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THE THEORY OF CHARACTERS OF COMMUTATIVE HAUSDORFF BICOMPACT SEMIGROUPS

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Let S be a Hausdorff bicompact commutative semigroup. By a character we mean a complex- valued continuous function $\chi(x)$ defined on S and satisfying $\chi(a)\,\chi(b) = \chi(ab)$ for every couple $a, b \in S$. In contrary to the paper [5] the absolute value $|\chi(x)|$ need not be 1. The set of all characters of S forms a new semigroup which will be denoted by S^*. The purpose of the paper is to find the structure of S^*. A series of general theorems concerning this problem is given. Under some additional suppositions about S the structure of S^* is fully described.

Let S be a commutative Hausdorff bicompact semigroup. By a character of S we mean a continuous complex — valued function $\chi(x)$ defined on S satisfying the relation $\chi(a)\,\chi(b) = \chi(ab)$ for all $a, b \in S$.

Every semigroup has two trivial characters: the zero character χ_0 and the unit character χ_1. By the zero character we mean the function identically zero on S. By the unit character we mean the function $\chi_1(x)$ identically 1 on S.

In the paper [5] the set of all characters assuming only values of absolute value unity was studied. We shall show that these are exactly all characters vanishing nowhere on S. The set of these characters forms in an obvious manner a group. The structure of this group and its relation to S has been described in the paper [5].

In this paper we shall use the term character in the wider sense defined above. This is in accordance with the terminology used in the papers [7], [8], [9], where — of course — only finite semigroups were treated.

Instead of the term character it would be perhaps more convenient to use the term "multiplicative functional". I do not use this notion to avoid eventual confusion with the notion of the "linear multiplicative functional" which is commonly used in the theory of Banach spaces. (See f. i. HILLE [3], 126—128 or Канторович-Вулих-Пинскер [4], 262—270.[1])

[1]) *Hewitt* and *Zuckerman* [1], [2] use the term "semicharacter" and exclude the zero character from their considerations.

Suppose that χ_α, χ_β are two characters of S. With the usual definition of multiplication the product $\chi_\alpha . \chi_\beta$ is clearly again a character of S. The set of all characters of S forms a semigroup S^*. The purpose of this paper is to prove a number of theorems concerning the structure of S^*. For finite semigroups this problem can be regarded — to a great extent — as solved by the results of the papers [7], [8], [9]. With respect to the far more general suppositions the results of the present paper are, of course, of another kind. Nevertheless much of the important features can be transfered from the finite to the bicompact case.

Remark to the notations. In all the paper the symbol $A \subset B$ (in distinction to $A \subseteqq B$) denotes that A is a proper subset of B.

1

In this section we quote briefly some results concerning the structure of bicompact semigroups which are used in the following. The detailed proofs can be found in the paper [6].

Throughout the paper S denotes a commutative Hausdorff bicompact semigroup. Such a semigroup contains always at least one idempotent. The symbol $\{e_\alpha \mid \alpha \in \Lambda\}$ denotes the set of all idempotents $\in S$. Let be $a \in S$, $A = \{a, a^2, a^3, \ldots\}$. The closure $\bar{A}$ contains one and only one idempotent e_α. We shall say that a *belongs to the idempotent* e_α. If a belongs to e_α, b belongs to e_β, then ab belongs to $e_\alpha e_\beta$. The set of all elements $\in S$ belonging to e_α forms a semigroup which will be denoted by P_α. We shall call this semigroup "*maximal semigroup belonging to the idempotent* e_α".[2]) The semigroup S can be written as a sum of disjoint maximal semigroups: $S = \sum_{\alpha \in \Lambda} P_\alpha$.

To every idempotent e_α there exists further a unique "*maximal group*" G_α containing e_α as unit element. It is clearly $G_\alpha \subseteqq P_\alpha$. The group G_α is closed in S and it holds $P_\alpha . e_\alpha = G_\alpha$.

The element $a \in P_\alpha$ is called regular if $ae_\alpha = a$. It follows from the formulae just written that those and only those elements $\in P_\alpha$ are regular which are contained in G_α.

The maximal semigroups P_α are, in general, neither closed nor open in S.

Let us remark further: Let be $a \in P_\alpha$. Write again $A = \{a, a^2, a^3, \ldots\}$ and let e_α be the idempotent $\in \bar{A}$. Then the greatest group contained in $\bar{A}$ (and having e_α as unit element) is $\bar{A}e_\alpha$. Therefore $\bar{A}e_\alpha \subseteqq \bar{A} \cap G_\alpha$.

2

An *ideal* J of S is a set J satisfying the relation $aJ \subseteqq J$ for every $a \in S$. An ideal is called a *prime ideal* if $S - J$ is a semigroup. It is useful to call also the

[2]) For brevity we shall omit the words "belonging to the idempotent e_α". The idempotent to which the elements $\in P_\alpha$ belong will be always stated in the index.

empty set $\varnothing$ and the whole semigroup S prime ideals of S. (These trivial prime ideals are both open and closed.)

The purpose of this section is to find the structure of open prime ideals of a semigroup S.

Lemma 2,1. *Every open prime ideal $J \neq \varnothing$ of S is a class sum of (disjoint) maximal semigroups*: $J = \sum_{\eta} P_{\eta}$.

Proof. It is sufficient to prove: if an open prime ideal J has a non-empty intersection with the maximal semigroup P_α, then P_α is contained in J.

Suppose $b \in P_\alpha \cap J \neq \varnothing$. Let e_α be the idempotent $\in P_\alpha$. Since $b \in J$, we have $bS \subseteqq J$, especially $be_\alpha \in J$. The element be_α is a regular element $\in S$, therefore $be_\alpha \in G_\alpha$. The relation $be_\alpha \in J$ implies $Sbe_\alpha \subseteqq J$, i. e., $\{G_\alpha + \ldots\}\, be_\alpha \subseteqq J$. Hence $G_\alpha \subseteqq J$ and $e_\alpha \in J$.

According to the supposition $S - J$ is a closed semigroup. Suppose that there exists an element $c \in P_\alpha$, but c non $\in J$, i. e., $c \in S - J$. Then it holds also $\{c, c^2, c^3, \ldots\} \subseteqq S - J$ and $\overline{\{c, c^2, c^3, \ldots\}} \subseteqq S - J$. Therefore the idempotent to which c belongs, i. e. ,e_α, is contained in $S - J$. This is a contradiction to $e_\alpha \in J$ and Lemma 2,1 is proved.

Remark. Every prime ideal is not necessarily a sum of maximal semigroups. Further the converse of Lemma 2,1 is, in general, not true: a prime ideal which is a sum of maximal semigroups need not be open.

An example to the first assertion is furnished by the multiplicative semigroup of all complex numbers $|z| \leqq 1$ the topology being the usual topology in the plane. Here two idempotents exist: $z = 0$ and $z = 1$. The maximal semigroups are $P_0 = \{z \mid |z| < 1\}$ and $P_1 = \{z \mid |z| = 1\}$. The one-element set $\{0\}$ is clearly a prime ideal of S, but it is not a sum of maximal semigroups.

An example to the second assertion is furnished by the set of all real numbers of a closed interval $\langle a, b\rangle$ with the ordinary topology on the real line and the multiplication $a \odot b = \mathrm{Min}\,(a, b)$. Each element forms a maximal semigroup. Choose a c with $a < c < b$. The set $\langle a, c\rangle$ is a prime ideal that is a sum of maximal semigroups. It is clearly not open.

In the following we need some facts about the set E of all idempotents of a commutative semigroup S. In E we can define a partial ordering by the statement: $e_\alpha \leqq e_\beta$ whenever $e_\alpha e_\beta = e_\alpha$. To every couple $e_\gamma, e_\delta \in E$ we can construct the lattice-theoretical intersection by putting $e_\gamma \wedge e_\delta = e_\gamma e_\delta$. Hence the idempotents form a so called semi-lattice. If E contains only a finite number of idempotents it is clear that there exists a unique least idempotent in E (namely the product of all idempotents $\in S$). It is essential for our purposes that it can be proved (see [5]) that in a commutative Hausdorff bicompact semigroup there exists always a unique least idempotent. In the following this theorem will be used several times.

Let $J \neq S$ be an arbitrary open prime ideal of S. According to Lemma 2,1 the semigroup $S - J$ is a sum of maximal semigroups $S - J = \sum_{\xi} P_{\xi}$. Since $S - J$ is closed, and therefore bicompact, there exists in $S - J$ a least idempotent e_{α}. For any idempotent $e_{\xi} \in S - J$ we have $e_{\xi} e_{\alpha} = e_{\alpha}$. For any idempotent $e_{\eta} \in J \neq \emptyset$ we have $e_{\alpha} e_{\eta} \in e_{\alpha} J \subseteqq J$ and therefore certainly $e_{\alpha} e_{\eta} \neq e_{\alpha}$. Hence the idempotents $\in J$ are precisely those for which $e_{\alpha} e_{\eta} \neq e_{\alpha}$ holds.

Now we prove conversely:

Lemma 2,2. *Let e_{α} be an arbitrary idempotent $\in S$. Let $\{e_{\xi}\}$ be the totality of all idempotents $\in S$ for which $e_{\alpha} e_{\xi} = e_{\alpha}$ holds. Construct the sum $\sum_{\xi} P_{\xi}$ and the sum $S - \sum_{\xi} P_{\xi} = \sum_{\eta} P_{\eta}$, where e_{η} runs through all idempotents for which $e_{\eta} e_{\alpha} \neq e_{\alpha}$ holds. Then*

a) $Q = \sum_{\xi} P_{\xi}$ is a closed semigroup;

b) $J = \sum_{\eta} P_{\eta}$ is an open prime ideal of S.

Proof. 1) Let $a \in Q$, $b \in Q$. We then have $a \in P_{\xi_1}$, $b \in P_{\xi_2}$, i. e. a belongs to some idempotent e_{ξ_1}, b belongs to some idempotent e_{ξ_2}. The element ab belongs to the idempotent $e_{\xi_1} . e_{\xi_2}$. Denoting $e_{\xi_1} . e_{\xi_2} = e_{\xi_3}$ we have $ab \in P_{\xi_3}$. Since $e_{\xi_1} e_{\alpha} = e_{\alpha}$, $e_{\xi_2} e_{\alpha} = e_{\alpha}$, we have also $e_{\xi_1} e_{\xi_2} e_{\alpha} = e_{\xi_1} e_{\alpha} = e_{\alpha}$, i. e., $e_{\xi_3} e_{\alpha} = e_{\alpha}$. Therefore $P_{\xi_3} \subseteqq Q$, i. e. $ab \in Q$. This proves that Q is a semigroup.

2) We prove that J is a prime ideal. If $J = \emptyset$, there is nothing to prove. Suppose therefore $J \neq \emptyset$. Let be $a \in J$, hence $a \in P_{\eta}$ and a belongs to the idempotent e_{η}. Let s be an arbitrary element $\in S$ belonging to the idempotent $e_{\sigma} \in S$. Then as belongs to the idempotent $e_{\eta} e_{\sigma}$. This idempotent is contained in J. For if it were $e_{\eta} e_{\sigma} e_{\alpha} = e_{\alpha}$, it would be also $e_{\eta}(e_{\eta} e_{\sigma} e_{\alpha}) = e_{\eta} e_{\alpha}$, i. e., $e_{\eta} e_{\sigma} e_{\alpha} = e_{\eta} e_{\alpha}$, $e_{\eta} e_{\alpha} = e_{\alpha}$ and e_{η} would be contained in Q which is a contradiction. We have therefore $as \in J$ for every $a \in J$, $s \in S$. This proves that J is an ideal of S. With respect to the fact proved sub 1) we get that J is, moreover, a prime ideal.

3) It remains to show that Q is closed. Let us remark first: if G_{α} is the maximal group belonging to the idempotent e_{α}, then $Qe_{\alpha} = G_{\alpha}$. We have $Qe_{\alpha} = (\sum_{\xi} P_{\xi}) e_{\alpha} = \sum_{\xi} P_{\xi} e_{\alpha}$. Every element $\in P_{\xi} e_{\alpha}$ belongs to the idempotent $e_{\xi} e_{\alpha} = e_{\alpha}$. Therefore $P_{\xi} e_{\alpha} \subseteqq P_{\alpha}$. Further every element $\in P_{\xi} e_{\alpha}$ is regular since for every xe_{α}, $x \in P_{\xi}$ the relation $(xe_{\alpha}) e_{\alpha} = xe_{\alpha}$ holds. Hence it is even $P_{\xi} e_{\alpha} \subseteqq G_{\alpha}$ for all $P_{\xi} \subseteqq Q$. In the relation $Qe_{\alpha} \subseteqq G_{\alpha}$ the sign of equality must hold since we know that for the maximal semigroup P_{α} the relation $P_{\alpha} e_{\alpha} = G_{\alpha}$ is valid.

Now we prove that Q is closed. This follows indirectly. Suppose $\overline{Q} - Q \neq \emptyset$, hence $\overline{Q} \cap J \neq \emptyset$. Let be $v \in \overline{Q} \cap J$. Let us consider the element $v . e_{\alpha}$. It is contained in J (since we have even $v . S \subseteqq J$). Since G_{α} is closed, and therefore $S - G_{\alpha}$ open, there exists a neighbourhood $U(ve_{\alpha})$ such that $U(ve_{\alpha}) \cap G_{\alpha} = \emptyset$.

Further there exist neighbourhoods $U(v)$, $U(e_\alpha)$ such that $U(v) \,.\, U(e_\alpha) \subseteqq U(ve_\alpha)$ and therefore $U(v) \,.\, U(e_\alpha) \cap G_\alpha = \emptyset$. Since $v \in \bar{Q}$ there exists an $x \in Q$ with $x \in U(v)$. Hence for $x \in Q$ we have $xe_\alpha \cap G_\alpha = \emptyset$. This is a contradiction to $Qe_\alpha = G_\alpha$. This proves Lemma 2,2.

It follows from Lemma 2,1 and Lemma 2,2 the following

Theorem 2,1. *All open prime ideals $J \neq S$, $J \neq \emptyset$ of a commutative Hausdorff bicompact semigroup can be obtained in the following manner. Choose an idempotent $e_\alpha \in S$. Find all idempotents e_η (if such exist) with $e_\eta e_\alpha \neq e_\alpha$. Construct the corresponding maximal semigroups P_η. Then $J = \sum_\eta P_\eta$ is an open prime ideal of S.*

Remark. If we choose for e_α the least idempotent $\in S$, then the set of idempotents $\{e_\eta\}$ is vacuous. In this case we shall put formally $J = \emptyset$.

According to the Theorem 2,1 and the remark just made there corresponds to every idempotent $e_\alpha \in S$ an open prime ideal J_α. Conversely, to every open prime ideal $J_\alpha \neq S$ there corresponds a unique idempotent e_α (namely the least idempotent of the bicompact semigroup $S - J_\alpha$). This one-to-one correspondence between the idempotents and the open prime ideals $\neq S$ will be denoted by

$$J_\alpha \longleftrightarrow e_\alpha \,. \tag{1}$$

It is easy to prove that the sum of two open prime ideals J_α, J_β is the open prime ideal $J_\alpha + J_\beta$. Further it holds:

Lemma 2,3. *Let J_α, J_β be two open prime ideals of S. Then the intersection $J_\alpha \cap J_\beta$ contains a unique open prime ideal J_γ with the following properties*[3]*):*

a) $J_\gamma \subseteqq J_\alpha \cap J_\beta$,

b) there does not exist an open prime ideal X of S satisfying the relation $J_\gamma \subset X \subset J_\alpha \cap J_\beta$.

Proof. The prime ideal $\emptyset$ satisfies the condition a). Hence the set of open prime ideals satisfying a) is non-vacuous.

If the intersection $J_\alpha \cap J_\beta$ is a prime ideal, it is sufficient to put $J_\gamma = J_\alpha \cap J_\beta$. This is certainly the case if either $J_\alpha \subseteqq J_\beta$ or $J_\beta \subseteqq J_\alpha$. We show that if none of these relations holds, then the intersection $J_\alpha \cap J_\beta$ is not a prime ideal. Indeed, under the suppositions just made there exist two elements a, b such that $a \in J_\alpha$, a non $\in J_\beta$ and $b \in J_\beta$, b non $\in J_\alpha$. It holds a non $\in J_\alpha \cap J_\beta$, b non $\in J_\alpha \cap J_\beta$, but $ab \in aJ_\beta \subseteqq J_\beta$, $ab \in bJ_\alpha \subseteqq J_\alpha$. Thus $ab \in J_\alpha \cap J_\beta$ and therefore $J_\alpha \cap J_\beta$ cannot be a prime ideal. The proof shows at the same time that the element ab cannot be contained in any prime ideal which is itself entirely contained in $J_\alpha \cap J_\beta$.

