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Czechoslovak Mathematical Journal, Vol. 5 (1955), No. 4, 532–545

Persistent URL: http://dml.cz/dmlcz/100169

Чехословацкий математический журнал, т. 5 (80) 1955

TWO REMARKS ON WEAK COMPACTNESS

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(Received June 11, 1955.)

It is the purpose of the present paper to show that a method developed by the author in [6] may be applied without essential modifications to obtain generalizations of theorem (2,1) of [6] and of a recent result of E. E. Floyd and V. L. Klee.

According to a wellknown theorem [1], a closed convex subset B of a complete convex topological linear space is weakly compact if and only if every decreasing sequence of nonvoid closed convex subsets $C_n \subset B$ has a nonempty intersection. In a recent paper [3], E. E. Floyd and V. L. Klee have established a similar result with the weaker assumption that C_n be closed linear manifolds. In a paper devoted to the investigation of the substance of theorems of the Eberlein type, we have developed a method of proof which is particularly well adapted to the discussion of questions concerning intersections of linear manifolds. It is the purpose of the present paper to show that the method mentioned may be applied without essential modifications to obtain a still more general result, which includes also that of [6]; the proof is considerably simpler. The simplification of the proof is only a formal one, however.

The paper is divided into two sections. In the first part, we prove a theorem analogous to that of [6] for sets which fulfil a certain condition (see condition (C) below) concerning countable intersections of hypervarieties. This result is valid even for nonconvex sets. In the second section we prove a similar theorem for convex sets. The additional assumption of convexity enables us to replace condition (C) by a still weaker one. The main idea of the proof rests the same, a technical difficulty occuring here is overcome by a simple trick shaped after the example of [3].

The notion of an almost continuous functional has been introduced in [5]. This paper and [6] should be consulted as far as terminology and notations are concerned.

§ 1.

If X is a linear space, we shall denote by $F(X)$ the linear space of all linear forms defined on X (the *dual algébrique* in the terminology of J. Dieudonné). If f is a nonzero linear form defined on X and β a given real number, we shall denote by $H(f; \beta)$ the set of all $x \in X$ for which $f(x) = \beta$. A set of this form will be called a hypervariety, the term hyperplane being reserved for hypervarieties of the form $H(f; 0)$. We shall write frequently $Z(f)$ for $H(f; 0)$.

If X is a convex topological linear space, we shall denote by $L(X)$ the space of all linear functionals defined on X. If X is a given convex topological linear space, we denote by R the space $F(L(X))$ equipped with the weak topology corresponding to $L(X)$. The space X (taken in its weak topology) may be considered as a subspace of R.

If B is an arbitrary bounded subset of X, it is easy to see that the closure of B in R is compact. This fact will be frequently used and will not be repeated hereafter.

Now let a convex topological linear space X be given. We shall be concerned with subsets $B \subset X$ which fulfil one of the following two conditions.

(C) *If $r \in R$ lies in the closure of B, and $y_1, y_2, \ldots,$ is an arbitrary sequence of linear functionals on X, then there exists a point $b \in B$ such that $by_j = ry_j$ for every j.*

(H) *If H_n is a sequence of closed hypervarieties in X such that $B \cap H_1 \cap \cap \ldots \cap H_n$ is nonvoid for every natural n, then there exists a point $b \in B$ which belongs to every H_n.*

First of all it is easy to see that both conditions (C) and (H) are consequences of a condition analogous to (H) but with hypervarieties replaced by closed halfspaces (i. e. sets of the form $\underset{x}{\mathrm{E}}[xy \geqq \beta]$ with $y \in L(X)$, $y \neq 0$ and β a real number). This condition is fulfilled e. g. in the case that B is weakly pseudocompact. The relation between properties (C) and (H) is more complicated.

(1,1) *Let $B \subset X$ be a bounded set with property* (C). *Then B fulfills condition* (H).

Proof. Let y_n be a sequence of linear functionals on X and let β_n be a sequence of real numbers such that, for every natural m, there exists a $b_m \in B$ such that $b_m y_j = \beta_j$ for $j = 1, 2, \ldots, m$. Let R be the space $F(L(X))$ equipped with the weak topology corresponding to $L(X)$. Since B is bounded, the closure of B in R is compact. Let B_m be the closure (in R) of the set consisting of the points $b_m, b_{m+1}, \ldots$. There exists an $r \in R$ which lies in every B_m. We have $B_m y_m = \beta_m$ so that $ry_m = \beta_m$. Since r lies in the closure of B, there exists a $b \in B$ which belongs to $H(y_m; ry_m)$ for every m. This completes the proof.

