$, $f(x) < y \leqq g(x)$ we have

$$F_n(x, y) = \int_{f(x)}^{y} f_n(x, t)\, dt \to \int_{f(x)}^{y} f_0(x, t)\, dt = \frac{\mu(A_x^m \cap (f(x), y))}{\gamma(x)} = \Phi(x, y)\,.$$

Choose any $y > g(x)$ and put $\delta = \mu((g(x), y) - A_x^m)$. By assumption, $\delta > 0$; if $n > \frac{1}{\delta}$, then $\mu((g(x), \psi_n(x)) - A_x^m) = \frac{1}{n} < \delta$, whence $\psi_n(x) < y$, $F_n(x, y) = 0 = \Phi(x, y)$. We see that $F_n(z) \to \Phi(z)$ for all z; as $0 \leqq F_n \leqq 1$, it follows that $P_m(F_n) \to P_m(\Phi)$. By (59) and (58) we get $P_m(F_n) = \int_A f_n(z)\, dz \to$ measure of Z, which completes the proof.

32. Lemma. *Let A be a Borel subset of E_m, $A \in \mathfrak{A}_m$. For each $x \in E_{m-1}$ let G_x be the open subset of E_1, which was defined in lemma 28. (See also footnote[19])). Let $\varphi(x)$ be the number of components of G_x. (If G_x has infinitely many components, we put, of course, $\varphi(x) = \infty$.) Then φ is a Borel function and $\int_{E_{m-1}} \varphi(x)\, dx < \infty$.*

Proof. We define the functions $f_1, g_1, f_2, g_2, \ldots$ on the space E_{m-1} by (43) and put $Z_n = E[x; f_n(x) < \infty]$. We then choose a natural number n and write $f = f_n$, $g = g_n$, $Z = Z_n$ in lemma 31. (See lemma 28.) The corresponding function $\Phi = \Phi_n$ has the following properties: $\Phi_n(x, f_n(x)) = 0$, $\Phi_n(x, g_n(x)) = 1$, $\Phi_n(x, y)$ is a linear function of y for $f_n(x) \leqq y \leqq g_n(x)$, if $x \in Z_n$; $\Phi_n(z)$ vanishes for the remaining z. If c_n is the characteristic function of Z_n, then by the preceding lemma

$$P_m(\Phi_n) = \int_{E_{m-1}} c_n(x)\, dx.$$

Now put $\Psi(z) = \sum_{n=1}^{\infty} \Phi_n(z)$. Since for each z there is at most one non-zero member in the series, we have $0 \leqq \Psi \leqq 1$, and consequently

$$\sum_{n=1}^{\infty} P_m(\Phi_n) = P_m(\Psi) \leqq \|A\|_m .$$

Further we see that $x \in Z_n$, if and only if G_x has at least n components, whence $\varphi(x) = \sum_{n=1}^{\infty} c_n(x)$, $\int_{E_{m-1}} \varphi(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} \int_{E_{m-1}} c_n(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} P_m(\Phi_n) \leqq \|A\|_m$, which completes the proof.

33. Theorem. *Given a set $A \in \mathfrak{A}_m$, there exists a Borel subset K of E_{m-1} with the following properties:*

1) *$E_{m-1} - K$ has measure zero.*

2) *For each $x \in K$ there exist a (unique) non-negative integer r and real numbers*

$$a_1 < b_1 < \ldots < a_r < b_r \tag{60}$$

such that A_x^m is equivalent[13]) *to $\bigcup_{j=1}^{r} (a_j, b_j)$. If we put $r = \varphi(x)$, then φ is a Borel function on K and*

$$2 \int_{E_{m-1}} \varphi(x)\, \mathrm{d}x = \|A\|_m . \tag{61}$$

3) *Let A_- (resp. A_+, resp. A_0) be the set of all $z = [z_1, \ldots, z_m] \in E_m$ such that $x = [z_1, \ldots, z_{m-1}] \in K$ and $z_m \in \{a_1, \ldots, a_r\}$ (resp. $z_m \in \{b_1, \ldots, b_r\}$, resp. $z_m \in \bigcup_{j=1}^{r} (a_j, b_j)$), where a_j, b_j correspond to x. Then A_-, A_+, A_0 are Borel sets.*

Proof. First, let A be a Borel set and let $\varphi(x)$ be the number of components of G_x^A (see lemma 28, footnote [18])). By lemma 32, φ is a summable Borel function; consequently, $K_A = E[x; \varphi(x) < \infty]$ is a Borel set and $E_{m-1} - K_A$ has measure zero.

Let C be a bounded open convex set such that $\bar{A} \subset C$; put $B = C - A$. On account of (13b) (theorem 9) the set B also belongs to $\mathfrak{A}_m$. Let K_B be the set of all $x \in E_{m-1}$ such that G_x^B has only a finite number of components; put

$$K_0 = K_A \cap K_B\,, \quad G_x = G_x^A \cup G_x^B \quad (x \in E_{m-1})\,.$$

For each $x \in K_0$ we can write

$$G_x = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \cup \ldots \cup (\alpha_s, \beta_s)\,, \tag{62}$$

where $\alpha_1 < \beta_1 \leqq \alpha_2 < \beta_2 \leqq \ldots \leqq \alpha_s < \beta_s$ $(s \geqq 0)$, and put

$$p_j(x) = \alpha_j\,, \quad q_j(x) = \beta_j \quad \text{for} \quad j \leqq s\,,$$
$$p_j(x) = \infty\,, \quad q_j(x) = -\infty \quad \text{for} \quad j > s$$

$(x \in K_0)$. As $G_x^A \cap G_x^B = \emptyset$, we have $\langle a, b\rangle \subset G_x$ if and only if either $\langle a, b\rangle \subset G_x^A$ or $\langle a, b\rangle \subset G_x^B$. By lemma 28, $E[x; \langle a, b\rangle \subset G_x^A] = M_{a,b}^A$, $E[x; \langle a, b\rangle \subset G_x^B] =$

$= M^B_{a,b}$ are Borel subsets of E_{m-1}; it follows that $E[x; \langle a, b\rangle \subset G_x] = M^A_{a,b} \cup \cup M^B_{a,b}$ is a Borel set too ($a < b$). According to lemma 27, p_j, q_j are Borel functions.