In the following suppose that $J_\alpha \cap J_\beta$ is not a prime ideal. If $J_\alpha \cap J_\beta$ contains only the prime ideal $\emptyset$ there is nothing to prove. In the other case consider an arbitrary collection $\{J_\lambda\}_{\lambda \in \Lambda}$ of open prime ideals contained in $J_\alpha \cap J_\beta$,

[3]) Since $J_\alpha J_\beta \subseteqq J_\alpha \cap J_\beta$ the intersection is always non-vacuous.

and simply ordered by set-inclusion. Each of the J_λ has the property that it does not contain the element ab. The set $\bigcup_{\lambda \varepsilon \Lambda} J_\lambda = J'$ is clearly again an open prime ideal[4]) of S not containing the element ab. Hence every by inclusion simply ordered system of open prime ideals of S contained in $J_\alpha \cap J_\beta$ has an upper bound (i. e. such an open prime ideal which contains each prime ideal of the system). It follows from Zorn's lemma that there is at least one maximal open prime ideal contained in $J_\alpha \cap J_\beta$. We prove that there exists exactly one such prime ideal. Suppose that J_γ, J_δ are two different maximal open prime ideals. The set $J_\gamma + J_\delta$ is again an open prime ideal of S contained in $J_\alpha \cap J_\beta$. There cannot be $J_\gamma + J_\delta = J_\alpha \cap J_\beta$ since $J_\alpha \cap J_\beta$ is not a prime ideal. It would be therefore $J_\gamma + J_\delta \subset J_\alpha \cap J_\beta$. The proper subideal $J_\gamma + J_\delta$ contains J_γ, J_δ as proper subsets. This is a contradiction to the maximality of J_γ and J_δ. This proves Lemma 2,3.

Define in the system of all open prime ideals of S partially ordered by inclusion the lattice-theoretical operations by the relations: $J_\alpha \vee J_\beta = J_\alpha + J_\beta$ and $J_\alpha \wedge J_\beta = J_\gamma$, where J_γ is the unique maximal open prime ideal contained in $J_\alpha \cap J_\beta$. Then it holds clearly:

Lemma 2,4. *With respect to the operations just introduced the set of all open prime ideals of S forms a lattice.*

We prove further:

Lemma 2,5. *Let in the correspondence* (1) *be $J_\alpha \longleftrightarrow e_\alpha$, $J_\beta \longleftrightarrow e_\beta$. If $J_\alpha \subset J_\beta$, then $e_\alpha < e_\beta$ and conversely.*

Proof. Suppose $J_\alpha \subset J_\beta$. According to Lemma 2,2 e_α is the least idempotent $\epsilon\, S - J_\alpha$, e_β is the least idempotent $\epsilon\, S - J_\beta$. Since $S - J_\alpha \supset S - J_\beta$ it holds clearly $e_\beta \geqq e_\alpha$. The equality $e_\alpha = e_\beta$ would imply $J_\alpha = J_\beta$. Hence we have $e_\beta > e_\alpha$.

Suppose on the other hand $e_\alpha < e_\beta$. According to Lemma 2,1 the prime ideal J_α is the sum of maximal semigroups P_γ belonging to those idempotents e_γ for which $e_\gamma e_\alpha \neq e_\alpha$ holds. The prime ideal J_β is the sum of maximal semigroups P_γ belonging to those idempotents e_γ for which $e_\gamma e_\beta \neq e_\beta$ holds. If for some e_γ $e_\gamma e_\alpha \neq e_\alpha$ holds, it holds also $e_\gamma e_\beta \neq e_\beta$. For if we had $e_\gamma e_\beta = e_\beta$, it would be also $e_\gamma e_\beta e_\alpha = e_\beta e_\alpha$, i. e. $e_\gamma e_\alpha = e_\alpha$, contrary to the assumption. This implies $J_\alpha \subseteqq J_\beta$. The element e_α satisfies $e_\alpha \,.\, e_\alpha = e_\alpha$ and $e_\alpha e_\beta \neq e_\alpha$. Hence $e_\alpha \in J_\beta$ but e_α non $\in J_\alpha$. Thus $J_\alpha \subset J_\beta$. This proves Lemma 2,5.

It follows from the results obtained immediately:

Theorem 2,2. *The semi-lattice of all open prime ideals $\neq S$ and the semi-lattice of idempotents $\epsilon\, S$ are isomorphic.*

Remark. By the semi-lattice operation in the system of all open prime ideals $\neq S$ we mean here the operation $J_\alpha \wedge J_\beta$ introduced above. Evidently: the relations $J_\alpha \longleftrightarrow e_\alpha$, $J_\beta \longleftrightarrow e_\beta$ imply $J_\alpha \wedge J_\beta \longleftrightarrow e_\alpha \,.\, e_\beta$.

[4]) For if c, d non $\epsilon\, J'$, we have $c\,.\,d$ non $\epsilon\, J_\lambda$ for all λ and therefore $c\,.\,d$ non $\epsilon\, J'$.

3

In this section we prove some theorems concerning the properties of a fixed chosen character.

Let $\chi(x)$ be a character of S. Then the functions $\chi(x)^n$ ($n \geqq 1$ an integer), $\bar{\chi}(x)$, $|\chi(x)|$ are also characters of S. If α is any complex number, $\Re(\alpha) > 0$, the function $|\chi(x)|^\alpha = e^{\alpha \log |\chi(x)|}$ is again a character of S. (By the logarithm we mean here the real logarithm.)

If e is an idempotent, then $e^2 = e$ implies $\chi(e)^2 = \chi(e)$. Hence it holds either $\chi(e) = 0$ or $\chi(e) = 1$.

Lemma 3,1. *For every $b \in S$ and every χ it holds always $|\chi(b)| \leqq 1$.*

Proof. Suppose $|\chi(b)| = c > 1$. For every integer $n \geqq 1$ we have $b^n \in S$ and $\chi(b^n) = [\chi(b)]^n$. Hence $|\chi(b^n)| = c^n > c > 1$. Since every continuous function on a bicompact Hausdorff space is bounded in absolute value, the last relation constitutes an obvious contradiction. This proves our lemma.

Lemma 3,2. *Let χ be a character of the semigroup S. Let $\mathfrak{p}$ be the set of all elements $a \in S$ for which $\chi(a) = 0$ holds. Let $\mathfrak{p}$ be non-vacuous. Then $\mathfrak{p}$ is a closed prime ideal of S.*

Proof. 1) Let be $a \in \mathfrak{p}$, $s \in S$. Then the element as satisfies the relation $\chi(as) = \chi(a)\,\chi(s) = 0\,.\,\chi(s) = 0$, i. e., $as \in \mathfrak{p}$. Hence $\mathfrak{p}$ is an ideal.

2) Let be $a \in S - \mathfrak{p}$, $b \in S - \mathfrak{p}$, i. e. $\chi(a) \neq 0$, $\chi(b) \neq 0$. Then it holds $\chi(ab) \neq 0$, i. e. $ab \in S - \mathfrak{p}$. This shows that $S - \mathfrak{p}$ is a semigroup, i. e. $\mathfrak{p}$ is a prime ideal.

3) The function χ being continuous, the set of all $a \in S$ for which $\chi(a) = 0$ is closed. Hence $\mathfrak{p}$ is closed, which completes the proof.

Remark. The set $\mathfrak{p}$ may be empty. This is the case, f. i., for the unit character $\chi_1(x)$. But the Lemma holds formally even in this case since we consider $\varnothing$ to be a closed prime ideal.

Lemma 3,3. *Let χ be a character of S. Let Q be the set of all $a \in S$ for which $|\chi(a)| = 1$ holds. Let Q be non-vacuous. Then Q is a closed semigroup.*

Proof. a) Let be $a, b \in Q$, i. e., $|\chi(a)| = |\chi(b)| = 1$. Then $|\chi(ab)| = |\chi(a)|\,.\,|\chi(b)| = 1$. Hence Q is a semigroup.

b) The function $|\chi|$ being continuous, the set of all $a \in S$ with $|\chi(a)| = 1$ is closed. Hence Q is closed, which completes the proof.

Remark. The set Q may be, of course, empty. Let, for instance, S be the multiplicative semigroup of real numbers of the closed interval $\langle 0, \frac{1}{2} \rangle$ with the ordinary topology on the real axis. Then the real character $\chi(x) = x$ attains on S its maximum $\frac{1}{2}$.

Lemma 3,4. *The set Q from Lemma 3,3 is non vacuous if and only if there exists at least one idempotent e for which $\chi(e) = 1$.*

Proof. a) If for some idempotent $\chi(e) = 1$, then Q is non-vacuous since there is $e \in Q$.

b) Suppose that Q is non-vacuous, i. e. there exists an $a \in Q$ with $|\chi(a)| = 1$. Let us put $A = \{a, a^2, a^3, \ldots\}$. For every integer $n \geqq 1$ we have clearly $|\chi(a^n)| = 1$. Let e be the idempotent $\in \bar{A}$. If we had $\chi(e) = 0$, there would exist for every $\varepsilon > 0$ such a neighbourhood $U_\varepsilon(e)$ of e that for every $x \in U_\varepsilon(e)$ $|\chi(x)| < \varepsilon$ holds. Since $e \in \bar{A}$ there exists an integer $m > 0$ such that $a^m \in U_\varepsilon(e)$. For a^m we have $|\chi(a^m)| = 1$. This constitutes an obvious contradiction. Therefore $\chi(e) = 1$, q. e. d.

Lemma 3,5. *Let $\chi(x)$ be a character of S. Let J be the set of all $a \in S$ for which $|\chi(a)| < 1$ holds. Then J is an open prime ideal of S.*

Proof. If $J = \varnothing$, the Lemma is true. Let therefore $J \neq \varnothing$. If $a \in J$, $s \in S$, we have $|\chi(as)| = |\chi(a)| \cdot |\chi(s)| < 1 \cdot 1 = 1$, i. e., $as \in J$; thus J is an ideal. According to Lemma 3,3 the set $S - J$, i. e., the set of all a with $|\chi(a)| = 1$, is a closed semigroup. Hence J is an open prime ideal, q. e. d.

Remark. J again can be empty. This is the case, for instance, if χ is the unit character χ_1.

According to Lemma 2,1 J and Q are class-sums of maximal semigroups of S. For every idempotent $e_\eta \in J$ we have $\chi(e_\eta) = 0$. For every idempotent $e_\xi \in Q$ we have $\chi(e_\xi) = 1$.

Let for some element $b \in S$ be $|\chi(b)| = 1$. If b belongs to the idempotent e_β it holds also $|\chi(m)| = 1$ for every $m \in P_\beta$ since with b the whole set P_β is contained in Q.

Let for some $c \in S$ be $|\chi(c)| < 1$. Suppose that c belongs to the idempotent e_γ. Then we have necessarily $\chi(e_\gamma) = 0$ and the whole set P_γ is contained in J. In this case we can add a further conclusion. Let G_γ be the maximal group belonging to the idempotent e_γ. Then for every $d \in G_\gamma$ $de_\gamma = d$ and $\chi(d) = \chi(d) \cdot \chi(e_\gamma) = \chi(d) \cdot 0 = 0$. Hence — if $\mathfrak{p}$ retains the meaning from Lemma 3,2 — $\mathfrak{p}$ contains not only every idempotent $\in J$ but also every maximal group contained in J.

Summarily we obtain the following result:

Theorem 3,1. *Let χ be a fixed character of S. Let J be the set of all $a \in S$ with $|\chi(a)| < 1$ and Q the set of all $a \in S$ with $|\chi(a)| = 1$. Then we can write the decomposition of S into two disjoint summands $S = J + Q$, where J is an open prime ideal and Q a closed semigroup. Both J and Q are class-sums of maximal semigroups of S. J is the class-sum of all maximal semigroups belonging to the idempotents e_η for which $\chi(e_\eta) = 0$ holds. Q is the class-sum of all maximal semigroups belonging to the idempotents e_ξ for which $\chi(e_\xi) = 1$ holds. The set $\mathfrak{p}$ of all $a \in S$ for which $\chi(a) = 0$ holds is a closed prime ideal containing all maximal groups which are contained in J.*

Remark 1. The situation is schematically shown in the fig. 1.

Remark 2. J is vacuous if and only if for every $a \in S$ $|\chi(a)| = 1$. Q is vacuous if and only if for every idempotent $e_\eta \in S$ $\chi(e_\eta) = 0$.

Remark 3. One shows on simple examples that there need not be $P_\xi \subseteqq \mathfrak{p}$; this means $\mathfrak{p}$ need not be a class-sum of maximal semigroups.

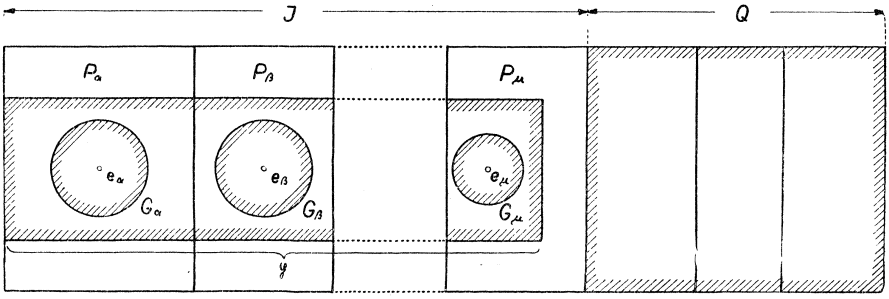

Fig. 1.

Remark 4. The set $J - \mathfrak{p}$ is clearly open in S. Further it is a semigroup. For if $a \in J - \mathfrak{p}$, $b \in J - \mathfrak{p}$, i. e., $0 < |\chi(a)| < 1$, $0 < |\chi(b)| < 1$, we have also $0 < |\chi(ab)| < 1$, i. e., $ab \in J - \mathfrak{p}$. Hence $J - \mathfrak{p}$ is an open sub-semigroup of S without idempotents.

4

The question arises how far a character is determined by the sets $\mathfrak{p}$, J, Q. It is clear that if $\mathfrak{p}$ is given J, Q are uniquely determined. On the other hand one shows on examples that if J is given $\mathfrak{p}$ is — in general — not uniquely determined.

Problem A. *Let $\mathfrak{p} \neq \varnothing$ be a closed prime ideal of the semigroup S. Does there exist a character χ of S which vanishes just on $\mathfrak{p}$?*

The answer to this question is, in general, *negative*. We show this on the following simple example. Let S be the semigroup of real numbers of the interval $\langle 0, 1\rangle$ with the ordinary topology on the real line. Let the composition be defined by $a \odot b = \mathrm{Min}\,(a, b)$. Then the one-element set $\{0\}$ is a closed prime ideal of S.[5]) Since every element $a \in S$ is idempotent, we have for every $a \in S$ either $\chi(a) = 1$ or $\chi(a) = 0$. It follows from the continuity of χ that the only characters are: the unit character χ_1 and the zero character χ_0. But none of them vanishes just on $\{0\}$.

Remark. If a character χ vanishes just on $\mathfrak{p}$, so does $\chi\bar{\chi}$. Hence there exists also a real character vanishing just on $\mathfrak{p}$. If $Q \neq \varnothing$, this real character is iden-

[5]) Indeed, we get $0 \odot s = \mathrm{Min}\,(0, s) = 0$ for every $s \in S$ and $a \odot b = \mathrm{Min}\,(a, b) \neq 0$ for $a \neq 0$, $b \neq 0$.

tically 1 on the whole semigroup Q. (For we have $\chi\bar{\chi} \geqq 0$ and on Q $|\chi\bar{\chi}| = 1$, whence $\chi\bar{\chi} = 1$ on Q.)

The example just discussed leads to the following definition:

Definition 4,1. *A prime ideal $\mathfrak{p}$ of the semigroup S is called a generating prime ideal if*

a) $\mathfrak{p}$ is closed in S,

b) there exists at least one character of S which vanishes just on $\mathfrak{p}$.

Remark. The prime ideal $\emptyset$ is always a generating prime ideal since the unit character vanishes nowhere. Similarly, the prime ideal S is a generating prime ideal since the zero character is the (unique) character which vanishes on S.

Problem B. *Let $\mathfrak{p}$ be a generating prime ideal of the semigroup S. Let $\mathfrak{S}_\mathfrak{p}$ be the set of all characters of S which vanish just on $\mathfrak{p}$. What can be said about the structure of the set $\mathfrak{S}_\mathfrak{p}$?*

The set $\mathfrak{S}_\mathfrak{p}$ is a semigroup. Indeed, if χ_α and χ_β vanish just on $\mathfrak{p}$, so does $\chi_\alpha\chi_\beta$. The set of real characters which vanish just on $\mathfrak{p}$ form a sub-semigroup of $\mathfrak{S}_\mathfrak{p}$.