The following example shows that the assumption that B be bounded is essential in the preceding lemma.

Let X be the space of all sequences x of real numbers such that $\lim x_n = 0$. In X, we introduce the norm $|x| = \max |x_n|$ so that X becomes a complete normed linear space.

For every natural n we define the unit vector e_n as follows

$$e_{nn} = 1, \; e_{nk} = 0 \text{ for } k \neq n \, .$$

Further, let $f_k \in L(X)$ be defined by the relation $e_i f_k = \delta_{ik}$. For every natural n, let $b_n = \sum_{j=1}^{n} ie_i$. The set consisting of all b_n will be denoted by B. Let R be the space $F(L(X))$ equipped with the weak topology corresponding to $L(X)$.

We are going to show that B is closed in R. To see that, take an $r \in R$ which belongs to the closure of B. For every $b \in B$, we have $bf_1 = 1$. It follows that $rf_1 = 1$. For every $k > 1$, the set Bf_k consists of exactly two points 0 and k. It follows that, for every $k > 1$, we have either $rf_k = 0$ or $rf_k = k$.

Let $z \in L(X)$ be defined as the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} f_k$. Since $b_n z = \sum_{k=1}^{n} \frac{1}{k}$, the set Bz has no limit point. It follows that there exists a natural n such that $rz = b_n z$. Let $k > n$. There exists a point $b \in B$ such that

$$|(b - r)\, z| < \frac{1}{2(n+1)}\, , \; |(b - r)\, f_j| < \tfrac{1}{2} \text{ for } 1 \leqq j \leqq n \, , \, |(b - r)\, f_k| < \tfrac{1}{2} \, .$$

From the first of these relations, we obtain that $b = b_n$. Since $b_n f_k = 0$ for $k > n$, we obtain that $rf_k = 0$, and, at the same time, that $rf_j = j$ for $1 \leqq j \leqq$ $\leqq n$. We have thus shown that $rf_k = b_n f_k$ for every k. Now let y be an arbitrary member of $L(X)$. We are going to show that $ry = b_n y$. Let $\alpha_j = e_j y$ and let us define y_0 by the relation

$$y = \alpha_1 f_1 + \ldots + \alpha_n f_n + y_0 \, .$$

Since $b_n y_0 = 0$, it follows that $ry = b_n y + ry_0$. Suppose that $ry_0 \neq 0$. Then there exists a point $b \in B$ such that

$$|(b - r)\, y_0| < \tfrac{1}{2}|ry_0| \, , \quad |(b - r)\, f_{n+1}| < \tfrac{1}{2} \, .$$

Since $rf_{n+1} = 0$, it follows from the second relation that $bf_{n+1} = 0$ so that b is one of the elements $b_1, b_2, \ldots, b_n$. Every one of these points fulfills, however, the relation $b_i y_0 = 0$, which is a contradiction with the first inequality. Hence $ry_0 = 0$ which concludes the proof.

Since B is closed in R, condition (C) is fulfilled trivially for B. On the other hand, we have

$$b_n \in B \cap H(f_1; 1) \cap \ldots \cap H(f_n; n)$$

for every natural n and, at the same time, the intersection of the sequence $H(f_n; n)$ is empty.

Now we are able to give the proof of the main result of this section. We shall begin with some simple remarks. They will be given without proof here since, besides being trivial, they have been discussed in detail in [6].

Let $r \in R$ be such an element that, for every sequence $y_1, y_2, \ldots$ of linear functionals on X, there exists an $x \in X$ which fulfills $xy_j = ry_j$ for every natural j. It is easy to see that r behaves as a continuous function as far as those properties are concerned which involve countable sets of arguments only. Especially it follows that r is bounded and attains its maximum value on every weakly compact subset of $L(X)$.

The proof of the following theorem is based on the same idea as that used in [6].

(1,2) Theorem. *Let $B \subset X$ be a bounded set with property (C). Let $r \in R$ belong to the closure of B. Then r is an almost continuous functional on $L(X)$.*

Proof. Let U be an arbitrary neighbourhood of zero in X. We are going to show that there exists a sequence $b_n \in B$ with the following property. If $y \in U^*$ and $b_n y = 0$ for every n, then $ry = 0$. The construction of the sequence b_n will proceed by a simple induction.

The set U^* being weakly compact, there exists a point $y_1 \in U^*$ such that

$$ry_1 = \max ry\,, \quad y \in U^*\,.$$

Since B has property (C), a point $b_1 \in B$ may be found such that $b_1 y_1 = ry_1$.