If $A^m_x = \emptyset$, then $G^A_x = \emptyset$, $B^m_x = C^m_x$; since C is open and convex, we have $G_x = = G^B_x = C^m_x$ and the number s in (62) does not exceed 1. If $A^m_x \neq \emptyset$, put $\sigma = = \sup(\bar{A})^m_x$, $\iota = \inf(\bar{A})^m_x$, $C^m_x = (\gamma, \delta)$. As $\bar{A} \subset C$, we get $\gamma < \iota \leqq \sigma < \delta$; evidently $(\gamma, \iota) \cup (\sigma, \delta) \subset B^m_x$, whence $(\gamma, \iota) \cup (\sigma, \delta) \subset G^B_x \subset G_x$, $\alpha_1 \leqq \gamma$, $\delta \leqq \beta_s$. The obvious relations $G^A_x \subset \overline{A^m_x}$, $G^B \subset \overline{B^m_x}$, $A^m_x \cup B^m_x = C^m_x$ imply $G_x \subset \subset \langle\gamma, \delta\rangle$, whence $\gamma \leqq \alpha_1$, $\beta_s \leqq \delta$. We see that $\alpha_1 = \gamma$, $\beta_s = \delta$, $(\alpha_1, \beta_1) \cup \cup (\alpha_s, \beta_s) \subset G^B_x$ and that $s \geqq 1$.

Put $N_1 = E[x; -\infty < q_1(x) < p_2(x) < \infty]$. We shall prove that N_1 has measure zero. If $q_1(x) \leqq t < y \leqq p_2(x)$, then neither $(t, y) \subset G^A_x$ nor $(t, y) \subset G^B_x$; we have thus

$$\mu((t, y) - A^m_x) > 0 \tag{63}$$

and, since $(t, y) \subset C^m_x$,

$$\mu((t, y) \cap A^m_x) > 0\,. \tag{64}$$

We choose an arbitrary natural number n and define functions $\Phi_1, \ldots, \Phi_n$ on E_m as follows: We select firstly a point $x \in N_1$, an integer j ($1 \leqq j \leqq n$) and put

$$a = q_1(x) + (j-1)\,.\,\frac{p_2(x) - q_1(x)}{n+1}\,, \quad b = q_1(x) + j\,.\,\frac{p_2(x) - q_1(x)}{n+1}\,;$$

then for $a < y \leqq b$ put

$$\Phi_j(x, y) = \frac{\mu(A^m_x \cap (a, y))}{\mu(A^m_x \cap (a, b))}\,,$$

for other points $[x, y]$ write $\Phi_j(x, y) = 0$. According to (63), (64) and lemma 31, $P_m(\Phi_j)$ equals the measure of N_1; since $0 \leqq \sum_{j=1}^{n} \Phi_j \leqq 1$, we get $P_m(\sum_{j=1}^{n} \Phi_j) \leqq \|A\|_m$, whence

$$n\,.\,(\text{measure of } N_1) = \sum_{j=1}^{n} P_m(\Phi_j) \leqq \|A\|_m\,.$$

As n was an arbitrary natural number, the measure of N_1 is zero.

By similar reasoning we see that the measure of the set $N_2 = E[x; -\infty < < q_2(x) < p_3(x) < \infty]$ is zero; and so on. Put $K = K_0 - \bigcup_{j=1}^{\infty} N_j$. K is obviously a Borel set, the measure of its complement is zero and we have $\beta_1 = \alpha_2, \ldots \ldots, \beta_{s-1} = \alpha_s$ for each $x \in K$; evidently $(\alpha_1, \beta_1) \subset G^B_x$, $(\alpha_2, \beta_2) \subset G^A_x, \ldots, (\alpha_s, \beta_s) \subset G^B_x$. If $s > 0$, then s is odd, so that $r = \frac{s-1}{2}$ is an integer $\geqq 0$. If we put $r = 0$ for $s = 0$, we see that in both cases A^m_x is equivalent to $\bigcup_{j=1}^{r} (\alpha_{2j}, \beta_{2j})$ ($x \in K$) and in (60) we can write $a_j = \alpha_{2j}$, $b_j = \beta_{2j}$.

As p_j, q_j are Borel functions, A_+, A_-, A_0 are Borel sets. The function $F = c_+ - c_-$, where c_+ (resp. c_-) is the characteristic function of A_+ (resp. A_-), obviously fulfils the relation ${}^mF(x)$[20]) $= 2r = 2\varphi(x)$ for each $x \in K$. From (32) and 14., 4) we see that $2 \int_{E_{m-1}} \varphi(x)\,dx = \int_{E_{m-1}} {}^mF(x)\,dx = P_m(F) \leqq \|A\|_m$. Since (by theorem 20) $\|A\|_m \leqq 2 \int_{E_{m-1}} \varphi(x)\,dx$, we have $2 \int_{E_{m-1}} \varphi(x)\,dx = \|A\|_m$. The theorem is thus proved for Borel sets $A \in \mathfrak{A}_m$.

Now let A be an arbitrary set of $\mathfrak{A}_m$. There exists a Borel set $\hat{A}$ which is equivalent to A. Then $\|A\|_m = \|\hat{A}\|_m$ (see (13a)); the sets $\hat{A}^m_x$, A^m_x are equivalent for almost all $x \in E_{m-1}$. Hence there exists a Borel set N with measure zero such that the sets A^m_x, $\hat{A}^m_x$ are equivalent for all $x \in E_{m-1} - N$; moreover, we can find a set $\hat{K}$ which fulfils the conditions of our theorem, if we write $\hat{A}$ in place of A. Putting $K = \hat{K} - N$, we see that the proof is complete.

34. Theorem. *If $A \in \mathfrak{A}$, there exists a Borel set $B \subset D$, which has measure zero and fulfils the condition $p(B) = p(D)$ (D is the boundary and p is the surface measure of A).*

Proof. Let K, A_+, A_- be the sets from theorem 33. Then $B_m = A_+ \cup A_-$ is obviously a Borel set of measure zero. If f is a bounded Borel function on D such that $f(z) = 0$ for all $z \in B_m$, ve have ${}^mf(x)$[20]) $= 0$ for each $x \in K$, whence $P_m(f) = \int_{E_{m-1}} {}^mf(x)\,dx = 0$. If we analogously define the sets $B_1, \ldots, B_{m-1}$, we see that we can put $B = \bigcup_{j=1}^{m} B_j$.