It can be shown on simple examples that the semigroup $\mathfrak{S}_\mathfrak{p}$ need not have idempotents. One shows also on examples that the structure of $\mathfrak{S}_\mathfrak{p}$ may be very various. At first sight there is little hope to find a detailed description of the structure of $\mathfrak{S}_\mathfrak{p}$ at least in the general case (i. e. without any supplementary suppositions concerning S).

The following theorem shows that a character $\epsilon\ \mathfrak{S}_\mathfrak{p}$ will be known as soon as we shall know its values on an arbitrary "small" ideal of S which is not entirely contained in $\mathfrak{p}$.

Theorem 4,1. *Let $\mathfrak{p}$ be a generating prime ideal of S. Let $\mathfrak{q}$ be any ideal of S such that $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} \neq \mathfrak{q}$. Then two characters $\epsilon\ \mathfrak{S}_\mathfrak{p}$ which assume the same values on $\mathfrak{q}$ are identical, i. e. assume the same values on the whole semigroup S.*

Proof. Let u be an arbitrary element $\epsilon\ \mathfrak{q} - \mathfrak{p}$, a any element $\epsilon\ S$. Let χ_α, χ_β be two characters $\epsilon\ \mathfrak{S}_\mathfrak{p}$ which assume the same values in every element $\epsilon\ \mathfrak{q}$.

According to the supposition we have

$$\chi_\alpha(u) = \chi_\beta(u) \neq 0 \,. \tag{*}$$

Further we have $a \,.\, u \in S\mathfrak{q} \subseteqq \mathfrak{q}$ and therefore (again according to the supposition) $\chi_\alpha(au) = \chi_\beta(au)$. The relation

$$\chi_\alpha(a) \cdot \chi_\alpha(u) = \chi_\beta(a) \cdot \chi_\beta(u)$$

implies (with respect to (*)) $\chi_\alpha(a) = \chi_\beta(a)$ for every $a \in S$, q. e. d.

Remark. Theorem 4,1 holds also under the more general assumption that $\mathfrak{q}$ is any ideal which is not entirely lying in $\mathfrak{p}$, i. e. such that $\mathfrak{q} - (\mathfrak{q} \cap \mathfrak{p}) \neq \emptyset$. In the proof of Theorem 4,1 it is sufficient to choose $u \in \mathfrak{q} - (\mathfrak{q} \cap \mathfrak{p})$. This justifies the remark before the statement of Theorem 4,1.

It holds also conversely:

Theorem 4,2. *Let $\mathfrak{q}$ be any ideal of S. Let ψ be a character of the semigroup $\mathfrak{q}$ not vanishing everywhere on $\mathfrak{q}$. Then there exists one and only one complex-valued function $\chi(x)$ defined on S such that*

a) χ is a character of the semigroup S,

b) for $x \in \mathfrak{q}$ $\chi(x) = \psi(x)$ holds.

Proof 1) Let $u \in \mathfrak{q}$ be an element with $\psi(u) \neq 0$. We show that there exists at most one character of S satisfying the assertions of Theorem 4,2. Suppose there exist two such characters χ_α and χ_β. Let a be any element $\in S$. Then $a \,.\, u \in a\mathfrak{q} \subseteqq \mathfrak{q}$. Hence $\psi(au) = \chi_\alpha(au) = \chi_\beta(au)$, $\chi_\alpha(a)\,\chi_\alpha(u) = \chi_\beta(a) \,.\, \chi_\beta(u)$, $\chi_\alpha(a)\,\psi(u) = \chi_\beta(a)\,\psi(u)$, i. e., $\chi_\alpha(a) = \chi_\beta(a)$ for every $a \in S$, q. e. d.

2) For every $a \in S$ define $\chi(a)$ by the relation

$$\chi(a) = \frac{\psi(au)}{\psi(u)} \,. \tag{2}$$

Since $au \in a\mathfrak{q} \subseteqq \mathfrak{q}$, $\psi(au)$ is defined and the right hand side has a meaning. For $a \in \mathfrak{q}$ we have clearly $\chi(a) = \psi(a)$. The function $\chi(a)$ has the following property

$$\chi(a) \,.\, \chi(b) = \frac{\psi(au)}{\psi(u)} \,.\, \frac{\psi(bu)}{\psi(u)} = \frac{\psi(abu^2)}{\psi(u) \,.\, \psi(u)} = \frac{\psi(abu \,.\, u)}{\psi(u) \,.\, \psi(u)} =$$

$$= \frac{\psi(abu) \,.\, \psi(u)}{\psi(u) \,.\, \psi(u)} = \frac{\psi(abu)}{\psi(u)} = \chi(ab) \,.$$

The function $\chi(x)$ is obviously continuous at $x = a \in S$, which completes the proof.

Remark 1. The proof shows that for the validity of this theorem the assumption of bicompactness is not necessary.

Remark 2. One can also show immediately that the value of $\chi(a)$ does not depend on the choice of the element u. If v is an other element $\in \mathfrak{q}$ with $\psi(v) \neq 0$, we have

$$\psi(au) \,.\, \psi(v) = \psi(auv) = \psi(av \,.\, u) = \psi(av) \,.\, \psi(u) \,,$$

whence

$$\frac{\psi(au)}{\psi(u)} = \frac{\psi(av)}{\psi(v)} \,, \quad \text{q. e. d.}$$

In the following we need a fact that is based on the results of the paper [5] and which need not be proved therefore in extenso here.

Let $\mathfrak{p}$ be a generating prime ideal of S. Suppose $Q \neq \varnothing$. We know that Q is closed and therefore a bicompact semigroup in the relative topology. Let e_0 be the least idempotent $\in Q$ and G_0 the maximal group belonging to the idempotent e_0. Every character $\psi(x)$ of the semigroup Q with values of absolute value unity can be obtained in the following manner. We take a suitable non-zero character $\varphi(x)$ of the group G_0 and we put

$$\psi(a) = \varphi(ae_0) \quad \text{for every} \quad a \in Q\,.^{6)} \tag{3}$$

The function $\psi(x)$ is the continuation of the character $\varphi(x)$ to the whole semigroup Q. The set of all characters of the semigroup Q with values of absolute value unity is algebraicaly isomorphic[7]) to the group of non-zero characters of the group G_0.

This has the following consequence:

Every character $\chi \in \mathfrak{S}_\mathfrak{p}$ induces on Q a character ψ of the semigroup Q. The function $\psi(x)$ assumes only values of absolute value unity. One can find these characters ψ only among the characters ψ defined by (3).

The following question arises:

Problem C. *Let $\mathfrak{p}$ be a generating prime ideal of the semigroup S. Suppose $Q \neq \emptyset$. Let $\psi(x)$ be any character of Q with $|\psi(x)| = 1$ on the whole Q. Does there exist a character $\chi \in \mathfrak{S}_\mathfrak{p}$ which satisfies the relation $\chi(x) = \psi(x)$ on Q?*

A simple example shows that there can exist also an infinity of such characters. Let S be the multiplicative semigroup of real numbers of the interval $\langle 0, 1\rangle$ with the ordinary topology on the real line. Let be $\mathfrak{p} = \{0\}$. Then $Q = \{1\}$. Let the character of Q be defined by $\psi(1) = 1$. Then every character of the form $x^\alpha = \chi(x)$, $\alpha > 0$ real, assumes on Q the value 1.

At this writing I cannot decide whether really to every $\psi(x)$ there corresponds at least one such $\chi(x)$. Almost trivial is only that such a character exists always to the unit character ψ_1 of Q. For if χ is any character $\in \mathfrak{S}_\mathfrak{p}$ then $\chi\bar{\chi}$ induces on Q the unit character of Q. It is also evident that the set of all characters $\in \mathfrak{S}_\mathfrak{p}$ which induce on Q the unit character ψ_1 forms a sub-semigroup $(\mathfrak{S}_\mathfrak{p})^{(1)}$, $(\mathfrak{S}_\mathfrak{p})^{(1)} \subseteqq \mathfrak{S}_\mathfrak{p}$.

Theorem 4,1 has the following consequence which we formulate explicitly as a theorem.

Theorem 4,3. *Suppose that $J \neq \mathfrak{p}$. Then two characters $\in \mathfrak{S}_\mathfrak{p}$ which equal on J equal on the whole semigroup S.*

Remark 1. Theorem 4,3 does not hold if $J = \mathfrak{p}$ (i. e. if J is closed). In this case we have $S = \mathfrak{p} + Q$. If $\psi(x)$ is an arbitrary character of Q with $|\psi| = 1$ on Q, then all functions of the form

$$\chi(x) = \begin{cases} 0 \text{ for } x \in \mathfrak{p}\,, \\ \psi(x) \text{ for } x \in Q\,, \end{cases}$$

are equal on $\mathfrak{p}$ but different as characters of S. (The functions are really characters of S since S is then a non-connected semigroup and the continuity is guaranteed by the continuity of ψ on Q.)

[6]) It is easy to prove that ae_0 is really contained in the group G_0.

[7]) Note that we have not introduced a topology into the set of characters of S.

Remark 2. If J is not closed, we can prove directly that a character $\chi \in \mathfrak{S}_p$ is already determined by its values on J by means of the following Lemma which itself is sometimes useful.

Lemma 4,1. *Suppose that J is not closed. Then the maximal group G_0 belonging to the least idempotent $e_0 \in S - J$ satisfies the relation $G_0 \subseteqq \bar{J} \cap Q$.*

Proof. Since J is not closed, we have $S - J \neq \emptyset$. Further $\bar{J} \cap Q$ is non-vacuous. The intersection $\bar{J} \cap Q$ is closed, hence it is a bicompact semigroup. It has at least one idempotent e. This idempotent e is contained in Q. At the same time e is contained in the ideal $\bar{J}$. If e belongs to an ideal so do also all idempotents $\leqq e$. (See [7], p. 233.) But e_0 is the least idempotent $\in Q$, hence $e_0 \in \bar{J}$. If any element of a group is contained in an ideal, then the maximal group containing this element is contained in the same ideal. Hence $G_0 \subseteqq \bar{J}$. This proves Lemma 4,1.

Proof of the remark 2. With respect to the continuity of the characters the values of χ on $\bar{J}$ are determined by the values of χ on J. In particular, the values of χ on the group G_0 are already determined by the values of χ on J. But as soon as we know the values of χ on G_0 we know (according to the remarks after Theorem 4,2 above) the values of χ on the whole semigroup S. This proves our assertion.

5

In this section we shall study some examples. First we prove a general theorem in which it is not necessary to suppose that S is bicompact.

Theorem 5,1. *Suppose T_1, T_2 are two commutative semigroups with unit elements e_1, e_2. Let $T = T_1 \times T_2$ be their direct product. Suppose that T_1^*, T_2^* are the semigroups of characters of T_1 and T_2 respectively. Denote by N the set of couples $\{[\overset{0}{\chi_1}, \chi_2], [\chi_1, \overset{0}{\chi_2}]\}$, where $\overset{0}{\chi_1}$, $\overset{0}{\chi_2}$ are the zero characters of T_1 and T_2 and χ_1 runs through all characters $\in T_1^*$, χ_2 through all characters $\in T_2^*$.*[8]*) Then the semigroup of characters of T is isomorphic to the difference semigroup*[9]*) $T_1^* \times T_2^*/N$. In formulae*

$$T^* \cong T_1^* \times T_2^*/N\ .$$

Proof. The elements of $T_1 \times T_2$ are all couples $x = [x_1, x_2]$, $x_1 \in T_1$, $x_2 \in T_2$. Elements of $T_1^* \times T_2^*$ are the couples $\chi = [\chi_1, \chi_2]$, $\chi_1 \in T_1^*$, $\chi_2 \in T_2^*$. Since the set of couples N is an ideal of $T_1^* \times T_2^*$ the difference semigroup is defined.

Let us remark in advance: for a non-zero character χ_1 of the semigroup T_1 we have $\chi_1(e_1) = 1$. For suppose $\chi_1(e_1) = 0$. The relation $x = e_1 x$ (valid for all

[8]) In this Theorem (and its proof) χ_1 does exceptionally not mean the unit character.

[9]) The difference semigroup T/N is a semigroup which is formed in essential so that we identify the elements of an ideal N with a new zero element while the other elements retain (up to an isomorphism) their original meaning. (The concept was introduced by D. Rees. See also [9], p. 308.)

$x \in T_1$) implies $\chi_1(x) = \chi_1(e_1)\, \chi_1(x) = 0$. Hence χ_1 would be the zero character, contrary to the supposition. Similarly, for a non-zero character χ_2 of the semigroup T_2 we have $\chi_2(e_2) = 1$.

a) Let us assign to every element $\chi = [\chi_1, \chi_2] \in T_1^* \times T_2^*$ the following function defined on T

$$\hat{\chi}(x) = \hat{\chi}([x_1, x_2]) = \chi_1(x_1) \cdot \chi_2(x_2) .$$

The function $\hat{\chi}$ is clearly a character of the semigroup T.

The function $\hat{\chi}$ is the zero character of T if and only if $[\chi_1, \chi_2] \in N$. For if $\hat{\chi}$ is the zero character the relation $\hat{\chi}(x) = \chi_1(x_1) \cdot \chi_2(x_2) = 0$ must hold for every $x = [x_1, x_2] \in T$. Substituting $x_1 = e_1$, $x_2 = e_2$ we get $\chi_1(e_1) \cdot \chi_2(e_2) = 0$. This implies that either χ_1 is the zero character of T_1, or χ_2 is the zero character of T_2, or both take place. Hence $[\chi_1, \chi_2] \in N$.

We shall show that (in the above correspondence) to two different elements $\chi' \neq \chi$ of the set $T_1^* \times T_2^* - N$ there correspond two different (non-zero) characters of the semigroup T. Let be $\chi = [\chi_1, \chi_2]$, $\chi' = [\chi_1', \chi_2']$ and $\chi \neq \chi'$. If we had $\hat{\chi}(x) = \hat{\chi}'(x)$, $\chi_1(x_1) \cdot \chi_2(x_2) = \chi_1'(x_1) \cdot \chi_2'(x_2)$ would hold for every $x = [x_1, x_2] \in T$.

In particular, for the couple $[e_1, x_2]$ we have $\chi_1(e_1) \cdot \chi_2(x_2) = \chi_1'(e_1) \cdot \chi_2'(x_2)$. Since $\chi_1(e_1) = \chi_1'(e_1) = 1$, we get $\chi_2(x_2) = \chi_2'(x_2)$, i. e., the characters χ_2, χ_2' of the semigroup T_2 are identical. One proves similarly that χ_1 and χ_1' are two identical characters of the semigroup T_1. Hence we have $\chi = \chi'$, contrary to the supposition.

Evidently, to the product of two elements $\chi = [\chi_1, \chi_2]$, $\chi' = [\chi_1', \chi_2']$ there corresponds the character $\hat{\chi} \cdot \hat{\chi}'$ of the semigroup T and the product $\hat{\chi} \cdot \hat{\chi}'$ is the zero character of T if and only if at least one of the characters is contained in N.[10])

b) It remains to prove that every non-zero character $\hat{\chi}$ of the semigroup T can be obtained by the above method.

Let us remark first: every element $x = [x_1, x_2] \in T$ can be written in the form of the product $x = [x_1, e_2] \cdot [e_1, x_2]$. Further the set of all elements $[x_1, e_2]$, $x_1 \in T_1$ is a sub-semigroup of the semigroup $T_1 \times T_2$ and it is isomorphic to the semigroup T_1.

Let now $\hat{\chi}$ be an arbitrary non-zero character of the semigroup $T = T_1 \times T_2$. The character $\hat{\chi}$ induces on the sub-semigroup $\{[x_1, e_2] \mid x_1 \in T_1\}$ a character $\hat{\psi}_1$ and on the sub-semigroup $\{[e_1, x_2] \mid x_2 \in T_2\}$ a character $\hat{\psi}_2$. As already remarked we have

$$\{[x_1, e_2] \mid x_1 \in T_1\} \cong T_1 , \quad \{[e_1, x_2] \mid x_2 \in T_2\} \cong T_2 ,$$

[10]) For if $\hat{\chi}\hat{\chi}'$ is the zero character of T, we have in particular $\hat{\chi}\hat{\chi}'([e_1, e_2]) = 0$, hence $\hat{\chi}([e_1, e_2]) \cdot \hat{\chi}'([e_1, e_2]) = 0$, i. e., $\chi_1(e_1) \cdot \chi_2(e_2) \cdot \chi_1'(e_1) \cdot \chi_2'(e_2) = 0$. If now, f. i., $\chi_1(e_1) = 0$, we have $\chi_1 = \chi_1^0$, hence $[\chi_1, \chi_2] \in N$. Similarly the other possibilities.

and these isomorphisms are realised by the orderings

$$[x_1, e_2] \in T_1 \times T_2 \longleftrightarrow x_1 \in T_1\,, \qquad [e_1, x_2] \in T_1 \times T_2 \longleftrightarrow x_2 \in T_2\,.$$

Put

$$\chi_1(x_1) = \hat{\psi}_1([x_1, e_2])\,, \qquad \chi_2(x_2) = \hat{\psi}_2([e_1, x_2])\,.$$

We have thus defined a character χ_1 of the semigroup T_1 and a character χ_2 of the semigroup T_2. For the given character $\hat{\chi}(x)$ we can write now

$$\hat{\chi}(x) = \hat{\chi}([x_1, x_2]) = \hat{\chi}([x_1, e_2] \,.\, [e_1, \; x_2]) = \hat{\chi}([x_1, e_2]) \,.\, \hat{\chi}([e_1, x_2]) =$$
$$= \hat{\psi}_1([x_1, e_2]) \,.\, \hat{\psi}_2([e_1, x_2]) = \chi_1(x_1) \,.\, \chi_2(x_2)\,,$$

i. e.

$$\hat{\chi}([x_1, x_2]) = \chi_1(x_1) \,.\, \chi_2(x_2)\,.$$

Hence every non-zero character $\hat{\chi}$ of the semigroup $T_1 \times T_2$ can be obtained by the construction described sub a). This proves our Theorem .

Remark. In the proof we did not use explicitly the continuity of the characters. Indeed, the theorem holds also if T^* (T_1^*, T_2^*) denotes the set of all complex-valued functions defined on T (T_1, T_2) and satisfying $\chi(ab) = \chi(a) \,.\, \chi(b)$ for all $a, b \in T$ (T_1, T_2). If of course χ_1, χ_2 are supposed to be continuous, so is $\hat{\chi}$ and conversely, and the modifications needed are obvious.

Example 5,1. Let S be the multiplicative semigroup of real numbers of the closed interval $\langle 0, 1\rangle$ with the ordinary topology on the real axis. We wish to find the set S^*.

We wish to find the set of all continuous complex-valued functions satisfying in $\langle 0, 1\rangle$ the functional equation

$$f(x_1) \,.\, f(x_2) = f(x_1 x_2)\,. \tag{4}$$

Consider first the open interval $(0, 1)$. Put $x = e^{-t}$. Then t runs through the open interval $(0, \infty)$. The equation (4) takes the form

$$f(e^{-t_1}) \,.\, f(e^{-t_2}) = f(e^{-(t_1+t_2)}), \qquad 0 < t_1\,, t_2 < \infty\,.$$

Put $f(e^{-t}) = g(t)$. We have $g(t_1 + t_2) = g(t_1) \,.\, g(t_2)$. It is known[11]) that if $g(t)$ is supposed to be continuous (or even only measurable), then we have either a) $g(t) \equiv 0$ in $(0, \infty)$ or $b)$ $g(t) = e^{\gamma t}$, where γ is any complex number.

The relation $g(t) \equiv 0$ leads to $f(x) \equiv 0$ in $(0,1)$.

The relation $g(t) = e^{\gamma t}$ gives $f(e^{-t}) = e^{\gamma t}$, i. e., $f(x) = e^{\delta \log x}$, where $\delta = -\gamma$ is any complex number. Write $\delta = \alpha + i\beta$ (α, β real). Then it is clear that

$$f(x) = e^{(\alpha+i\beta)\log x} = x^{\alpha+i\beta}$$

is a continuous solution of (4) in the open interval $(0, 1)$.

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[11]) See f. i. HILLE [3], p. 162—167, where analogous much deeper problems are treated or MAAK [12], p. 69—71, where a detailed discussion of our functional equation is given.

If $\alpha < 0$ $\lim\limits_{x=0+} x^{\alpha+i\beta}$ does not exist. If $\alpha = 0$ $\lim\limits_{x=0+} x^{i\beta}$ does not exist unless $\beta = 0$. Then $f(x) \equiv 1$ in $(0, 1)$ and $\lim\limits_{x=0+} f(x) = 1$. This gives $f(x) \equiv 1$ in $\langle 0, 1\rangle$. If $\alpha > 0$ $\lim\limits_{x=0+} x^{\alpha+i\beta} = 0$ and $f(x)$ is continuous in $\langle 0, 1\rangle$ if we define $f(0) = 0$.

Hence all characters of the semigroup S are:

a) $\chi(x) \equiv 0$ for all $x \in S$;

b) $\chi(x) \equiv 1$ for all $x \in S$;

c) $\chi(x) = \begin{cases} x^{\alpha+i\beta},\ \alpha > 0 & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$

Our semigroup contains only three closed prime ideals: $\varnothing$, S and $\{0\}$. Each of them is a generating prime ideal. $\mathfrak{S}_\varnothing$ and $\mathfrak{S}_s$ contain only one element (character). The semigroup $\mathfrak{S}_0$ contains exactly all functions defined sub c).

Hence $S^* = \mathfrak{S}_s + \mathfrak{S}_\varnothing + \mathfrak{S}_0$. The semigroup $\mathfrak{S}_0$ is isomorphic to the additive semigroup of vectors of the open right half-plane.

Remark: Our semigroup contains a further prime ideal $\langle 0, 1)$. But this is not closed.

Let us remark further: since $\langle 0, \varepsilon)$, $\varepsilon > 0$ is an ideal of S containing $\{0\}$, it is sufficient (according to Theorem 4,1) to consider in solving (4) only the interval $\langle 0, \varepsilon)$. This is really used in the calculations concerning the form of $g(t)$ not explicitly stated here.

Example 5,2. We have to find the characters of the multiplicative semigroup S of complex numbers $|z| \leq 1$, the topology being the ordinary topology on the plane.

Consider first the semigroup S_0 of complex numbers $0 < |z| \leq 1$. Every $z \in S_0$ can be written uniquely in the form $z = |z| \cdot e^{i\varphi}$, $0 \leq \varphi < 2\pi$. Hence we can write $S_0 \cong L \times K$, where L is the semigroup of real numbers of the interval $(0, 1\rangle$ and K the group of complex numbers with $|z| = 1$. It follows from the example 5,1 that L^* contains the function identically zero on L and all functions $f = x^{\gamma+i\beta}$ with an arbitrary complex number $\gamma + i\beta$. It is well-known that the characters of K are the functions $f_n(e^{i\varphi}) = e^{in\varphi}$, $n \gtreqless 0$ an integer and, of course, the function identically zero on K.[12]) According to Theorem 5,1 the set of non-zero characters of the semigroup S_0 is given by the set of functions

$$f(z) = |z|^{\gamma+i\beta} \cdot e^{in\varphi} = |z|^{\alpha+i\beta} \cdot z^n\,, \qquad \alpha = \gamma - n\,,$$

where $\alpha + i\beta$ is an arbitrary complex number and n an integer.

To find S^* it is sufficient to take those of them that can be continuously extended to $z = 0$.

If $\alpha + n > 0$ let us put

$$\chi(z) = \begin{cases} |z|^{\alpha+i\beta} z^n & \text{for } z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases}$$

These are clearly characters of S.

If $\alpha + n < 0$ $f(z)$ cannot be continuously extended to $z = 0$.

If $\alpha + n = 0$ we have for $z \neq 0$ $f(z) = |z|^{i\beta} \cdot e^{i\varphi n}$. For $\beta \neq 0$ $\lim_{|z|=0} |z|^{i\beta} \cdot e^{i\varphi n}$ does not exist. If $\beta = 0$ then $f(z) = \frac{z^n}{|z|^n}$. If $n \neq 0$ $\lim_{z=0} \frac{z^n}{|z|^n}$ does not exist. For $n = 0$ we get $f(z) \equiv 1$.

As the result we obtained: all characters of the semigroup S are

a) $\chi(z) = 0$ for all $z \in S$;

b) $\chi(z) = 1$ for all $z \in S$;

c) $\chi(z) = \begin{cases} 0 \text{ for } z = 0, \\ |z|^{\alpha + i\beta} \cdot z^n \text{ for } z \neq 0 \ (n \text{ an integer}, n + \alpha > 0, \beta \text{ real}). \end{cases}$

Example 5,3. Let S be the set of couples of real numbers $[a, b]$, $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$. Let the multiplication be defined by $[a_1 b_1,] \cdot [a_2, b_2] = [a_1 a_2, b_1 b_2]$. We may regard S as a subset of the plane. Let the topology be the ordinary topology in the plane. S is then a Hausdorff bicompact semigroup. We wish to find S^*.

There exist four idempotents: $e_1 = [0,0]$, $e_2 = [1, 0]$, $e_3 = [0, 1]$, $e_4 = [1, 1]$. The maximal semigroups are:

$$P_1 = \{[x, y] | 0 \leqq x < 1, 0 \leqq y < 1\},\ P_2 = \{[1, y] | 0 \leqq y < 1\},$$
$$P_3 = \{[x, 1] | 0 \leqq x < 1\},\ P_4 = \{[1, 1]\}.$$

The maximal groups shrink into the idempotents: $G_i = \{e_i\}$, $i = 1, \ldots, 4$.

There exist five closed prime ideals[13]):

$$\mathfrak{p}_1 = \{[x, 0] \mid 0 \leqq x \leqq 1\},\quad \mathfrak{p}_2 = \{[0, y] \mid 0 \leqq y \leqq 1\}\quad \mathfrak{p}_3 = \mathfrak{p}_1 + \mathfrak{p}_2,\quad \mathfrak{p}_\varnothing = \varnothing,$$
$$\mathfrak{p}_s = S.$$

Each of them is a generating prime ideal.

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[12]) See, f. i., Понтрягин [11], p. 252. If χ is a non-zero character of K, a any element $\in K$, we have necessarily $|\chi(a)| = 1$. Hence our definition of characters coincides with that used in the theory of groups. Proof. Suppose that for a non-zero character an dan element $a \in K$ $|\chi(a)| = c < 1$. This would imply $|\chi(a^n)| = c^n < 1$. The set $\{a, a^2, a^3, \ldots\}$ contains a unique idempotent. This is necessarily the element $z = 1$. With respect to the continuity of χ there cannot hold $\chi(1) = 1$, hence $\chi(1) = 0$. But then the relation $b = b \cdot 1$ (valid for every $b \in K$) implies $\chi(b) = \chi(b) \cdot \chi(1) = 0$, i. e. χ is the zero character of K, contrary to the supposition.

[13]) There exist also other prime ideals in S. For instance the sets

$$\{[x, y] \mid 0 \leqq x < 1,\ 0 \leqq y \leqq 1\},\ \{[x, y] \mid 0 \leqq x \leqq 1,\ 0 \leqq y < 1\},\ \{[x, y] \mid xy \neq 1\}.$$

But these prime ideals are not closed and cannot be therefore a priori generating prime ideals.

Put

$$\varphi_{\alpha\beta}(x, y) = \begin{cases} y^{\alpha+i\beta} \text{ for } y \neq 0\,, \\ 0 \text{ for } y = 0\,. \end{cases} \qquad \psi_{\alpha\beta}(x, y) = \begin{cases} x^{\alpha+i\beta} \text{ for } x \neq 0\,, \\ 0 \quad \text{ for } x = 0\,. \end{cases}$$

$$\chi_{\alpha,\beta,\gamma,\delta}(x, y) = \begin{cases} x^{\alpha+i\beta} \,.\, y^{\gamma+i\delta} \text{ for } xy \neq 0\,, \\ 0 \qquad \text{ for } xy = 0\,. \end{cases}$$

Then the semigroups $\mathfrak{S}_{\mathfrak{p}}$ are given by the following sets:
$\mathfrak{S}_{\mathfrak{p}_1} = \{\varphi_{\alpha\beta}(x, y) \mid \alpha > 0, \beta \text{ real}\}$, $\quad \mathfrak{S}_{\mathfrak{p}_2} = \{\psi_{\alpha\beta}(x, y) \mid \alpha > 0, \beta \text{ real}\}$,
$\mathfrak{S}_{\mathfrak{p}_3} = \{\chi_{\alpha\beta\gamma\delta}(x, y) \mid \alpha > 0, \gamma > 0, \beta, \delta \text{ real}\}$,
$\mathfrak{S}_{\emptyset} = \{\text{the function} \equiv 1 \text{ on } S\}$, $\quad \mathfrak{S}_s = \{\text{the function} \equiv 0 \text{ on } S\}$.
It is therefore

$$S^* = \mathfrak{S}_{\emptyset} + \mathfrak{S}_{\mathfrak{p}_1} + \mathfrak{S}_{\mathfrak{p}_2} + \mathfrak{S}_{\mathfrak{p}_3} + \mathfrak{S}_s\,.$$

Example 5,4. Let Q be the Hilbert cube, i. e. the set of all points $x = [x_1, x_2, x_3, \ldots]$ of the Hilbert space for which $0 \leqq x_n \leqq 2^{-n}$. The topology is given by means of the distance function $\varrho(x, y)$ of the Hilbert space. Let the product of two points $x = [x_1, x_2, x_3, \ldots]$ and $y = [y_1, y_2, y_3, \ldots]$ be defined by $x \odot y = [x_1y_1, x_2y_2, x_3y_3, \ldots]$. One proves easily that Q is a bicompact semigroup.[14])

Denote by $\mathfrak{p}_i$ $(i = 1, 2, \ldots)$ the set of all points $\in Q$ of the form $[x_1, x_2, \ldots \ldots, x_{i-1}, 0, x_{i+1}, \ldots]$, $0 \leqq x_n \leqq 2^{-n}$ for $n \neq i$ (the sides of the Hilbert cube). Every $\mathfrak{p}_i$ is a closed prime ideal of Q. For the product of two points is contained in $\mathfrak{p}_i$ if and only if at least one of them has its i-th coordinate equal to zero.

[14]) Since Q is a bicompact space (see Alexandrov [13], p. 300) we have only to show that the multiplication is continuous. To this end it is sufficient to prove that for every $\varepsilon > 0$ there exist neighbourhoods $U(x)$, $U(y)$ such that for every $\bar{x} \in U(x)$, $\bar{y} \in U(y)$, $\varrho(x \odot y, \bar{x} \odot \bar{y}) < \varepsilon$. We shall show that it is sufficient to choose for $U(x)$, $U(y)$ the set of all points contained in Q and satisfying $\varrho(x, \bar{x}) < \frac{\varepsilon}{2}$ and $\varrho(y, \bar{y}) < \frac{\varepsilon}{2}$ respectively. Let us choose $\bar{x}, \bar{y}$ in the described manner. Then we have

$$\varrho(x \odot y, \bar{x} \odot y) = \left[\sum_{i=1}^{\infty} (\bar{x}_i y_i - x_i y_i)^2\right]^{\frac{1}{2}} \leqq \left[\sum_{i=1}^{\infty} (\bar{x}_i - x_i)^2 \,.\, \sum_{i=1}^{\infty} y_i^2\right]^{\frac{1}{2}} = \varrho(\bar{x}, x) \,.\, \sqrt{\sum_{i=1}^{\infty} y_i^2}\,.$$

Similarly we have $\varrho(\bar{x} \odot y, \bar{x} \odot \bar{y}) \leqq \varrho(\bar{y}, y) \,.\, \sqrt{\sum_{i=1}^{\infty} \bar{x}_i^2}$. Since for every point $x \in Q$

$$\sum_1^{\infty} x_n^2 \leqq \sum_1^{\infty} 2^{-2n} = \tfrac{1}{3},$$

we get

$$\varrho(x \odot y, \bar{x} \odot \bar{y}) \leq \varrho(\bar{x} \odot \bar{y}, \bar{x} \odot y) + \varrho(\bar{x} \odot y, x \odot y) \leqq \varrho(\bar{y}, y) \,.\, \sqrt{\sum_1^{\infty} \bar{x}_n^2} + $$

$$+ \varrho(\bar{x}, x) \sqrt{\sum_1^{\infty} y_n^2} < \frac{\varepsilon}{2} \frac{1}{\sqrt{3}} + \frac{\varepsilon}{2} \frac{1}{\sqrt{3}} < \varepsilon, \text{ q. e. d.}$$

Our semigroup has *only one idempotent* (namely $[0, 0, 0, \ldots]$) but it contains an *infinity* of closed prime ideals $\mathfrak{p}_i$. A sum of a finite number of them $P_{i_1, i_2, \ldots, i_k} = \mathfrak{p}_{i_1} + \mathfrak{p}_{i_2} + \ldots + \mathfrak{p}_{i_k}$ is again a closed prime ideal.