35. Theorem. *Let A, B be bounded measurable sets. Then*

$$\max(\|A \cup B\|, \|A \cap B\|, \|A - B\|) \leqq \|A\| + \|B\| . \tag{65}$$

Proof. This relation holds of course, if $\|A\| + \|B\| = \infty$. We may therefore suppose that $A, B \in \mathfrak{A}$. We shall prove only that

$$\|A \cap B\| \leqq \|A\| + \|B\| ; \tag{66}$$

the proof for $\|A \cup B\|$ and $\|A - B\|$ is similar. Let K, A_-, A_0, A_+ (resp. r, a_j, b_j) be the sets (resp. numbers) from theorem 33; in an analogous manner, taking only B instead of A, we form sets L, B_-, B_0, B_+ and numbers s, c_j, d_j. Put $C = A \cap B$, $M = K \cap L$; we can obviously suppose that $K = L = M$. For each $x \in M$ the set C^m_x is equivalent to a set $\bigcup_{j=1}^{t} (\alpha_j, \beta_j)$, where $\alpha_1 < \beta_1 < \ldots < \alpha_t < \beta_t$ (t integer $\geqq 0$). Now we define in an evident way the sets C_+, C_-; we get

$$C_+ = (A_+ \cap (B_+ \cup B_0)) \cup (B_+ \cap (A_+ \cup A_0)) ,$$
$$C_- = (A_- \cap (B_- \cup B_0)) \cup (B_- \cap (A_- \cup A_0)) ,$$

---

[20]) See (31).

so that C_+, C_- are Borel sets and

$$C_+ \subset A_+ \cup B_+ , \quad C_- \subset A_- \cup B_- ,$$
$$C_+ \cap A_- = C_- \cap A_+ = C_+ \cap B_- = C_- \cap B_+ = \emptyset .$$

The set $(A_+)^m_x$ (resp. $(B_+)^m_x$, resp. $(C_+)^m_x$) has r (resp. s, resp. t) elements; from $(C_+)^m_x \subset (A_+)^m_x \cup (B_+)^m_x$ we see that $t \leqq r + s$. By theorem 33, the functions $\varphi(x) = r$, $\psi(x) = s$ are summable; on account of theorem 20 we get $C \in \mathfrak{A}_m$. By similar reasonings, $C \in \mathfrak{A}_1, \ldots, C \in \mathfrak{A}_{m-1}$, whence $C \in \mathfrak{A}$.

Let $\nu = [\nu_1, \ldots, \nu_m]$ be the normal vector of the set C. Put $A_D = A_+ \cup A_-$, $B_D = B_+ \cup B_-$, $C_D = C_+ \cup C_-$ and define the functions f, g on E_m as follows: $f(z) = \nu_m(z)$ for $z \in (A_D - B_D) \cap C_D$, $f(z) = 0$ elsewhere; $g(z) = \nu_m(z)$ for $z \in B_D \cap C_D$, $g(z) = 0$ elsewhere. For $z \in C_D$ we have either $z \in A_D - B_D$, $f(z) = \nu_m(z)$, $g(z) = 0$, or $z \in B_D$, $f(z) = 0$, $g(z) = \nu_m(z)$; in both cases

$$\nu_m(z) = f(z) + g(z) . \tag{67}$$

If $z \in A_+ - C_+ = A_+ - C_D$ or $z \in C_+ - A_+ = C_+ - A_D$, then $f(z) = 0$. For each $x \in E_{m-1}$ we have therefore

$$\sum_{y \in (A_+)^m_x} f(x, y) = \sum_{y \in (C_+)^m_x} f(x, y) ,$$

i. e.

$$\sum_{j=1}^{r} f(x, b_j) = \sum_{j=1}^{t} f(x, \beta_j) .$$

By similar reasoning,

$$\sum_{j=1}^{s} g(x, d_j) = \sum_{j=1}^{t} g(x, \beta_j) ,$$

whence (see (67))

$$\sum_{j=1}^{r} f(x, b_j) + \sum_{j=1}^{s} g(x, d_j) = \sum_{j=1}^{t} (f(x, \beta_j) + g(x, \beta_j)) = \sum_{j=1}^{t} \nu_m(x, \beta_j) ;$$

analogously

$$\sum_{j=1}^{r} f(x, a_j) + \sum_{j=1}^{s} g(x, c_j) = \sum_{j=1}^{t} \nu_m(x, \alpha_j) .$$

We have therefore

$$\sum_{j=1}^{r} (f(x, b_j) - f(x, a_j)) + \sum_{j=1}^{s} (g(x, d_j) - g(x, c_j)) = \sum_{j=1}^{t} (\nu_m(x, \beta_j) - \nu_m(x, \alpha_j)) . \tag{68}$$

Now we write $f = v_m$, $g = w_m$ and define in an obvious way the functions $v_1, \ldots, v_{m-1}, w_1, \ldots, w_{m-1}$. Thus we have defined two Borel vectors $v = [v_1, \ldots, v_m]$, $w = [w_1, \ldots, w_m]$. Since either $v_i(z) = 0$ or $v_i(z) = \nu_i(z)$, we see that $|v| \leqq |\nu| \leqq 1$, whence $P(A, v) \leqq \|A\|$; analogously $P(B, w) \leqq \|B\|$. According to (68), theorem 20 implies

$$P_m(A, v_m) + P_m(B, w_m) = P_m(A \cap B, \nu_m) .$$

Similar equalities hold for $1, \ldots, m-1$; it follows that

$$\|A \cap B\| = P(A \cap B, \nu) = P(A, v) + P(B, w) \leqq \|A\| + \|B\|,$$

which completes the proof.

36. Lemma. *Suppose that $A, B_1, \ldots, B_n \in \mathfrak{A}$ and that $B_i \cap B_j = \emptyset$ for $1 \leqq i < j \leqq n$. Then*

$$\sum_{j=1}^{n} \|A \cap B_j\| \leqq \|A\| \sqrt{m} + \sum_{j=1}^{n} \|B_j\|.$$

Proof. By theorem 35, $A \cap B_j \in \mathfrak{A}$ for $j = 1, \ldots, n$; let ν^j be the normal vector of $A \cap B_j$. Given an index j $(1 \leqq j \leqq n)$, we can attach vectors v^j, w^j to the sets A, B_j in the same way as we attached vectors v, w to the sets A, B in the proof of theorem 35. We consider the m-th component v_m of the vector $v = \sum_{j=1}^{n} v^j$ and the sets $A_0, (B_1)_0, \ldots, (B_n)_0$, defined as in theorem 33, 3). If $v_m(x, y) \neq 0$, then y is a boundary point of $(A_0)_x^m$ and an interior point of some — naturally exactly one — set $((B_j)_0)_x^m$, so that $v_m(x, y) = v_m^j(x, y) = \nu_m^j(x, y)$. It follows that $|v_m(x, y)| \leqq 1$; by similar reasonings, $|v_j(x, y)| \leqq 1$ for $j = 1, \ldots, m-1$, whence $|v| \leqq \sqrt{m}$. The rest of the proof is the same as in the preceding case.

Remark. If the sets B_j are open and if $v_m(z) = v_m^j(z) \neq 0$, then for other indices i we have either $v_i(z) = 0$ or $v_i(z) = \nu_i^j(z)$, whence $|v(z)| \leqq 1$ and consequently

$$\sum_{j=1}^{n} \|A \cap B_j\| \leqq \|A\| + \sum_{j=1}^{n} \|B_j\|. \tag{69}$$

The same relation holds of course, if e. g. the boundaries of B_j have measure zero.