Every prime ideal $\mathfrak{p}_i$ and every prime ideal of the form $P_{i_1, i_2, \ldots, i_k}$ is a generating prime ideal. For instance the function $f(x) = f([x_1, x_2, x_3, \ldots]) = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k}$ is a character of Q vanishing just on $P_{i_1, i_2, \ldots i_k}$.

The set $\mathfrak{S}_{\mathfrak{p}_k}$ is the totality of all functions $\chi_{\alpha + i\beta}(x)$ of the following form

$$\chi_{\alpha+i\beta}(x) = \begin{cases} 0 & \text{for } x_k = 0, \\ x_k^{\alpha+i\beta} & \text{for } x_k \neq 0, \end{cases}$$

where $\alpha + i\beta$ is a complex number with $\alpha > 0$.

The set $\mathfrak{S}_{P_{i_1, i_2, \ldots, i_k}}$ is the totality of all functions of the form

$$\chi_{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k}(x) = \begin{cases} 0 & \text{for} \quad x_{i_1} \cdot x_{i_2} \ldots x_{i_k} = 0, \\ x_{i_1}^{\alpha_1 + i\beta_1} \cdot x_{i_2}^{\alpha_2 + i\beta_2} \cdot \ldots \cdot x_{i_k}^{\alpha_k + i\beta_k} & \text{for } x_{i_1} \cdot x_{i_2} \ldots x_{i_k} \neq 0, \end{cases}$$

where $\alpha_j + i\beta_j$ are complex numbers with $\alpha_j > 0$ $(j = 1, 2, \ldots, k)$.

The set $P = \sum_{i=1}^{\infty} \mathfrak{p}_i$ is clearly a prime ideal of Q. We show that P is dense in Q. It is sufficient to prove that in every neighbourhood of the point $x = [x_1, x_2, x_3, \ldots] \in Q$ there exists at least one point $y = [y_1, y_2, y_3, \ldots] \in \mathfrak{p}_i$ for a suitable i. Let $U(x)$ be a neighbourhood of x consisting of all $z \in Q$ with $\varrho(x, z) < \varepsilon < \frac{1}{2}$. If at least one of the coordinates of x is zero, then $x \in P$ and there is nothing to prove. Suppose therefore that $x_i \neq 0$ for every $i = 1, 2, \ldots$. Find an n such that $\frac{1}{2^n} < \varepsilon$ and consider the point $y = [x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots] \neq x$. It is $y \in P$ and $\varrho(x, y) = \sqrt{x_n^2} \leqq \sqrt{2^{-2n}} < \varepsilon$, which proves our assertion.

Since $\overline{P} = Q$ and we have clearly $P \neq Q$, the prime ideal P is not closed and therefore it cannot be a generating prime ideal.

Remark: One proves similarly that an arbitrary sum of an infinite number of the prime ideals $P' = \sum_{k=1}^{\infty} \mathfrak{p}_{i_k}$ is dense in Q and P' cannot be therefore a generating prime ideal.[14a])

[14a]) I am indebted to prof. E. Hewitt for the following remark to this example. The following general theorem holds. Let S be a bicompact Hausdorff semigroup. Let A be a collection (of continuous) characters of S, that contains with every pair of characters their product, contains the conjugate of a character along with a character, and also contains the function identically 1. Then, suppose further that if there exists a character on S that assumes distinct values at x and y in S, then there is a character χ in A such that $\chi(x) \neq \chi(y)$. Then A consists of all characters of S. This theorem can be applied to the semigroup Q of our Example 5,4. Every non-zero character $\neq \chi_1$ of S is a finite product of characters of the form $\chi(x) = x_i^\gamma$, where γ is a complex number with positive real part.

Let $\mathfrak{P}$ be the set of all generating prime ideals of the semigroup S. $\mathfrak{P}$ is not vacuous since $\emptyset$ and S are contained in it. Introduce in $\mathfrak{P}$ a partial ordering with respect to the inclusion.

The set $\mathfrak{P}$ has the property that with every two elements $\mathfrak{p}_1$ and $\mathfrak{p}_2$ also their sum $\mathfrak{p}_1 + \mathfrak{p}_2$ belongs to $\mathfrak{P}$.[15])

If in every intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ there existed a unique maximal closed (and generating) prime ideal (eventually equal to $\emptyset$), then the set $\mathfrak{P}$ would form a lattice with the obvious operations $\wedge$, $\vee$. This is the case if the number of generating prime ideals is finite.[16]) In general it holds only:

Lemma 6,1. *With respect to the inclusion the partially ordered set $\mathfrak{P}$ of generating prime ideals forms a semi-lattice with the greatest element S. The lattice operation is here the sum of sets.*

If $\mathfrak{p}_\beta \subseteqq \mathfrak{p}_\alpha$, then $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$.[17]) For every character $\chi \in \mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta}$ vanishes clearly just on $\mathfrak{p}_\alpha$. Conversely, if $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, every character $\chi = \chi_\alpha \cdot \chi_\beta$ ($\chi_\alpha \in \mathfrak{S}_{\mathfrak{p}_\alpha}$, $\chi_\beta \in \mathfrak{S}_{\mathfrak{p}_\beta}$) vanishes just on $\mathfrak{p}_\alpha$ [18]), hence we have necessarily $\mathfrak{p}_\beta \subseteqq \mathfrak{p}_\alpha$.

Let us introduce in the system of sets $\mathfrak{S} = \{\mathfrak{S}_{\mathfrak{p}_\alpha}\}$[19]) a binary relation $\leqq$ by means of the following definition:

let be $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\beta}$ whenever $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$ holds. (5)

We have introduced thus a partial ordering. For

a) $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\alpha} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, hence $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\alpha}$.

b) Let be $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\beta}$, $\mathfrak{S}_{\mathfrak{p}_\beta} \leqq \mathfrak{S}_{\mathfrak{p}_\gamma}$. Then $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \leqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, i. e., $\mathfrak{p}_\alpha \supseteqq \mathfrak{p}_\beta$ and $\mathfrak{S}_{\mathfrak{p}_\beta} . \mathfrak{S}_{\mathfrak{p}_\gamma} \subseteqq \mathfrak{S}_{\mathfrak{p}_\beta}$, i. e., $\mathfrak{p}_\beta \supseteqq \mathfrak{p}_\gamma$. We have therefore $\mathfrak{p}_\alpha \supseteqq \mathfrak{p}_\gamma$, hence $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\gamma} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, i. e., $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\gamma}$.

c) $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\beta}$, $\mathfrak{S}_{\mathfrak{p}_\beta} \leqq \mathfrak{S}_{\mathfrak{p}_\alpha}$ imply $\mathfrak{p}_\alpha \supseteqq \mathfrak{p}_\beta$, $\mathfrak{p}_\beta \supseteqq \mathfrak{p}_\alpha$, hence $\mathfrak{p}_\alpha = \mathfrak{p}_\beta$ and $\mathfrak{S}_{\mathfrak{p}_\alpha} = \mathfrak{S}_{\mathfrak{p}_\beta}$.

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[15]) The set $\mathfrak{p}_1 + \mathfrak{p}_2$ is clearly a closed prime ideal. Further, if χ_1 vanishes just on $\mathfrak{p}_1$ and χ_2 vanishes just on $\mathfrak{p}_2$, then $\chi_1\chi_2$ vanishes just on $\mathfrak{p}_1 + \mathfrak{p}_2$. Hence $\mathfrak{p}_1 + \mathfrak{p}_2$ is a generating prime ideal. We have seen in the Example 5,4 that this theorem need not hold for an infinite number of summands.

[16]) By means of Zorn's Lemma one proves easily the existence of a unique maximal prime ideal contained in $\mathfrak{p}_1 \cap \mathfrak{p}_2$. But it is not possible (without further assumptions) to prove that this prime ideal is closed and generating.

[17]) The multiplication of complexes means here multiplication of sets of characters (elements of S^*).

[18]) Otherwise there would hold namely $\mathfrak{p}_\beta + \mathfrak{p}_\alpha \supset \mathfrak{p}_\alpha$ and the characters $\in \mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta}$ would vanish just on $\mathfrak{p}_\alpha + \mathfrak{p}_\beta \neq \mathfrak{p}_\alpha$.

[19]) We write $\mathfrak{S}$ instead of S^* to make clear that the elements of $\mathfrak{S}$ are semigroups of characters. In the set-theoretical sense we have, of course, $S^* = \sum_\alpha \mathfrak{S}_{\mathfrak{p}_\alpha}$.

Since $\mathfrak{p}_\alpha \subseteqq \mathfrak{p}_\beta \Leftrightarrow \mathfrak{S}_{\mathfrak{p}_\beta} \leqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, we get with respect to the one-to-one correspondence $\mathfrak{p}_\alpha \longleftrightarrow \mathfrak{S}_{\mathfrak{p}_\alpha}$ the following result:

Theorem 6,1. *The semi-lattice of generating prime ideals partially ordered by inclusion and the system of sets $\{\mathfrak{S}_{\mathfrak{p}_\alpha}\}$ partially ordered by* (5) *are anti-isomorphic.*

Remark 1. In this anti-isomorphism there corresponds to the prime ideal $\varnothing$ the group $\mathfrak{S}_\varnothing$ (having the unit character χ_1 of S as the unit element). To the prime ideal S there corresponds the group $\mathfrak{S}_s = \{\chi_0\}$ (containing only one element, namely the zero character χ_0 of S). The element $\mathfrak{S}_\varnothing$ is the greatest and the element $\mathfrak{S}_s$ the least element of the partially ordered system $\mathfrak{S}$.

Remark 2. If $\mathfrak{S}_{\mathfrak{p}_\beta} \,.\, \mathfrak{S}_{\mathfrak{p}_\gamma} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, then $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\beta}$ and $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\gamma}$. For the last relation implies $\mathfrak{p}_\beta + \mathfrak{p}_\gamma = \mathfrak{p}_\alpha$, hence $\mathfrak{p}_\beta \subseteqq \mathfrak{p}_\alpha$, therefore $\mathfrak{S}_{\mathfrak{p}_\alpha} \,.\, \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha}$, i. e., $\mathfrak{S}_{\mathfrak{p}_\alpha} \leqq \mathfrak{S}_{\mathfrak{p}_\beta}$. Analogously for the second "inequality".

We resume the obtained results concerning the structure of the semigroup S^*. We know that $S^* = \sum\limits_\alpha \mathfrak{S}_{\mathfrak{p}_\alpha}$, where the summands are disjoint, i. e., for $\mathfrak{p}_\alpha \neq \mathfrak{p}_\beta$ we have $\mathfrak{S}_{\mathfrak{p}_\alpha} \cap \mathfrak{S}_{\mathfrak{p}_\beta} = \varnothing$. Two of the summands ($\mathfrak{S}_\varnothing$ and $\mathfrak{S}_s$) are always groups.

Each of the semigroups $\mathfrak{S}_{\mathfrak{p}_\alpha}$ has a further property: in $\mathfrak{S}_{\mathfrak{p}_\alpha}$ the cancellation law holds. Let $\chi_\alpha, \chi'_\alpha, \chi''_\alpha$ be three elements $\epsilon\, \mathfrak{S}_{\mathfrak{p}_\alpha}$. Suppose that $\chi_\alpha\chi'_\alpha = \chi_\alpha\chi''_\alpha$, i. e. for every $a \,\epsilon\, S$ $\chi_\alpha(a) \,.\, \chi'_\alpha(a) = \chi_\alpha(a) \,.\, \chi''_\alpha(a)$. For $a \,\epsilon\, S - \mathfrak{p}_\alpha$ $\chi_\alpha(a) \neq 0$, hence $\chi'_\alpha(a) = \chi''_\alpha(a)$. For $a \,\epsilon\, \mathfrak{p}_\alpha$ $\chi'_\alpha(a) = \chi''_\alpha(a) = 0$. Hence for every $a \,\epsilon\, S$ holds $\chi'_\alpha(a) = \chi''_\alpha(a)$, i. e., $\chi'_\alpha = \chi''_\alpha$.

Remark. The semigroup S^* has the property that it can be written as a sum of disjoint sub-semigroups and in each of them the cancellation law holds. Semigroups having these properties form a very special type of semigroups. A finite semigroup having these properties is a sum of disjoint (maximal) groups. This follows from the fact that a finite semigroup in which the cancellation law holds is a group. Hence in the finite case S^* is a sum of disjoint groups (see [7]).

If S is connected we get further information about the sets $\mathfrak{S}_{\mathfrak{p}_\alpha}$ by means of the following lemma:

Lemma 6,2. *Let S be connected. Then there exist two and only two idempotent characters: the unit character χ_1 and the zero character χ_0.*

Proof. Let χ be an idempotent character. It is sufficient to show that in the decomposition $S = J + Q$ formed by χ (in the sense of Theorem 3,1) either $J = \varnothing$ or $Q = \varnothing$. For if $J = \varnothing$ we have $S = \varnothing$ and $\chi(x) = \chi_1(x)$; if $Q = \varnothing$ $S = J$, hence $\chi(x) = \chi_0(x)$.

Suppose $\chi \neq \chi_0$, $\chi \neq \chi_1$, $S = J + Q$, $J \neq \varnothing$. Since S is connected and J open, it follows from the supposition $Q \neq \varnothing$ necessarily $\bar{J} \cap Q \neq \varnothing$. Let be $a \,\epsilon\, \bar{J} \cap Q$ and $U(a)$ any neighbourhood of a. There holds $\chi(a) = 1$ but for some $b \,\epsilon\, U(a) \cap J \neq \varnothing$ $\chi(b) = 0$. (An idempotent character can assume only two values: 0 and 1.) This gives a contradiction with the continuity of the function χ. Hence $Q = \varnothing$. This proves our lemma.

If S is connected none of the semigroups $\mathfrak{S}_{\mathfrak{p}_\alpha}$, $\mathfrak{p}_\alpha \neq \varnothing$, $\mathfrak{p}_\alpha \neq S$, can be a group since according to Lemma 6,2 $\mathfrak{S}_{\mathfrak{p}_\alpha}$ does not contain an idempotent. At the same time it follows that $\mathfrak{S}_{\mathfrak{p}_\alpha}$ cannot be finite since — according to the remark above — it would then be a group, which is impossible.[20])

We have proved the following theorem:

Theorem 6,2. *The semigroup S^* can be written as a class-sum of disjoint semigroups $S^* = \sum_\alpha \mathfrak{S}_{\mathfrak{p}_\alpha}$. In each of the semigroups $\mathfrak{S}_{\mathfrak{p}_\alpha}$ the cancellation law holds. Two of the semigroups $\mathfrak{S}_{\mathfrak{p}_\alpha}$, namely $\mathfrak{S}_\varnothing$ and $\mathfrak{S}_s$, are always groups. If S is connected each of the remaining semigroups has an infinity of elements and none of them is a group.*

Consider the product $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta}$. We know that $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_{\mathfrak{p}_\alpha + \mathfrak{p}_\beta}$. Suppose that there exist two generating prime ideals $\mathfrak{p}_\alpha$, $\mathfrak{p}_\beta$ such that $\mathfrak{p}_\alpha + \mathfrak{p}_\beta = S$. Then $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} \subseteqq \mathfrak{S}_s = \{\chi_0\}$, i. e. S^* contains zero divisors.Conversely, suppose that there exist zero divisors in S^* and let be $\chi_\beta \cdot \chi_\alpha = \chi_0$. Put $\mathfrak{p}_\alpha = \{x \mid x \in S, \chi_\alpha(x) = 0\}$, $\mathfrak{p}_\beta = \{x \mid x \in S, \chi_\beta(x) = 0\}$. Then $\mathfrak{p}_\beta + \mathfrak{p}_\alpha$ is the set of all $x \in S$ for which $\chi_\alpha\chi_\beta(x)$ is zero. According to the supposition this set is the whole S. Hence $\mathfrak{p}_\alpha + \mathfrak{p}_\beta = S$. We have the following result: *The semigroup S^* has zero divisors if and only if S contains two different generating prime ideals $\mathfrak{p}_\alpha$, $\mathfrak{p}_\beta$ such that $\mathfrak{p}_\alpha + \mathfrak{p}_\beta = S$.*

A maximal generating prime ideal is a prime ideal $\mathfrak{q} \neq S$ having the following properties: a) $\mathfrak{q}$ is a generating prime ideal, b) there does not exist a generating prime ideal $\mathfrak{p}$ with $\mathfrak{q} \subset \mathfrak{p} \subset S$.