37. Theorem. *Let $A_1, A_2, \ldots$ be measurable subsets of E_m and let the set $A = \bigcup_{n=1}^{\infty} A_n$ be bounded. Then*

$$\|A\| \leqq \sum_{n=1}^{\infty} \|A_n\|; \tag{70}$$

if, moreover, $A_1 \subset A_2 \subset \ldots$, then

$$\|A\| \leqq \liminf_{n \to \infty} \|A_n\|. \tag{71}$$

Proof. First suppose that $A_1 \subset A_2 \subset \ldots$ If $v \in \mathfrak{V}_A$,[21]) we have $\|v\|_{A_n} \leqq 1$ and therefore $\int_{A_n} \operatorname{div} v(x)\, dx \leqq \|A_n\|$ for $n = 1, 2, \ldots$ Making $n \to \infty$, we obtain $\int_A \operatorname{div} v(x)\, dx \leqq \liminf_{n\to\infty} \|A_n\|$, whence (71) easily follows.

---

[21]) See definition 2.

Returning to the general case put $B_n = A_1 \cup \ldots \cup A_n$. By theorem 35 (and by induction), $\|B_n\| \leqq \sum_{j=1}^{n} \|A_j\|$, whence, according to (71), $\|A\| = \|\bigcup_{n=1}^{\infty} B_n\| \leqq \leqq \liminf_{n\to\infty} \|B_n\| \leqq \sum_{j=1}^{\infty} \|A_j\|$.

38. Definition. Suppose that the sets $A, A_1, A_2, \ldots$ fulfil the following conditions:

1) $A \in \mathfrak{A}$;

2) $A_n \in \mathfrak{A}$, $\bar{A}_n \subset A^0$ for $n = 1, 2, \ldots$;

3) the relation $P(A_n, v) \to P(A, v)$ holds for each continuous vector v on $\bar{A}$.

Then we write $A_n \overset{+}{\to} A$.

39. Theorem. *If $A_n \overset{+}{\to} A$ and if the Lebesgue integral $\int_A f(x)\,\mathrm{d}x$ (finite or infinite) exists, then*

$$\lim_{n\to\infty} \int_{A_n} f(x)\,\mathrm{d}x = \int_A f(x)\,\mathrm{d}x\,. \tag{72}$$

Proof. If we put $v(x) = [0, \ldots, 0, x_m]$ for each $x = [x_1, \ldots, x_m] \in E_m$, then the relation $P(B, v) =$ measure of B holds for every set $B \in \mathfrak{A}$. According to 38., 3) we thus obtain measure of $A_n \to$ measure of A, from which the assertion easily follows.

40. Theorem. *Suppose that $A_n \overset{+}{\to} A$. Let the vector v be continuous on $\bar{A}$; let v and the function f be associated on A^0. Then*

$$P(A, v) = \lim_{n\to\infty} \int_{A_n} f(x)\,\mathrm{d}x\,; \tag{73}$$

if, moreover, the Lebesgue integral $\int_A f(x)\,\mathrm{d}x$ exists, we have

$$P(A, v) = \int_A f(x)\,\mathrm{d}x\,. \tag{74}$$

Proof. In virtue of theorem 24 we have $P(A_n, v) = \int_{A_n} f(x)\,\mathrm{d}x$ for $n = 1, 2, \ldots$, whence (73) follows at once. The relation (74) is an immediate consequence of (73) and (72).

41. Definition. If $\emptyset \neq D \subset E_m$ and if $\varepsilon > 0$, let $\Omega(D, \varepsilon)$ be the set of all points $x \in E_m$, whose distance from D is less than ε; if $D = \emptyset$, put $\Omega(D, \varepsilon) = \emptyset$. Let $\mathfrak{N}$ be the system of all bounded sets $A \subset E_m$ which have the following property: If D is the boundary of A, then the function $\dfrac{\text{measure of } \Omega(D, \varepsilon)}{\varepsilon}$ of the variable ε is bounded in $(0, 1)$. The boundary of each set $A \in \mathfrak{N}$ obviously has measure 0.

42. Theorem. $\mathfrak{N} \subset \mathfrak{A}$.

Proof. Let D be the boundary of the set $A \in \mathfrak{N}$. For each $x \in E_{m-1}$ put $f_n(x) = \mu((G_n)_x^m)$, where $G_n = \Omega\left(D, \frac{1}{n}\right)$; let $\varphi(x)$ denote the number of components of $(\bar{A})_x^m$. We shall prove that the relation

$$2\varphi(x) \leqq \liminf_{n\to\infty} n \,.\, f_n(x) \tag{75}$$

holds for each $x \in E_{m-1}$. This is obvious if $\varphi(x) = 0$. If now $\varphi(x) > 0$ (this includes the case $\varphi(x) = \infty$), choose a natural number $n_0 \leqq \varphi(x)$. There exist numbers $a_1 < a_2 < \ldots < a_{n_0}$ such that the points $[x, a_j]$ belong to the boundary of $\bar{A}$ and so to D also. If $\frac{1}{n} < \frac{a_{k+1} - a_k}{2}$ $(k = 1, \ldots, n_0 - 1)$, then no two of the intervals $\left(a_j - \frac{1}{n}, a_j + \frac{1}{n}\right)$ have common points and $(G_n)_x^m \supset \bigcup_{j=1}^{n_0} \left(a_j - \frac{1}{n}, a_j + \frac{1}{n}\right)$; hence $f_n(x) \geqq n_0 \,.\, \frac{2}{n}$, $n \,.\, f_n(x) \geqq 2n_0$, from which (75) follows at once.

Since $A \in \mathfrak{N}$, there exists a finite constant C such that n . (measure of G_n) $< C$ and therefore $\int_{E_{m-1}} n \,.\, f_n(x)\, \mathrm{d}x < C$ holds for each n, whence

$$\int_{E_{m-1}} \liminf n \,.\, f_n(x)\, \mathrm{d}x \leqq C\,. \tag{76}$$

On account of (75), (76) and of theorem 20 we get $\bar{A} \in \mathfrak{A}_m$; obviously also $\bar{A} \in \mathfrak{A}_1, \ldots, \bar{A} \in \mathfrak{A}_m$, so that $\bar{A} \in \mathfrak{A}$. The relation $A \in \mathfrak{A}$ follows from the fact that D has measure zero.

43. Theorem. *Given a set $A \in \mathfrak{N}$, there exist A_n such that $A_n \overset{+}{\to} A$.*

Proof. Let h be a fixed positive number, $h \leqq 1$. Let $\mathfrak{K}$ be the system of all cubes $\langle n_1 h, (n_1 + 1)\, h\rangle \times \ldots \times \langle n_m h, (n_m + 1)\, h\rangle$, where $n_1, \ldots, n_m$ are integers; let $\mathfrak{K}_0$ (resp. $\mathfrak{K}_D$) be the system of all $K \in \mathfrak{K}$ such that $K \subset A^0$ (resp. $K \cap D \neq \emptyset$, where D is the boundary of A). If we put $L = \bigcup \mathfrak{K}_0$, $M = \bigcup \mathfrak{K}_D$, we evidently have $A \subset L \cup M$. Write $\mathfrak{K}_D = \{K_1, \ldots, K_s\}$. Since $K_i^0 \cap K_j^0 = \emptyset$ $(i \neq j)$ and $\bigcup_{j=1}^{s} K_j^0 \subset \Omega(D, h\sqrt{m})$, we get

$$s \,.\, h^m \leqq \text{measure of } \Omega(D, h\sqrt{m})\,.$$

On account of the fact that A belongs to $\mathfrak{N}$, there exists a finite constant C such that measure of $\Omega(D, \delta) \leqq C\delta$ for each $\delta \in (0, \sqrt{m})$, whence

$$s \,.\, h^m \leqq Ch\sqrt{m}\,.$$

Since $\|K_j\| = 2mh^{m-1}$ $(j = 1, \ldots, s)$, we obtain

$$\sum_{j=1}^{s} \|K_j\| = 2msh^{m-1} \leqq 2Cm^{\frac{3}{2}} = C_1\,. \tag{77}$$

It follows from 14., remark 2, that the relation

$$P(A, v) = P(L, v) + \sum_{j=1}^{s} P(A \cap K_j, v) \tag{78}$$

holds for every continuous vector v on $\bar{A}$; by (69) (remark to lemma 36) and (77) we get

$$\sum_{j=1}^{s} \|A \cap K_j\| \leqq \|A\| + \sum_{j=1}^{s} \|K_j\| \leqq \|A\| + C_1 . \tag{79}$$

If $B \in \mathfrak{A}$ and if v is a constant vector, then evidently $P(B, v) = 0$. Therefore, if v is a continuous vector on the boundary D_B of B and if $|v(x) - v(y)| \leqq \eta$ for arbitrary points $x, y \in D_B$, the relation $|P(B, v)| \leqq \|B\| \eta$ is valid. Now let v be a continuous vector on $\bar{A}$. If we put $\omega = \sup |v(x) - v(y)|$, where $x, y \in \bar{A}$, $|x - y| \leq h\sqrt{m}$, we get $|P(A \cap K_j, v)| \leqq \omega \|A \cap K_j\|$ for $j = 1, \ldots, s$. From (78) and (79) then follows the relation

$$|P(A, v) - P(L, v)| \leqq \sum_{j=1}^{s} |P(A \cap K_j, v)| \leqq \omega \sum_{j=1}^{s} \|A \cap K_j\| \leqq \omega(\|A\| + C_1) .$$

Since v is uniformly continuous, we have $\omega \to 0$ for $h \to 0+$. If we put $h = \dfrac{1}{n}$ and write $A_n = L$ $(n = 1, 2, \ldots)$, we see that $P(A_n, v) \to P(A, v)$ and so $A_n \underset{\rightarrow}{+} A$.

Remark. Let v be a continuous vector on $\bar{A}$, where $A \in \mathfrak{N}$; let v and the function f be associated on A^0. Theorems 39, 40, 42 and 43 show that $P(A, v)$ is then, in a certain sense, a "mean value" of the integral $\int_A f(x) \, dx$.

44. Definition. If $v^1, \ldots, v^{m-1}$ are vectors (on a given set), then the *outer product* of $v^1, \ldots, v^{m-1}$ is such a vector w, that for each v the scalar product $v \, . \, w$ is equal to the determinant with rows $v^1, \ldots, v^{m-1}, v$. (For instance, the m-th component of w is the determinant

$$\begin{vmatrix} v_1^1, & \ldots, & v_{m-1}^1 \\ \ldots & \ldots & \ldots \\ v_1^{m-1}, & \ldots, & v_{m-1}^{m-1}, \end{vmatrix},$$

and so on.)

We say that a vector v on an open set $G \subset E_m$ is *solenoidal*, if there exist functions $\varphi_1, \ldots, \varphi_{m-1}$ of the class C_1 on G such that v is the outer product of the vectors $\operatorname{grad} \varphi_1, \ldots, \operatorname{grad} \varphi_{m-1}$.

45. Lemma. *Let $\varphi_1, \ldots, \varphi_{m-1}$ be functions of the class C_2 on an open set $G \subset E_m$; let v be the outer product of the vectors* $\operatorname{grad} \varphi_1, \ldots, \operatorname{grad} \varphi_{m-1}$. *Then* $\operatorname{div} v(x) = 0$ *for each $x \in G$.*

Proof. Let M be the matrix with rows $\operatorname{grad} \varphi_1, \ldots, \operatorname{grad} \varphi_{m-1}$; omitting the r-th column of M, we get a matrix which we call M_r. If T_r is the determinant of

M_r, we have $\frac{\partial T_r}{\partial x_r} = T_{r1} + \ldots + T_{r,r-1} + T_{r,r+1} + \ldots + T_{rm}$, where T_{r1} is the determinant

$$\begin{vmatrix} \frac{\partial^2 \varphi_1}{\partial x_1 \, \partial x_r}, & \frac{\partial \varphi_1}{\partial x_2}, & \ldots, & \frac{\partial \varphi_1}{\partial x_{r-1}}, & \frac{\partial \varphi_1}{\partial x_{r+1}}, & \ldots, & \frac{\partial \varphi_1}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 \varphi_{m-1}}{\partial x_1 \, \partial x_r}, & \frac{\partial \varphi_{m-1}}{\partial x_2}, & \ldots, & \frac{\partial \varphi_{m-1}}{\partial x_{r-1}}, & \frac{\partial \varphi_{m-1}}{\partial x_{r+1}}, & \ldots, & \frac{\partial \varphi_{m-1}}{\partial x_m} \end{vmatrix}$$

and so on. If $r > 1$, we have, for instance,

$$T_{1r} = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_2}, & \ldots, & \frac{\partial \varphi_1}{\partial x_{r-1}}, & \frac{\partial^2 \varphi_1}{\partial x_r \partial x_1}, & \frac{\partial \varphi_1}{\partial x_{r+1}}, & \ldots, & \frac{\partial \varphi_1}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_{m-1}}{\partial x_2}, & \ldots, & \frac{\partial \varphi_{m-1}}{\partial x_{r-1}}, & \frac{\partial^2 \varphi_{m-1}}{\partial x_r \, \partial x_1}, & \frac{\partial \varphi_{m-1}}{\partial x_{r+1}}, & \ldots, & \frac{\partial \varphi_{m-1}}{\partial x_m} \end{vmatrix},$$

so that $T_{r1} = (-1)^{r-2} \, . \, T_{1r}$. Similarly it can be shown that $T_{rs} = (-1)^{r-s-1} \, . \, . \, T_{sr}$ for arbitrary indices r, s, where $r > s$.