If $\mathfrak{p}_\alpha$, $\mathfrak{p}_\beta$ are two different maximal generating prime ideals, then we have necessarily $\mathfrak{p}_\alpha + \mathfrak{p}_\beta = S$. Hence if S contains two different maximal generating prime ideals, then S^* has divisors of zero. If S has only a finite number of generating prime ideals there exists always at least one maximal generating prime ideal and every generating prime ideal can be embedded in a maximal generating prime ideal. This implies: *Suppose that S has only a finite number of generating prime ideals. Then S^* has not zero divisors if and only if S contains a unique maximal generating prime ideal.*

In general we can say only: if S has a unique maximal generating prime ideal in which every other generating prime ideal is contained, then S^* has not zero divisors.

Remark. In general it is not true that every generating prime ideal can be embedded in a maximal generating prime ideal. We shall show this on Exam-

[20]) If S is connected none of the semigroups $\mathfrak{S}_{\mathfrak{p}_\alpha}$ ($\neq \mathfrak{S}_\varnothing$, $\neq \mathfrak{S}_s$) can be bicompact in any topology. For it can be proved (see [10]) that a bicompact (commutative) semigroup in which the cancellation law holds is a group.

ple 5,4. The proof follows indirectly. We use the notations introduced in this example. Suppose that the semigroup $Q = S$ has a maximal generating prime ideal $\mathfrak{q}$. The prime ideal $\mathfrak{q}$ cannot contain all prime ideals $\mathfrak{p}_i$ $(i = 1, 2, 3, \ldots)$. For otherwise we have $P \subseteqq \mathfrak{q} \subset S$ and since $\overline{P} = S$ we have $\bar{\mathfrak{q}} = S$, i. e., $\bar{\mathfrak{q}} \neq \mathfrak{q}$ and $\mathfrak{q}$ would not be closed. Without loss of generality suppose $\mathfrak{p}_1$ non $\in \mathfrak{q}$. Since $\mathfrak{q}$ and $\mathfrak{p}_1$ are generating prime ideals $\mathfrak{q} + \mathfrak{p}_1$ would be also a generating prime ideal. With respect to the maximality of $\mathfrak{q}$ there must hold $\mathfrak{q} + \mathfrak{p}_1 = S$. Hence $\mathfrak{S}_{\mathfrak{q}} \cdot \mathfrak{S}_{\mathfrak{p}_1} = \{\chi_0\}$. Take the following character $\in \mathfrak{S}_{\mathfrak{p}_1}$: $\chi(x) = \chi([x_1, x_2, x_3, \ldots]) = x_1$. Then there must exist a character $\varphi(x) \in \mathfrak{S}_{\mathfrak{q}}$ such that for every point $x = [x_1, x_2, x_3, \ldots] \in Q$ $\varphi(x) \cdot x_1 = 0$. Consider the set R of all points $x \in S$ for which $x_1 \neq 0$, i. e., the set $x = [x_1, x_2, x_3, \ldots]$, $0 < x_1 \leqq \frac{1}{2}$, $0 \leqq x_n \leqq 2^{-n}$ for $n \geqq 2$. On R we have necessarily $\varphi(x) = 0$. But R is dense in S. With respect to the continuity we have $\varphi(x) \equiv 0$ on the whole S. Hence there would hold $\mathfrak{q} = S$. This is a contradiction. Our semigroup has not a maximal generating prime ideal.

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Let $\mathfrak{p}$ be a generating prime ideal. Let us put the question: under what conditions is the semigroup $\mathfrak{S}_{\mathfrak{p}}$ a *group?*

If $\mathfrak{S}_{\mathfrak{p}}$ is a group, then there must exist a continuous function χ which vanishes just on $\mathfrak{p}$ and is equal to 1 on $S - \mathfrak{p}$. (Since an idempotent character assumes only the values 0 and 1.) In our usual notations there must hold therefore $S - \mathfrak{p} = S - J$, hence $\mathfrak{p} = J$. But J is open. Therefore $\mathfrak{p}$ must be open.

Conversely, let $\mathfrak{p}$ be an arbitrary prime ideal which is both open and closed in S. The semigroup $Q = S - \mathfrak{p}$ is closed and bicompact. We know (see the remarks in section 4 above, or [5]) that the set of characters of Q with values of absolute value unity forms a group. Let us denote this group by $[Q^*]_1$.[21]) This group is non-vacuous since it contains at least the unit character of Q, i. e., the function $\psi_1(x) \equiv 1$ on Q.

Construct now a function $\chi(x)$ defined on S in the following manner. Choose a character $\psi(x) \in [Q^*]_1$ and put

$$\chi(x) = \begin{cases} 0 & \text{for } x \in \mathfrak{p}\ , \\ \psi(x) & \text{for } x \in Q\ . \end{cases} \tag{6}$$

This function is a character of S. (It is continuous since ψ is continuous on Q.) $\mathfrak{p}$ is therefore a generating prime ideal of S. In this manner we obtain all characters $\in \mathfrak{S}_{\mathfrak{p}}$. Further it is clear that $\mathfrak{S}_{\mathfrak{p}} \cong [Q^*]_1$, and since $[Q^*]_1$ is a group, $\mathfrak{S}_{\mathfrak{p}}$ is also a group.

Thus we have proved

Theorem 7,1. *The semigroup $\mathfrak{S}_{\mathfrak{p}}$ is a group if and only if $\mathfrak{p}$ is both open and*

[21]) We cannot write Q^* since Q itself has — in general — also other characters then those which vanish nowhere on Q.

closed. To every prime ideal $\mathfrak{p}$ of S which is both open and closed there exists a group of characters vanishing just on $\mathfrak{p}$. Every character $\epsilon\ \mathfrak{S}_{\mathfrak{p}}$ is of the form (6).

The theorem just proved implies:

Theorem 7,2. *The semigroup S^* is a class-sum of disjoint groups if and only if every generating prime ideal of S is both open and closed.*

In particular:

Corollary 7,2. *Let S be finite. Then S^* is a sum of disjoint groups.*

We know that in the decomposition $S^* = \sum_{\alpha} \mathfrak{S}_{\mathfrak{p}_\alpha}$ two summands $\mathfrak{S}_{\varnothing}$ and $\mathfrak{S}_s$ are always groups. With respect to Theorem 6,2 we get:

Theorem 7,3. *Let S be connected. Then S^* is a sum of groups if and only if S has only the two trivial generating prime ideals $\varnothing$ and S.*

An important type of semigroups to which we have just been led and that is often treated in the literature are semigroups that can be written as a sum of groups. They are also called "semigroups admitting relative inverses". In such a semigroup every maximal semigroup P_α is identical with the maximal group G_α. In such a semigroup every ideal is a sum of groups. Hence every closed prime ideal has the form $\mathfrak{p} = \sum_{\alpha} \mathrm{P}_\alpha$. If $\mathfrak{p}$ is a generating prime ideal, we have therefore $\mathfrak{p} = J$. A generating prime ideal is therefore open. Hence $\mathfrak{S}_{\mathfrak{p}}$ is a group. If S is connected, there exist, of course, only two (trivial) generating prime ideals. We have proved:

Theorem 7,4. *Let S be a* (*commutative bicompact*) *semigroup admitting relative inverses. Then S^* is always a sum of disjoint groups. If S is connected then there exist only two group-components and $S^* = \mathfrak{S}_{\varnothing} + \mathfrak{S}_s$ holds.*

Remark. It follows also from our previous results that if S is the sum of groups and $\chi \in S^*$, then $|\chi(x)| = 0$ or 1 for all $x \in S$.

If S is such that every generating prime ideal is both open and closed then the structure of S^* can be fully described. This will be the content of Theorems 7,5 and 7,6.

Definition 7,1. *Suppose that S is such that every generating prime ideal is both open and closed. An idempotent e_α will be called a generating idempotent if the ideal J_α belonging to e_α in the correspondence* (1) *$J_\alpha \longleftrightarrow e_\alpha$ of section 2 is closed.*

Let us assign to every generating idempotent e_α an idempotent character ε_α by the following definition:

$$\varepsilon_\alpha(x) = \begin{cases} 0 & \text{for } x \in J_\alpha\,, \\ 1 & \text{for } x \in S - J_\alpha\,. \end{cases}$$

It is clear that we obtain in this manner all non-zero idempotent characters $\epsilon\ S^*$. The set of generating idempotents can again be partially ordered by the statement $e_\alpha \leqq e_\beta \Leftrightarrow e_\alpha e_\beta = e_\alpha$. Analogously, the set of all idempotent characters can be partially ordered by the implication $\varepsilon_\alpha \varepsilon_\beta = \varepsilon_\alpha \Leftrightarrow \varepsilon_\alpha \leqq \varepsilon_\beta$.

If $e_\alpha < e_\beta$, then Theorem 2,2 implies $J_\alpha \subset J_\beta$. The equations

$$\varepsilon_\alpha(x) = \begin{cases} 0 & \text{for } x \in J_\alpha, \\ 1 & \text{for } x \in S - J_\alpha, \end{cases} \qquad \varepsilon_\beta(x) = \begin{cases} 0 & \text{for } x \in J_\beta, \\ 1 & \text{for } x \in S - J_\beta, \end{cases}$$

imply $\varepsilon_\alpha \varepsilon_\beta = \varepsilon_\beta$, i. e., $\varepsilon_\beta < \varepsilon_\alpha$. Hence the two partially ordered sets of idempotents are anti- isomorphic.[21a])

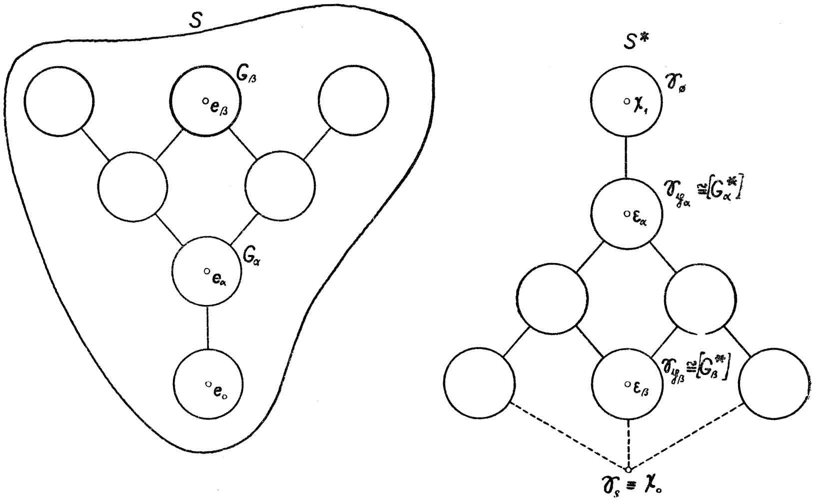

Fig. 2.

Let e_α be a generating idempotent, G_α the corresponding maximal group. The set of all characters vanishing just on J_α forms a group. This group is isomorphic to the group $[G_\alpha^*]$ of non-zero characters of the group G_α. Hence $\mathfrak{S}_{\mathfrak{p}_\alpha} \cong [G_\alpha^*]$. But $\mathfrak{S}_{\mathfrak{p}_\alpha}$ is exactly the group-component contained in S^* whose unit element is the idempotent character ε_α. Hence:

Theorem 7,5. *Let S be such a semigroup that every of its generating prime ideals is both open and closed. Then the partially ordered set of generating idempotents $\in S$ and the partially ordered set of non-zero idempotent characters $\in S^*$ are anti--isomorphic. If in this anti-isomorphism $e_\alpha \in S \longleftrightarrow \varepsilon_\alpha \in S^*$, then $\mathfrak{S}_{\mathfrak{p}_\alpha} \cong [G_\alpha^*]$, where $[G_\alpha^*]$ is the set of all non-zero characters of the maximal group G_α.*

[21a]) It should be noted that if e_α, e_β are generating idempotents, $e_\alpha e_\beta$ need not be a generating idempotent. If in our anti-isomorphism $e_\alpha \longleftrightarrow \varepsilon_\alpha$, $e_\beta \longleftrightarrow \varepsilon_\beta$ and $\varepsilon_\alpha \varepsilon_\beta \neq \chi_0$, then $\varepsilon_\alpha \varepsilon_\beta \longleftrightarrow e_\gamma$, where e_γ is the idempotent that in the correspondence (1) of section 2 corresponds to the ideal $J_\gamma = J_\alpha + J_\beta \neq S$. J_γ is a generating prime ideal which is both open and closed, hence e_γ is a generating idempotent. (e_γ is clearly the least idempotent contained in $S - (J_\alpha + J_\beta)$.) If $\varepsilon_\alpha \varepsilon_\beta = \chi_0$, then $J_\alpha + J_\beta = S$ and there exists no idempotent $\in S$ to which $\varepsilon_\alpha \varepsilon_\beta$ correspond.

Remark. The situation is schematically described (in a case with a finite number of idempotents) in fig. 2. Here e_0 denotes the least idempotent ϵS.

In the special case treated here Theorem 7,5 gives a full information concerning the "*gross structure*" of S^*. We shall obtain also further information concerning the "*fine structure*" of S^*. This means: we shall show how to obtain the product $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta}$ by means of characters of some subgroups $\subseteqq S$. In the proofs of the corresponding theorems we shall use analogous methods as in section VI of the paper [7], p. 241—244. Of course, we must take into consideration the topological suppositions.

Let $\mathfrak{S}_{\mathfrak{p}_\alpha}$, $\mathfrak{S}_{\mathfrak{p}_\beta}$ be two groups of S^* with ε_α, ε_β as unit elements. Let $\varepsilon_\alpha \varepsilon_\beta = \varepsilon_\gamma \neq \chi_0$.[21b]) Then

$$\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} = (\mathfrak{S}_{\mathfrak{p}_\alpha} \varepsilon_\alpha) . (\mathfrak{S}_{\mathfrak{p}_\beta} . \varepsilon_\beta) = (\mathfrak{S}_{\mathfrak{p}_\alpha} \varepsilon_\gamma) . (\mathfrak{S}_{\mathfrak{p}_\beta} \varepsilon_\gamma) .$$

Since $\varepsilon_\gamma \leqq \varepsilon_\alpha$, $\varepsilon_\gamma \leqq \varepsilon_\beta$, we have $e_\alpha \leqq e_\gamma$, $e_\beta \leqq e_\gamma$, hence $\mathfrak{p}_\alpha \subseteqq \mathfrak{p}_\gamma$, $\mathfrak{p}_\beta \subseteqq \mathfrak{p}_\gamma$.

Every character $\chi \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma$ is a character of S just vanishing on $\mathfrak{p}_\gamma$, hence $\mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma \subseteqq \mathfrak{S}_{\mathfrak{p}_\gamma}$. In particular we have $\varepsilon_\gamma \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma$. Further $\mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma$ is a group.[22]) Hence $\mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma$ and $\mathfrak{S}_{\mathfrak{p}_\beta} . \varepsilon_\gamma$ are subgroups of $\mathfrak{S}_{\mathfrak{p}_\gamma}$.

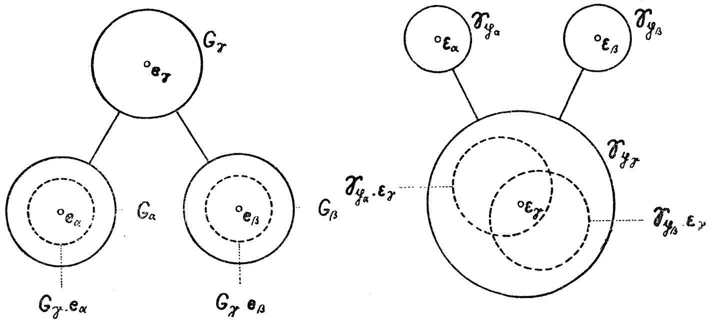

Fig. 3.

The product $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta}$ will be known as soon as we shall know the structure of the groups $\mathfrak{S}_{\mathfrak{p}_\alpha} . \varepsilon_\gamma$, $\mathfrak{S}_{\mathfrak{p}_\beta} . \varepsilon_\gamma$. This question is solved by the following theorem.

[21b]) Without loss of generality we may suppose $\varepsilon_\gamma \neq \chi_0$ (hence $\mathfrak{p}_\gamma \neq S$) since otherwise we have $\mathfrak{S}_{\mathfrak{p}_\alpha} . \mathfrak{S}_{\mathfrak{p}_\beta} = \{\chi_0\}$ and there is nothing more to investigate.