If we put $T_{rr} = 0$ ($r = 1, \ldots, m$), we obtain, finally,

$$\operatorname{div} v = \sum_{i=1}^{m} (-1)^{m+i} \frac{\partial T_i}{\partial x_i} = \sum_{i=1}^{m} (-1)^{m+i} \sum_{j=1}^{m} T_{ij} =$$

$$\sum_{i<j} (-1)^{m+i} \, . \, T_{ij} + \sum_{i>j} (-1)^{m+i} \, . \, T_{ij} = \sum_{i<j} (-1)^{m+i} \, . \, T_{ij} + \sum_{i<j} (-1)^{m+j} \, T_{ji} =$$

$$= \sum_{i<j} [(-1)^{m+i} + (-1)^{m+j} \, . \, (-1)^{j-i-1}] \, T_{ij} = 0 \, ,$$

since $[\ldots] = 0$ for all i, j.

46. Lemma. *Suppose that $A \in \mathfrak{A}$ and that f is a function of class C_1 on G, where G is open, $G \supset \bar{A}$. Let $\varphi_1, \ldots, \varphi_{m-1}$ be functions of class C_2 on G and let v be the outer product of the vectors* grad $\varphi_1, \ldots,$ grad φ_{m-1}. *Then*

$$P(A, fv) = \int_A v(x) \, . \, \operatorname{grad} f(x) \, \mathrm{d}x \, . \tag{80}$$

Proof. By lemma 45 we have $\operatorname{div} v = 0$; our assertion therefore follows immediately from the obvious relation $\operatorname{div}(fv) = f \operatorname{div} v + v \, . \operatorname{grad} f$.

47. Lemma. *Let f be a function of class C_1 on the open set $G \subset E_m$; let K be compact, $K \subset G$. Then there exist functions $f_1, f_2, \ldots$ of class C_∞ on E_m such that*

$$\lim_{n \to \infty} f_n(x) = f(x) \, , \quad \lim_{n \to \infty} \frac{\partial f_n(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} \quad (i = 1, \ldots, m)$$

uniformly on K.

Proof. There exists a function g of class C_1 on E_m which coincides with f in some neighbourhood of K (and vanishes outside a bounded set). Put

$$f_n(x) = \int_{E_m} g(t) \, \psi_n(x - t) \, \mathrm{d}t \quad (n = 1, 2, \ldots) \, ,$$

where ψ_n are the functions from notation 21. Since the relations

$$\frac{\partial f_n(x)}{\partial x_1} = \int_{E_m} g(t) \, \frac{\partial \psi_n(x-t)}{\partial x_1} \, \mathrm{d}t = \int_{E_m} \frac{\partial g(x-t)}{\partial x_1} \, \psi_n(t) \, \mathrm{d}t \,,$$

$$\frac{\partial^2 f_n(x)}{\partial x_1^2} = \int_{E_m} g(t) \, \frac{\partial^2 \psi_n(x-t)}{\partial x_1^2} \, \mathrm{d}t \,, \quad \text{etc.}$$

hold for $n = 1, 2, \ldots$ and for all $x \in E_m$, we see that the sequence $f_1, f_2, \ldots$ has the required properties.

48. Theorem. *Let f be a function of class C_1 on the open set $G \subset E_m$; let v be a solenoïdal vector on G. Then*

$$P(A, fv) = \int_A v(x) \,.\, \operatorname{grad} f(x) \, \mathrm{d}x \tag{81}$$

for each set $A \in \mathfrak{A}$, where $\bar{A} \subset G$.

Proof. Let v be the outer product of the vectors $\operatorname{grad} \varphi_1, \ldots, \operatorname{grad} \varphi_{m-1}$, where φ_j are functions of class C_1 on G. If $A \in \mathfrak{A}$, $\bar{A} \subset G$, there exist (in virtue of lemma 47) functions $\varphi_1^{(n)}, \ldots, \varphi_{m-1}^{(n)}$ $(n = 1, 2, \ldots)$ of class C_∞ on E_m whose derivatives of the first order converge uniformly on the set $\bar{A}$ to the corresponding derivatives of $\varphi_1, \ldots, \varphi_{m-1}$. The components of the vectors v^n, where v^n is the outer product of $\operatorname{grad} \varphi_1^{(n)}, \ldots, \operatorname{grad} \varphi_{m-1}^{(n)}$, therefore converge uniformly on $\bar{A}$ to the components of v. By lemma 46 we have $P(A, fv^n) = \int_A v^n(x) \,.\, \operatorname{grad} f(x) \,.\, \mathrm{d}x$ for $n = 1, 2, \ldots$; making $n \to \infty$, we obtain (81).

49. Definitions. If M is a matrix with elements a_{ik}, let M' be the matrix with elements $b_{ik} = a_{ki}$. If we consider vectors as columns, the scalar product of the vectors v, w can be written as the matrix product $v' \,.\, w = w' \,.\, v$.

If M is a square matrix, let $\operatorname{adj} M$ be the matrix with elements b_{ik}, where b_{ik} is the algebraic complement of a_{ki} in the matrix M. We have thus $M \,.\, \operatorname{adj} M = J \,.\, \det M$, where J is the unit matrix and $\det M$ is the determinant of M.

By the *norm* of the matrix M with elements a_{ik} we understand the quantity $|M| = \sqrt{\sum_{i,k} a_{ik}^2}$. If the product $M \,.\, N$ has a meaning, we have

$$|M \,.\, N| \leqq |M| \,.\, |N| \,. \tag{82}$$

If N is a vector, $|N|$ coincides with the usual norm.

Let φ be a mapping of the open set $G \subset E_m$ into E_m. We say that φ is *regular*, if $\varphi(x) = [\varphi_1(x), \ldots, \varphi_m(x)]$, where φ_j are functions of class C_1 (on the set G), and if the functional determinant of φ is distinct from zero in all points $x \in G$. By the *functional matrix* of φ we understand the matrix with rows $\operatorname{grad} \varphi_1, \ldots, \operatorname{grad} \varphi_m$.