[22]) To prove that $\mathfrak{S}_{\mathfrak{p}_\alpha} \varepsilon_\gamma$ is a group it is sufficient to show that for every $\chi' \varepsilon_\gamma \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha} \varepsilon_\gamma$ $(\chi' \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha})$ the equation $\xi . \chi' \varepsilon_\gamma = \varepsilon_\gamma$ has a solution $\xi \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha} \varepsilon_\gamma$. Find a $\chi'' \epsilon \mathfrak{S}_{\mathfrak{p}_\alpha}$ such that $\chi' \chi'' = \varepsilon_\alpha$ and put $\xi = \chi'' \varepsilon_\gamma$. Then it holds indeed $(\chi'' \varepsilon_\gamma)(\chi' \varepsilon_\gamma) = \chi' \chi'' \varepsilon_\gamma = \varepsilon_\alpha \varepsilon_\gamma = \varepsilon_\gamma$.

Theorem 7,6. *Suppose that in the anti-isomorphism of Theorem 7,5 the correspondences* $\varepsilon_\gamma \longleftrightarrow e_\gamma$, $\varepsilon_\beta \longleftrightarrow e_\beta$ *hold. Suppose* $e_\beta < e_\gamma$ *(hence* $\varepsilon_\beta > \varepsilon_\gamma$*). Then there holds*

$$\mathfrak{S}_{\mathfrak{p}_\beta} \,.\, \varepsilon_\gamma \cong [(G_\gamma e_\beta)^*] \,.$$

Proof. We have proved above

$$\mathfrak{S}_{\mathfrak{p}_\beta} \,.\, \varepsilon_\gamma \subseteqq \mathfrak{S}_{\mathfrak{p}_\gamma} \cong [G_\gamma^*] \,.$$

Since $e_\beta < e_\gamma$, we have $\mathfrak{p}_\beta \subset \mathfrak{p}_\gamma$. Every character $\epsilon\, \mathfrak{S}_{\mathfrak{p}_\beta}\varepsilon_\gamma$ induces on G_γ a character $\varphi(x)$ of the group G_γ. Of course, we obtain thus only some of the characters of G_γ.[23])

Consider the homomorphic mapping of the group G_γ into the group $G_\gamma e_\beta$:[24])[25])

$$x \,\varepsilon\, G_\gamma \to x e_\beta \,\epsilon\, G_\gamma e_\beta \subseteqq G_\beta \,. \tag{7}$$

Since $G_\gamma e_\beta$ is a continuous image of G_γ, $G_\gamma e_\beta$ is a bicompact semigroup. Hence (7) is an open homomorphic mapping of G_γ into $G_\gamma e_\beta$[26]). The kernel of this homomorphism is the set C of those $x \,\epsilon\, G_\gamma$ that are mapped into the element e_β, i. e., the set of those x for which $x e_\beta = e_\beta$. The set C is a closed subgroup of G_γ and the topological[26']) group $G_\gamma e_\beta$ is isomorphic to the topological group $G_\gamma \mid C$[27]). Therefore

$$G_\gamma e_\beta \cong G_\gamma \,/\, C \,,$$

where the isomorphism denotes an isomorphism of topological groups .

Write the decomposition of G_γ mod C in the form $G_\gamma = \sum\limits_\nu c_\nu C$, where one of the elements c_ν is e_γ. Then $(c_\nu C)\, e_\beta = c_\nu(C e_\beta) = c_\nu e_\beta$. Denoting $c_\nu e_\beta = b_\nu$ we get clearly $G_\gamma e_\beta = \sum\limits_\nu (c_\nu e_\beta) = \sum\limits_\nu b_\nu$, where one of the elements b_ν is e_β.

We w'sh to show now that *the set* $\mathfrak{S}_{\mathfrak{p}_\beta}\varepsilon_\gamma$ *is formed by those and only those of the characters* $\chi \,\epsilon\, \mathfrak{S}_{\mathfrak{p}_\gamma}$ *for which* $\chi(C) = 1$ *holds.*

---

[23]) Each of these characters $\varphi(x)$ has the special property that it can be extended to a character of S just vanishing on $\mathfrak{p}_\beta$.

[24]) The set $G_\gamma e_\beta$ is a group (subgroup of G_β). For, it has a unit element $e_\beta \,\epsilon\, G_\gamma e_\beta$, and if $c e_\beta \,\epsilon\, G_\gamma e_\beta$, $c \,\epsilon\, G_\gamma$ is any element $\epsilon\, G_\gamma e_\beta$, the equation $\xi \,.\, c e_\beta = e_\beta$ has a solution $\xi \,\varepsilon$ $\varepsilon\, G_\gamma e_\beta$.

[25]) The mapping is a homomorphism since if $x \to x e_\beta$, $y \to y e_\beta$, then $xy \to xy\, e_\beta =$ $= x e_\beta \,.\, y e_\beta$.

[26]) We use the theorem: a homomorphic mapping of a bicompact group into an other bicompact group is an open mapping. (See Понтрягин [11], p. 123.)

[26a]) It must be remarked that we can use the notion "topological group" since the following theorem is known: If G is a group and a bicompact Hausdorff space and if multiplication is continuous in both variables, then G is a topological group, that is x^{-1} is a continuous function of x. (This fact has independently been proved recently several times. See f. i. [14].)

[27]) See, f. i. Понтрягин [11], p. 121, Theorem 11.

a) Let $\chi_2 \in \mathfrak{S}_{\mathfrak{p}_\beta}$. Since $Ce_\beta = e_\beta$ and $\chi_2(e_\beta) = 1$, we get $\chi_2(Ce_\beta) = \chi_2(e_\beta)$, $\chi_2(C) \cdot \chi_2(e_\beta) = \chi_2(e_\beta)$, $\chi_2(C) = 1$. Every character $\in \mathfrak{S}_{\mathfrak{p}_\beta}\varepsilon_\gamma$ is of the form $\chi_2\varepsilon_\gamma$, $\chi_2 \in \mathfrak{S}_{\mathfrak{p}_\beta}$. Therefore $\chi_2\varepsilon_\gamma(C) = \chi_2(C) \cdot \varepsilon_\gamma(C) = 1 \cdot 1 = 1$. This shows that every character $\in \mathfrak{S}_{\mathfrak{p}_\beta}\varepsilon_\gamma$ has the value 1 on the whole set C.

b) Suppose, conversely, that χ_1 is any character $\in \mathfrak{S}_{\mathfrak{p}_\gamma}$ satisfying $\chi_1(C) = 1$. We shall show that $\chi_1 \in \mathfrak{S}_{\beta_\mathfrak{p}}\varepsilon_\gamma$.

The character χ_1 induces on G_γ a character $\varphi(x)$ of the group G_γ. For $x \in G_\gamma$ we have $\varphi(x) = \chi_1(x)$. Further

$$\varphi(c_\nu C) = \varphi(c_\nu) \cdot \varphi(C) = \varphi(c_\nu) .$$

Hence $\varphi(c_\nu)$ is (in the usual sense) a character of the factor group $G_\gamma \mid C$)[28]. Define a character $\psi(x)$ of the group $G_\gamma e_\beta$ in the following manner: if $b_\nu = c_\nu e_\beta$ let be $\psi(b_\nu) = \varphi(c_\nu)$. Since in the mapping $b_\nu \longleftrightarrow c_\nu$ the topological groups $G_\gamma \mid C$ and $G_\gamma e_\beta$ are isomorphic, $\psi(x)$ can be considered as a character of the group $G_\gamma e_\beta \subseteqq G_\beta$.

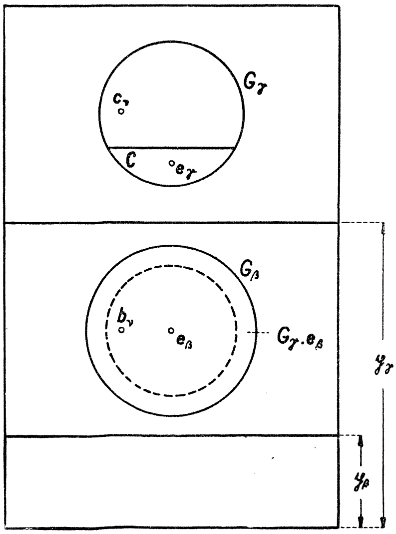

Fig. 4.

If $G_\gamma e_\beta = G_\beta$, $\psi(x)$ is a character of the whole group G_β. If $G_\gamma e_\beta \subset G_\beta$, it is known[29]) that there exists a continuation of the character $\psi(x)$ to the whole group G_β. This extended character will be denoted by $\psi'(x)$. (If the continuation is not necessary let $\psi'(x)$ mean $\psi(x)$.)

Construct a character $\chi_2 \in \mathfrak{S}_{\mathfrak{p}_\beta}$ that for $z \varepsilon G_\beta$ satisfies $\chi_2(z) = \psi'(z)$. This is possible since e_β is the least idempotent of the bicompact semigroup $S - \mathfrak{p}_\beta$. (See [5].) In particular for $x \in G_\gamma e_\beta$ $\psi'(x) = \psi(x) = \chi_2(x)$, hence $\psi(b_\nu) = \chi_2(b_\nu)$.

With respect to the definition of the function $\psi(x)$, $\varphi(c_\nu) = \psi(b_\nu)$ and therefore $\varphi(c_\nu) = \chi_2(b_\nu)$. For the set $c_\nu C$ (i. e. every element $\in c_\nu C$) we have

$$\chi_2(c_\nu C) = \chi_2(c_\nu C) \cdot \chi_2(e_\beta) = \chi_2(c_\nu)\, \chi_2(Ce_\beta) = \chi_2(c_\nu) \cdot \chi_2(e_\beta) = \chi_2(c_\nu e_\beta) = \chi_2(b_\nu) =$$
$$= \varphi(c_\nu) = \varphi(c_\nu C) .$$

Hence the character χ_2 assumes on G_γ the same values as the character $\varphi(x)$, i. e., the same values as the character χ_1.

The character $\chi_2\varepsilon_\gamma \in \mathfrak{S}_{\mathfrak{p}_\beta}\varepsilon_\gamma \subseteqq \mathfrak{S}_{\mathfrak{p}_\gamma}$ vanishes just on $\mathfrak{p}_\gamma$ and on the group G_γ (the maximal group belonging to the least idempotent $\in S - \mathfrak{p}_\gamma$) has the same

[28]) See Понтрягин [11], p. 248, Theorem 37.

[29]) See Понтрягин [11], p. 258, Theorem 42.

values as the character $\chi_1 \in \mathfrak{S}_{\mathfrak{p}_\gamma}$. Hence (according to the paper [5]) we have $\chi_1 = \chi_2 \cdot \varepsilon_\gamma$, hence $\chi_1 \in \mathfrak{S}_{\mathfrak{p}_\beta} \cdot \varepsilon_\gamma$. This proves our assertion.

We now complete easily the proof of Theorem 7,6.

Let $\varphi(x)$ be any character of the group G_γ which is equal 1 on C (i. e., any character of the factor group $G_\gamma \mid C$). Then the function

$$h(x) = \begin{cases} 0 & \text{for } x \in \mathfrak{p}_\gamma\,, \\ \varphi(xe_\gamma) & \text{for } x \in S - \mathfrak{p}_\gamma\,, \end{cases}$$

is a character $\in \mathfrak{S}_{\mathfrak{p}_\gamma}$ which is 1 on C. Every character with this property can be obtained in this manner by taking a suitable function φ. (See [5].). For two different characters φ_1, φ_2 we get two different characters h_1, h_2. This means: the set of all such characters, i. e., $\mathfrak{S}_{\mathfrak{p}_\beta} \cdot \varepsilon_\gamma$ is isomorphic[30]) to the group of non zero characters of $G_\gamma \mid C$. But $G_\gamma \mid C \cong G_\gamma e_\beta$, hence $\mathfrak{S}_{\mathfrak{p}_\beta} \varepsilon_\gamma \cong [(G_\gamma e_\beta)^*]$. This proves Theorem 7,6.

8

The purpose of this section is to establish conditions that a given closed prime ideal be generating. It should be remarked in advance that the necessary and sufficient condition stated below is of a very general nature and does not much help in concrete ("practical") examples.

Let $\chi \neq \chi_0$ be a character satisfying for at least one $a \in S$ the condition $|\chi(a)| \neq 1$. Then there exists necessarily an element $b \in S$ such that $|\chi(b)| = 0$. Such an element is, for instance, the idempotent e_α to which a belongs.

Since S is bicompact the continuous function $|\chi(x)|$ assumes on S its maximum M. There exists therefore an element $c \in S$ such that $|\chi(c)| = M$. Clearly $0 < M \leqq 1$. This maximum can be of course < 1. This is the case if and only if $Q = \varnothing$.

Lemma 8,1. *Let* $\max\limits_{x \in S} |\chi(x)| = M > 0$ *and for at least one* $z \in S$ $|\chi(z)| \neq 1$. *Let* m *be a real number such that* $0 < m \leqq M$ *and* J_m *the set of all elements* $\in S$ *for which* $|\chi(x)| < m$. *Then* J_m *is an open ideal in* S.

Proof. a) Let $a \in J_m$, i. e., $|\chi(a)| < m$. Let $s \in S$. Then $|\chi(as)| = |\chi(a)|\,|\chi(s)| < 1 \cdot m$, hence $as \in J_m$. The set J_m is an ideal.

b) The function $|\chi(x)|$ being continuous, the set of all $a \in S$ with $|\chi(a)| < m$ is open. Hence J_m is open, which completes the proof.

Lemma 8,2. *Under the same suppositions as in Lemma* 8,1 *let* H_m *denote the set of all* $x \in S$ *satisfying* $|\chi(x)| \leqq m$. *Then* H_m *is a closed ideal of* S.

Proof. a) Similarly as in Lemma 8,1 it follows that H_m is an ideal of S.

[30]) Here, of course, we mean an isomorphism of algebraic groups since we have not introduced a topology into S^*.

b) The function $|\chi(x)|$ is continuous. Therefore the set of all $a \in S$ with $|\chi(a)| \leqq m$ is closed. Hence H_m is closed, which completes the proof.

Remark 1. The Lemma remains valid also for $m = 0$ if we write (in our usual notation) $H_0 = \mathfrak{p}$.

Remark 2. For $m > 0$ we have clearly $J_m \subseteqq H_m$, hence $\bar{J}_m \subseteqq H_m$. In general we cannot say that $\bar{J}_m = H_m$. For let, e. g., S be finite. Then for every character of the form treated in Lemma 8,1 we have $M = 1$. For J_1 (i. e., all $x \in S$ satisfying $|\chi(x)| < 1$) we get $J_1 = \mathfrak{p} \neq \varnothing$, whereas $H_1 = S$. Since $\mathfrak{p} \subset S$, we have $H_1 \subset J_1$. Of course, one can show on simple examples that $\bar{J}_m = H_m$ is possible.

Lemma 8,3. *Let the suppositions of Lemma 8,1 be satisfied. Let $0 < m_1 < m_2 \leqq \leqq M$. Then $\bar{J}_{m_1} \subseteqq J_{m_2}$.*

Proof. Clearly $H_{m_1} \subseteqq J_{m_2}$ (since H_{m_1} is the set of all $a \in S$ satisfying $|\chi(a)| \leqq \leqq m_1$ and J_{m_2} the set of all $a \in S$ satisfying $|\chi(a)| < m_2$). With respect to the Remark 2 above we have $J_{m_1} \subseteqq \bar{J}_{m_1} \subseteqq H_{m_1} \subseteqq J_{m_2}$, q. e. d.

Remark 1. If S is connected, we have more precisely $\bar{J}_{m_1} \subset J_{m_2}$. Proof: Since $|\chi|$ is continuous, S connected and $|\chi|$ assumes on S the values 0 and M, there exists a $c \in S$ with $m_1 < |\chi(c)| = \dfrac{m_1 + m_2}{2} < m_2 < M$. Therefore $c \in J_{m_2}$ but c non $\bar{J}_{m_1}$. Hence $\bar{J}_{m_1} \subset J_{m_2}$.

Remark 2. The assumption of connectedness is, of course, essential. For a finite semigroup we have always for $0 < m_1 < m_2 < 1$, $J_{m_1} = \bar{J}_{m_1} = J_{m_2}$. Another instructive example is given by the multiplicative semigroup of the following complex numbers with the ordinary topology in the plane: $S = \{z \mid |z| \leqq \frac{1}{4}\} + \{z \mid \frac{1}{3} \leqq |z| \leqq \frac{1}{2}\} + \{z \mid |z| = 1\}$. S is a non-connected bicompact semigroup. Let $\chi(z) = z$; then for all m_1, m_2 with $\frac{1}{4} < m_1 < m_2 \leqq \frac{1}{3}$ $J_{m_1} = J_{m_2}$ holds.