50. Theorem. *Let φ be a one-to-one regular mapping of the open set $G \subset E_m$ into E_m; let M be the functional matrix of φ. Suppose that $A \in \mathfrak{A}$, $\bar{A} \subset G$. Then $\varphi(A) \in \mathfrak{A}$ and for every bounded Borel vector w on the boundary of $\varphi(A)$ we have*

$$P(\varphi(A), w) = P(A, v)\,, \tag{83}$$

where

$$v(x) = \operatorname{adj} M(x) \,.\, w(\varphi(x)) \,.\, \operatorname{sgn} \det M(x)\,. \tag{84}$$

Proof. We shall suppose firstly that $w = [w_1, \ldots, w_m] \in \mathfrak{B}_{\varphi(A)}$.[21] Let $s^1, \ldots, s^m$ be the columns of the matrix adj M . sgn det M. The vectors s^i are solenoidal; for example, s^1 is the outer product of the vectors grad $((-1)^{m+1} \,.\, \operatorname{sgn} \det M \,.\, \varphi_2)$, grad $\varphi_3, \ldots,$ grad φ_m (φ_i are the components of φ; since $\det M(x) \neq 0$ for every $x \in G$, sgn det M is constant in some neighbourhood of each point $x \in G$). Further, $v = \sum_{i=1}^{m} s^i f_i$, where $f_i(x) = w_i(\varphi(x))$. Theorem 48 gives $P(A, v) = \sum_{i=1}^{m} P(A, f_i s^i) = \sum_{i=1}^{m} \int_A s^i(x) \,.\, \operatorname{grad} f_i(x)\, \mathrm{d}x$. Since $(\operatorname{grad} f_i(x))' = (\operatorname{grad} w_i(y))' \,.\, M(x)$, where $y = \varphi(x)$, we have

$$s^i(x) \,.\, \operatorname{grad} f_i(x) = (\operatorname{grad} f_i(x))' \,.\, s^i(x) = (\operatorname{grad} w_i(y))' \,.\, M(x) \,.\, s^i(x)\,.$$

The components of s^i are algebraic complements of the elements from the i-th row of M, multiplied by sgn det M. Consequently, $M \,.\, s^i$ is a column with elements $0, \ldots, 0, |\det M|, 0, \ldots, 0$. We thus have

$$s^i(x) \,.\, \operatorname{grad} f_i(x) = \frac{\partial w_i(y)}{\partial y_i} \,.\, |\det M(x)|\,,$$

$$\sum_{i=1}^{m} s^i(x) \,.\, \operatorname{grad} f_i(x) = \operatorname{div} w(y) \,.\, |\det M(x)| \quad (\text{where } y = \varphi(x))\,,$$

so that

$$P(A, v) = \int_A \Big(\sum_{i=1}^{m} s^i(x) \,.\, \operatorname{grad} f_i(x)\Big)\, \mathrm{d}x =$$

$$= \int_A \operatorname{div} w(\varphi(x)) \,.\, |\det M(x)|\, \mathrm{d}x = \int_{\varphi(A)} \operatorname{div} w(y)\, \mathrm{d}y \tag{85}$$

(see, for instance, [2], p. 219, theorem 103). There exists a finite constant Ω such that $|\operatorname{adj} M(x)| < \Omega$ for each $x \in A$. As $|w(\varphi(x))| \leqq 1$, we get (see (82)) $|v(x)| \leqq |\operatorname{adj} M(x)| \,.\, |w(\varphi(x))| \leqq \Omega$, consequently $\int_{\varphi(A)} \operatorname{div} w(y)\, \mathrm{d}y = P(A, v) \leqq \|A\| \,.\, \Omega$, which implies $\|\varphi(A)\| \leqq \|A\| \,.\, \Omega$ and so $\varphi(A) \in \mathfrak{A}$.

From (85) we see that (83) holds for each vector whose components are polynomials. The completion of the proof is simple.

Remark. Since $\operatorname{sgn} \det M(x) \,.\, \operatorname{adj} M(x) = |\det M(x)| \,.\, M^{-1}(x)$ (where M^{-1} is the inverse matrix), the relation (84) can also be written as $v(x) = |\det M(x)| \,.\, M^{-1}(x) \,.\, w(y)$ or

$$w(y) = |\det M(x)|^{-1} \,.\, M(x) \,.\, v(x)\,. \tag{86}$$

Thus, if we "transform" the set A by means of the mapping φ, we must "transform" the vector v with the help of the formula (86).

51. Theorem. *If φ is a regular mapping of the open set $G \subset E_m$, then $\varphi(A) \in \mathfrak{A}$ for each $A \in \mathfrak{A}$, where $\bar{A} \subset G$.*

Proof. To each $x \in G$ there exists a cube $K = K(x)$ ($x \in K^0$) such that φ is a one-to-one mapping in some neighbourhood of K. If $A \in \mathfrak{A}$, $\bar{A} \subset G$, then there exist points $x_1, \ldots, x_q$ such that $A \subset \bigcup_{j=1}^{q} K(x_j)$. From theorems 35 and 50 we deduce that $A \cap K(x_j) \in \mathfrak{A}$, $\varphi(A \cap K(x_j)) \in \mathfrak{A}$ for $j = 1, \ldots, q$, and, finally, $\varphi(A) = \bigcup_{j=1}^{q} \varphi(A \cap K(x_j)) \in \mathfrak{A}$.

52. Theorem. *Let φ be a one-to-one regular mapping of the open set $G \subset E_m$ (into E_m). Put $N = (\operatorname{sgn} \det M \,.\, \operatorname{adj} M)'$ (so that $N' = |\det M| \,.\, M^{-1}$), where M is the functional matrix of φ. Suppose that $A \in \mathfrak{A}$, $\bar{A} \subset G$; let D (resp. p, resp. ν) be the boundary (resp. the surface measure, resp. the normal vector) of A. Put $\pi = N \,.\, \nu$, $\lambda(y) = \pi(x)$ (where $y = \varphi(x)$), $\hat{\nu} = \dfrac{\lambda}{|\lambda|}$ and*

$$\hat{p}(B) = \int_{\varphi^{-1}(B)} |\pi| \, \mathrm{d}p \tag{87}$$

for each Borel subset B of $\varphi(D)$. Then $\hat{p}$ (resp. $\hat{\nu}$) is the surface measure (resp. the normal vector) of $\varphi(A)$.

Proof. It follows easily from (87) that the relation

$$\int_{\varphi(D)} \hat{f} \, \mathrm{d}\hat{p} = \int_D f \, \mathrm{d}p \,, \quad \text{where} \quad f(x) = \hat{f}(\varphi(x)) \,.\, |\pi(x)| \,, \tag{88}$$

holds good for each bounded Borel function $\hat{f}$ on $\varphi(D)$. Let w be a bounded Borel vector on $\varphi(D)$ and let v be defined by (86). As $\hat{\nu}(y) = \dfrac{\pi(x)}{|\pi(x)|} = \dfrac{N(x) \,.\, \nu(x)}{|\pi(x)|}$, we have $w(y) \,.\, \hat{\nu}(y) = (\hat{\nu}(y))' \,.\, w(y) = \dfrac{(\nu(x))' . (N(x))' . M(x) . |\det M(x)|^{-1} . v(x)}{|\pi(x)|} = \dfrac{\nu(x) \,.\, v(x)}{|\pi(x)|}$, and consequently, on account of (88) and (83),

$$\int_{\varphi(D)} w \,.\, \hat{\nu} \, \mathrm{d}\hat{p} = \int_D v \,.\, \nu \, \mathrm{d}p = P(A, v) = P(\varphi(A), w) \,.$$

By theorem 17, $\hat{\nu}$ and $\hat{p}$ are the normal vector and the surface measure of $\varphi(A)$.