Definition 8,1. *We shall say that the set of ideals $\mathfrak{R} = \{Y_t, 0 \leqq t \leqq \delta, \delta > 0\}$ of S forms a χ-chain of ideals if to every real number t of the interval $\langle 0, \delta\rangle$ there corresponds an ideal $Y_t \in \mathfrak{R}$ such that the set of these ideals has the following properties:*

a) *Y_0 is a closed prime ideal,*
b) *for $0 < t \leqq \delta$ Y_t is an open ideal,*
c) *for $0 \leqq t_1 < t_2 \leqq \delta$ $\bar{Y}_{t_1} \subseteqq Y_{t_2}$ holds.*

Remark: For $t_1 \neq t_2$ we do not require $Y_{t_1} \neq Y_{t_2}$.

Definition 8,2. *Let $\mathfrak{R} = \{Y_t, 0 \leqq t \leqq \delta, \delta > 0\}$ be a χ-chain of ideals of S. Let $a \in Y_\delta$. Let t_a be the lower bound of real numbers τ for which $a \in Y_\tau$ holds. The number t_a will be called the index of the element a with respect to the χ-chain $\mathfrak{R}$. We shall write $t_a = \operatorname{ind}_{\mathfrak{R}} a$.*

For $a \in Y_0$ we have $\operatorname{ind}_{\mathfrak{R}} a = 0$.

In the following theorem we give a necessary and sufficient condition that a prime ideal $\mathfrak{p} \neq \emptyset$ be generating. We know that the prime ideal $\emptyset$ is always generating; therefore we restrict the considerations to the case $\mathfrak{p} \neq \emptyset$.

Theorem 8,1. *The necessary and sufficient condition that a prime ideal $\mathfrak{p} \neq \emptyset$ be generating is the fulfilment of the following condition. There exists a χ-chain of ideals of S $\mathfrak{R} = \{Y_t, 0 \leqq t \leqq \delta, \delta > 0\}$ such that*

a) $Y_0 = \mathfrak{p}$,

b) *for all $a, b \in Y_\delta$ holds* $\mathrm{ind}_{\mathfrak{R}}\, a \,.\, \mathrm{ind}_{\mathfrak{R}}\, b = \mathrm{ind}_{\mathfrak{R}}\, (ab)$.

Proof. 1. We show that the condition is necessary. Let $\mathfrak{p} \neq \emptyset$ be a generating prime ideal and χ a character vanishing just on $\mathfrak{p}$. Let us put $J_0 = \mathfrak{p}$ and for $0 < m \leqq M = \max\limits_{c \in S} |\chi(c)|$, $J_m = \{a \mid a \in S, |\chi(a)| < m\}$. Then Lemma 8,1 and 8,3 imply that $\mathfrak{R} = \{J_m, 0 \leqq m \leqq M, M > 0\}$ is a χ-chain of ideals of S.

We prove that $\mathrm{ind}_{\mathfrak{R}}\, a = |\chi(a)|$. Let $\mathrm{ind}_{\mathfrak{R}}\, a = \tau$ and $|\chi(a)| = \varrho$. According to the definition of the sets J_m and the index τ we have a) for $t' < \tau$ a non $\in J_{t'}$, i. e., $|\chi(a)| \geqq t'$, b) for $t'' > \tau$ $a \in J_{t''}$, i. e., $|\chi(a)| < t''$. Suppose first that $\varrho > \tau$. Choose $t'' = \dfrac{\varrho + \tau}{2} > \tau$. Then $\varrho = |\chi(a)| < \dfrac{\varrho + \tau}{2} < \varrho$, which is a contradiction. Suppose on the other hand that $\varrho < \tau$. Choose $t' = \dfrac{\varrho + \tau}{2} < \tau$. Then $\dfrac{\varrho + \tau}{2} \leqq |\chi(a)| = \varrho$, i. e., $\tau \leqq \varrho$, which is again a contradiction. Hence it is necessarily $|\chi(a)| = \varrho = \tau$.

Let now a, b be two arbitrary elements $\in S$. Then

$$\mathrm{ind}_{\mathfrak{R}}\, a \,.\, \mathrm{ind}_{\mathfrak{R}}\, b = |\chi(a)| \,.\, |\chi(b)| = |\chi(ab)| = \mathrm{ind}_{\mathfrak{R}}\, (ab)\,,$$

q. e. d.

2. We shall show that the condition is sufficient. With respect to Theorem 4,1 and 4,2 it is sufficient to prove the existence of a character on an arbitrary ideal containing $\mathfrak{p}$ as a proper subset. We shall show its existence on the ideal Y_δ.

Let $\mathfrak{R} = \{Y_t, 0 \leqq t \leqq \delta, \delta > 0\}$ be a χ-chain of ideals satisfying the conditions of Theorem 8,1. If $Y_0 = Y_t$ for a t with $0 < t \leqq \delta$, then Y_0 is both open and closed. According to Theorem 7,1 Y_0 is a generating prime ideal and there is nothing more to prove. In the following we can therefore suppose that $Y_t \supset Y_0$ for every $t > 0$, in particular $Y_\delta \supset Y_0$.

Construct a real function $\chi(x)$ defined on Y_δ in the following manner: if $a \in Y_\delta$ belongs with respect to the χ-chain $\mathfrak{R}$ to the index τ, let us put $\chi(a) = \tau$. (In particular for $b \in Y_0$ let us put $\chi(b) = 0$.)

We shall prove that this function $\chi(x)$ is a character of the semigroup Y_δ. To this end it is sufficient to prove that $\chi(a) \,.\, \chi(b) = \chi(ab)$ and that $\chi(x)$ is continuous on Y_δ.

Let be $\chi(a) = \tau$, $\chi(b) = \sigma$. According to the supposition ab belongs to the index $\tau\sigma$, hence $\chi(ab) = \tau\sigma$. This gives $\chi(ab) = \chi(a) \,.\, \chi(b)$.

It remains to show the continuity of χ. Let $\chi(a) = \tau$. We must prove that for every $\varepsilon > 0$ there exists a neighbourhood $U(a)$ such that for every $x \in U(a)$ $|\chi(x) - \tau| < \varepsilon$. Let be $0 < \tau < \delta$. Consider the ideals $Y_{\tau-\varepsilon}$, $Y_{\tau+\varepsilon}$. Clearly $a \in Y_{\tau+\varepsilon}$ and a non $\in \overline{Y}_{\tau-\varepsilon}$. (According to the supposition we have namely $\overline{Y}_{\tau-\varepsilon} \subseteqq Y_{\tau-\frac{\varepsilon}{2}}$ and a is not contained in $Y_{\tau-\frac{\varepsilon}{2}}$ since otherwise its index $\leqq \tau - \frac{\varepsilon}{2}$.) Thus $a \in Y_{\tau+\varepsilon} - \overline{Y}_{\tau-\varepsilon}$. Since this is an open set we can construct a neighbourhood $U(a)$ such that $U(a) \subseteqq Y_{\tau+\varepsilon} - \overline{Y}_{\tau-\varepsilon}$. For every $x \in U(a)$ the number $\mathrm{ind}_{\mathfrak{R}}\, x$ satisfies clearly $\tau - \varepsilon < \mathrm{ind}_{\mathfrak{R}}\, x < \tau + \varepsilon$, hence $|\mathrm{ind}_{\mathfrak{R}}\, x - \tau| < \varepsilon$, and $|\chi(x) - \chi(a)| < \varepsilon$, q. e. d.

If $\tau = 0$ it is sufficient to consider a neighbourhood contained in Y_ε, $\varepsilon > 0$. The modifications needed in the case $\tau = \delta$ are obvious. This completes the proof of Theorem 8,1.

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Резюме

ТЕОРИЯ ХАРАКТЕРОВ КОММУТАТИВНЫХ ХАУСДОРФОВЫХ БИКОМПАКТНЫХ ПОЛУГРУПП

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Характером полугруппы S называем непрерывную комплексную функцию $\chi(x)$, определенную на S и удовлетворяющую соотношению $\chi(ab) = \chi(a) \cdot \chi(b)$ для каждой пары $a, b \in S$. Изучением характеров конечных полугрупп занимался автор в работах [7], [8], [9]. Целью настоящей работы является обобщение разработанной там теории на случай хаусдорфовых бикомпактных полугрупп. В дальнейшем S означает всюду полугруппу указанного типа.

Множество всех характеров полугруппы S образует — при обычном определении умножения характеров — новую полугруппу S^*. Проблема состоит в исследовании строения полугруппы S^*.

В параграфе 1 коротко повторены некоторые результаты, касающиеся строения полугруппы S, которые были подробно доказаны в работе [6]. Пусть a — произвольный элемент $\in S$. Построим множество $A = \{a, a^2, a^3, \ldots\}$. Замыкание $\bar{A}$ содержит один и только один идемпотент e_α. Говорим, что a принадлежит к идемпотенту e_α. Множество всех $x \in S$, принадлежащих к одному идемпотенту e_α, образует полугруппу P_α, которую называем максимальной полугруппой, принадлежащей к идемпотенту e_α. Полугруппу S можно писать в виде суммы дизъюнктных слагаемых $S = \sum_\alpha P_\alpha$. Далее, к каждому идемпотенту e_α существует одна единственная максимальная группа $G_\alpha \subseteqq P_\alpha$, единицей которой является идемпотент e_α.

Множество всех идемпотентов $\in S$ можно частично упорядочить, если положить $e_\alpha \leqq e_\beta \Leftrightarrow e_\alpha \cdot e_\beta = e_\alpha$. Тогда это множество образует т. наз. полуструктуру, в которой операция $\wedge$ определяется соотношением $e_\alpha \wedge e_\beta = e_\alpha \cdot e_\beta$. Для нашей цели существенно то, что всякая хаусдорфова бикомпактная полугруппа имеет всегда лишь один наименьший идемпотент.

Идеалом полугруппы S называем множество I, удовлетворяющее условию $SI \subseteqq I$. Идеал I называем простым идеалом, если $S - I$ есть полугруппа. Выгодно считать простыми идеалами в S и пустое множество $\emptyset$, и всю полугруппу S. В параграфе 2 освящено строение открытых простых идеалов полугруппы S. Возьмем произвольный идемпотент $e_\alpha \in S$. Найдем все идемпотенты e_η, для которых $e_\alpha e_\eta \neq e_\alpha$. Построим все соответствующие максимальные полугруппы P_η. Тогда $I = \sum_\eta P_\eta$ есть открытый простой идеал полугруппы S. (Если e_α — наименьший идемпотент $\in S$, то $I = \emptyset$.) Таким образом можно получить любой из открытых простых идеалов $I \neq S$ полугруппы S.

Таким образом, каждому идемпотенту e_α соответствует одно-однозначно какой-то открытый простой идеал I_α. Относительно включения частично упорядоченная полуструктура открытых простых идеалов $\neq S$ и полуструктура всех идемпотентов $\in S$ взаимно изоморфны.

Пусть χ — характер полугруппы S. Для каждого $x \in S$ и для каждого χ $|\chi(x)| \leqq 1$.

Пусть χ — фиксированный характер. Множество $a \in S$ всех элементов $\in S$, для которых $|\chi(a)| < 1$, есть открытый простой идеал из S. Множество всех элементов $a \in S$, для которых $\chi(a) = 0$, есть замкнутый простой идеал полугруппы S.

Пусть, наоборот, $\mathfrak{p}$ — замкнутый простой идеал из S. Тогда, в общем, не существует характера полугруппы S, который равнялся бы нулю точно на $\mathfrak{p}$. Поэтому вводится следующее определение: простой идеал $\mathfrak{p}$ полугруппы S называем производящим, если а) $\mathfrak{p}$ есть замкнутый идеал в S, б) существует по крайней мере один характер χ полугруппы S, равный нулю точно на $\mathfrak{p}$. (Простые идеалы $\emptyset$ и S являются всегда производящими.)

Если $\mathfrak{p}$ означает производящий простой идеал, то символом $\mathfrak{S}\mathfrak{p}$ обозначим полугруппу всех характеров, которые равны нулю точно на $\mathfrak{p}$.

,,Грубое" строение полугруппы S^* можно теперь описать следующим образом: полугруппу S^* можно выразить в виде множественной суммы дизюнктных полугрупп $S^* = \sum_\alpha \mathfrak{S}\mathfrak{p}_\alpha$, где $\mathfrak{p}_\alpha$ пробегает все производящие простые идеалы полугруппы S. В каждой полугруппе $\mathfrak{S}_{\mathfrak{p}_\alpha}$ имеет место правило сокращения. Два из слагаемых, а именно $\mathfrak{S}_\Phi$ и $\mathfrak{S}_s$, суть всегда группы. Группа $\mathfrak{S}_\Phi$ является как раз группой характеров, которые на всем S удовлетворяют соотношению $|\chi(x)| = 1$. (Строение этой группы и ее отношение к полугруппе S были исследованы в работе [5].) Если S связно, то среди слагаемых $\mathfrak{S}\mathfrak{p}_\alpha$ не существует уже никаких других групп.

Возникает вопрос: когда $\mathfrak{S}\mathfrak{p}_\alpha$ будет группой? Полугруппа $\mathfrak{S}_{\mathfrak{p}_\alpha}$ является группой, если простой идеал $\mathfrak{p}_\alpha$ является открыто-замкнутым множеством. Наоборот: всякий открыто-замкнутый простой идеал $\mathfrak{p}_\alpha$ является производящим, и соответствующее $\mathfrak{S}\mathfrak{p}_\alpha$ есть группа. В этом случае можно дать более подробные сведения о строении группы $\mathfrak{S}\mathfrak{p}_\alpha$. Обозначим через e_α наименьший идемпотент замкнутой полугруппы $S - \mathfrak{p}_\alpha$, а через G_α — максимальную группу, принадлежащую к идемпотенту e_α. Тогда $\mathfrak{S}\mathfrak{p}_\alpha$ изоморфна группе характеров группы G_α, отличных от нуля.

Полугруппа S^* является суммой групп тогда и только тогда, если каждый производящий простой идеал полугруппы S есть открыто-замкнутое множество. Этот случай наступает всегда, когда имеем дело с конечными полугруппами. Далее, сюда относится каждая полугруппа, которая сама является суммой групп.

Идемпотент полугруппы e_α назовем производящим идемпотентом, если соответствующий ему открытый простой идеал является одновременно замкнутым простым идеалом. Справедлива следующая теорема: Пусть S — полугруппа, все производящие простые идеалы которой представляют открыто-замкнутые множества. Тогда частично упорядоченное множество производящих идемпотентов в S и частично упорядоченное множество отличных от нуля идемпотентных характеров полугруппы S^* антиизоморфны. (При этом частичное упорядочение в этих множествах определено аналогичным образом, как и выше.) Если же в этом антиизоморфизме будет $e_\alpha \in S \longleftrightarrow \varepsilon_\alpha \in S^*$, то $\mathfrak{S}\mathfrak{p}_\alpha$ изоморфна группе ненулевых характеров максимальной группы G_α, принадлежащей к идемпотенту e_α.

Множества $\mathfrak{S}\mathfrak{p}_\alpha$, $\mathfrak{S}\mathfrak{p}_\beta$ всегда удовлетворяют соотношению $\mathfrak{S}\mathfrak{p}_\alpha \cdot \mathfrak{S}\mathfrak{p}_\beta \subseteqq \mathfrak{S}\mathfrak{p}_\alpha + \mathfrak{p}_\beta$. Если же будем рассматривать только что описанный специальный случай (когда каждый производящий простой идеал — открыто-замкнутое множество), то строение произведения $\mathfrak{S}\mathfrak{p}_\alpha . \mathfrak{S}\mathfrak{p}_\beta$ можно охарактеризовать при помощи характеров определенных подгрупп полугруппы S. Это возможно, по-существу, на основании следующей теоремы: Пусть в только что указанном антиизоморфизме $\varepsilon_\gamma \longleftrightarrow e_\gamma$, $\varepsilon_\beta \longleftrightarrow e_\beta$ и $e_\beta < e_\gamma$; тогда $\mathfrak{S}\mathfrak{p}_\beta \varepsilon_\gamma$ есть группа, изоморфная группе характеров группы $G_\gamma e_\beta$.

В параграфе 5 рассматривается несколько примеров с целью показать всевозможные разновидности строения полугруппы S^*.

В параграфе 8 найдено необходимое и достаточное условие для того, чтобы данный замкнутый простой идеал был производящим.

Кроме приведенных выше результатов доказывается в работе целый ряд общих теорем, касающихся свойств характеров полугруппы S и далее теоремы, касающиеся строения полугруппы S^* и ее связи с полугруппой S.