53. Theorem. *Let φ be a one-to-one regular mapping of the open set $G \subset E_m$. Let the vector v and the function f be associated on G; let M be the functional matrix of φ. For each $y \in \hat{G} = \varphi(G)$ put*

$$\hat{f}(y) = \frac{f(x)}{|\det M(x)|} \,, \quad \hat{v}(y) = \frac{M(x) \,.\, v(x)}{|\det M(x)|} \quad (y = \varphi(x)) \,.$$

Then $\hat{v}$ and $\hat{f}$ are associated on $\hat{G}$.

Proof. Let $\hat{K}$ be a cube, $\hat{K} \subset \hat{G}$; put $K = \varphi^{-1}(\hat{K})$. According to theorem 50 we have $K \in \mathfrak{A}$ and it follows from theorem 24 that

$$P(K, v) = \int_K f(x)\, dx\,. \tag{89}$$

The functional determinant T of the mapping $\psi = \varphi^{-1}$ fulfils the relation $T(\varphi(x))\,.\,\det M(x) = 1$, so that

$$\int_K f(x)\, dx = \int_{\hat{K}} f(\psi(y))\,.\,|T(y)|\, dy = \int_{\hat{K}} \hat{f}(y)\, dy\,. \tag{90}$$

From theorem 50 (and relation (86)) we see that $P(K, v) = P(\hat{K}, \hat{v})$; relations (89), (90) show therefore that $P(\hat{K}, \hat{v}) = \int_{\hat{K}} \hat{f}(y)\, dy$, which completes the proof.

Remark. The reader may compare this paper with [6].

REFERENCES

[1] *V. Jarník:* Diferenciální počet, Praha 1953.

[2] *V. Jarník:* Integrální počet II, Praha 1955.

[3] *J. Mařík:* Vrcholy jednotkové koule v prostoru funkcionál na daném polouspořádaném prostoru, Časopis pro pěst. mat., *79* (1954), 3—40.

[4] *Ян Маржик* (Jan Mařík): Представление функционала в виде интеграла, Чехословацкий мат. журнал, *5* (80), 1955, 467—487.

[5] *J. Mařík:* Plošný integrál, Časopis pro pěst. mat., *81* (1956), 79—82.

[6] *Ян Маржик* (Jan Mařík): Заметка к теории поверхностного интеграла, Чехословацкий мат. журнал, *6* (81), 1956, 387—400.

[7] *S. Saks:* Theory of the integral, New York.

Резюме

ПОВЕРХНОСТНЫЙ ИНТЕГРАЛ

ЯН МАРЖИК (Jan Mařík), Прага.

(Поступило в редакцию 10/X 1955 г.)

Пусть m — натуральное число; пусть E_m — m-мерное евклидово пространство. Для всякого ограниченного измеримого множества $A \subset E_m$ положим $\|A\| = \sup \int_A \sum_{i=1}^{m} \frac{\partial v_i(x)}{\partial x_i}\, dx$, где $v_1, \ldots, v_m$ — многочлены такие, что $\sum_{i=1}^{m} v_i^2(x) \leqq 1$ для всех $x \in A$. Пусть $\mathfrak{A}$ — система всех ограниченных измеримых множеств A, для которых $\|A\| < \infty$. Теорема 18 тогда утверждает:

Пусть $A \in \mathfrak{A}$; пусть D — граница множества A. Тогда на системе $\mathfrak{B}$ всех борелевских подмножеств множества D существует мера p и на

множестве D существуют $\mathfrak{B}$-измеримые функции $v_1, \ldots, v_m$ такие, что $\sum_{i=1}^{m} v_i^2(x) = 1$ для каждого $x \in D$ и что

$$\int_D \sum_{i=1}^{m} v_i(x) \, . \, v_i(x) \, . \, \mathrm{d}p(x) = \int_A \sum_{i=1}^{m} \frac{\partial v_i(x)}{\partial x_i} \, \mathrm{d}x \, ,$$

если функции $v_1, v_2, \ldots, v_m, \frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \ldots, \frac{\partial v_m}{\partial x_m}$ непрерывны в некоторой окрестности множества $\overline{A}$.) Мера p этим определяется однозначно, а функции v_i ,,почти однозначно" (по отношению к мере p), и $p(D) = \|A\|$.*

Далее пишем $P(A, v_1, \ldots, v_m) = P(A, v) = \int_D \sum_{i=1}^{m} v_i \, . \, v_i \, \mathrm{d}p$. Теорема 20 содержит достаточное условие для того, чтобы данное множество принадлежало системе $\mathfrak{A}$; как видно из теоремы 33, это условие является и необходимым. Теоремы 9, 35, 37 показывают некоторые свойства системы $\mathfrak{A}$. Например, теорема 35 утверждает, что соединение и разность двух элементов из $\mathfrak{A}$ принадлежит также $\mathfrak{A}$.

Пусть теперь $A \in \mathfrak{A}$ и пусть φ — взаимно однозначное регулярное отображение какой-нибудь окрестности множества $\overline{A}$ в пространство E_m. Из теоремы 50 вытекает, что в этом случае $\varphi(A) \in \mathfrak{A}$ и что справедливо следующее предложение:

Пусть $v_1, \ldots, v_m$ — непрерывные функции на границе множества A; пусть M — функциональная матрица отображения φ и пусть v — столбец с элементами $v_1, \ldots, v_m$. Для каждого y, лежащего на границе множества $\varphi(A)$, положим $w(y) = |\det M(x)|^{-1} \, . \, M(x) \, . \, v(x)$, где $y = \varphi(x)$. Тогда $P(A, v) = P(\varphi(A), w)$.

Важным следствием отделов 38—43 является следующая теорема:

Пусть A — непустое ограниченное подмножество пространства E_m; пусть D — граница множества A. Для каждого $\varepsilon > 0$ пусть $\Omega(D, \varepsilon)$ будет множеством всех точек из E_m, расстояние которых от D меньше чем ε. Предположим далее, что функция (мера $\Omega(D, \varepsilon)$). ε^{-1} переменного ε ограничена для $0 < \varepsilon < 1$. Пусть функции $v_1, v_2, \ldots, v_m$ непрерывны на множестве $\overline{A}$ и пусть функции $\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \ldots, \frac{\partial v_m}{\partial x_m}$ непрерывны на множестве A°.) Тогда $A \in \mathfrak{A}$ и равенство*

$$P(A, v) = \int_A \sum_{i=1}^{m} \frac{\partial v_i(x)}{\partial x_i} \, \mathrm{d}x$$

справедливо, если существует интеграл Лебега в правой части.

О представлении поверхностного интеграла $P(A, v)$,,классическим способом" см. [6].

*) $\overline{A} = A \cup D$, $A^\circ = A - D$.