The set $U^* \cap Z(b_1)$ being weakly compact, there exists a point $y_2 \in U^* \cap$ $\cap\, Z(b_1)$ such that

$$ry_2 = \max ry\,, \quad y \in U^* \cap Z(b_1)\,.$$

Since B has property (C), a point $b_2 \in B$ may be found such that $b_2 y_1 = ry_1$ and $b_2 y_2 = ry_2$.

Suppose now that the elements $b_i \in B$ and $y_j \in U^*$ have been already defined for $1 \leqq i, j \leqq n$ so that the following relations are fulfilled

$$b_i y_j = ry_j \text{ for } j \leqq i\,, \quad b_i y_j = 0 \quad \text{for } j > i\,,$$

$$ry_j = \max ry\,, \quad y \in U^* \cap Z(b_1) \cap \ldots \cap Z(b_{j-1})\,.$$

We choose first a point $y_{n+1} \in U^* \cap Z(b_1) \cap \ldots \cap Z(b_n)$ such that

$$ry_{n+1} = \max ry\,, \quad y \in U^* \cap Z(b_1) \cap \ldots \cap Z(b_n)$$

and take a $b_{n+1} \in B$ which fulfills the relations $b_{n+1} y_j = ry_j$ for $1 \leqq j \leqq n + 1$. This completes the induction.

Let us show now that $\lim ry_n = 0$. Let y_0 be a weak limit point of the sequence y_n. If i is a fixed natural number, we have $b_i y_j = 0$ for $j > i$. It follows that $b_i y_0 = 0$ for every i. Let $H_0 = H(y_0; 0)$ and for every natural j, let $H_j =$ $= H(y_j; ry_j)$. For every natural m, the point b_m belongs to the intersection

$$B \cap H_0 \cap H_1 \cap \ldots \cap H_m\,.$$

Now B is bounded and fulfills condition (C). It follows from the preceding lemma that B has property (H). Hence there exists a point $b \in B \cap H_0$ which belongs to every H_j. Since y_0 is a weak limit point of y_n, the value by_0 is a limit point of the sequence by_n. Since $by_n = ry_n$ and ry_n is a nonincreasing sequence of nonnegative numbers, we have

$$0 = by_0 = \lim ry_n .$$

Now we are going to show that r is continuous on U^*. Let ε be an arbitrary positive number. Let us choose n so that $ry_n < \frac{1}{4}\varepsilon$. There are two cases possible.

1° We have $ry_n = 0$. In this case r may be expressed as a linear combination of $b_1, \ldots, b_n$ and the proof is complete.

2° We have $ry_n > 0$. Let $y \in U^*$. Let

$$z = \sum_{j=1}^{n} \frac{1}{ry_j} (b_j y - b_{j-1} y) \, y_j .$$

(For the sake of convenience, we put $b_0 = 0$.) Clearly, we have $b_i z = b_i y$ for $i = 1, 2, \ldots, n$ and, at the same time, $rz = b_n y$.

Let $\sigma = \max_{1 \leqq i \leqq n} |b_i y|$. A simple estimation of the coefficients shows that $z \in \frac{2n\sigma}{ry_n} U^*$, so that $y - z \in \left(1 + \frac{2n\sigma}{ry_n}\right) U^*$. We have $ry = rz + r(y - z) = b_n y + r(y - z)$. Since $b_i(y - z) = 0$ for $i = 1, 2, \ldots, n$, we have $|r(y - z)| \leqq \left(1 + \frac{2n\sigma}{ry_n}\right) ry_n$. We obtain thus the following estimate

$$|ry| \leqq \sigma + ry_n + 2n\sigma < \tfrac{1}{2}\varepsilon + 3n\sigma .$$

We have thus proved the following proposition: if $y \in U^*$ and $|b_i y| \leqq \frac{1}{6n} \varepsilon$ for $i = 1, 2, \ldots, n$, then $|ry| < \varepsilon$. This completes the proof.

We have thus shown that, in a complete space X, the closure of a bounded set with property (C) is weakly compact. It is natural to ask whether, perhaps, boundedness and property (C) are not sufficient for the set B to be closed, at least in the case of a normed space.

The following example shows that this question has to be answered in the negative.

(1,3) Let us denote by P the set of all real numbers $0 \leqq t \leqq 1$. Let us denote by X the linear space of all real functions x defined on P and such that, for every $\varepsilon > 0$, the set $\underset{t}{E}[t \in P, |x(t)| \geqq \varepsilon]$ is finite. It follows that every $x \in X$ is bounded so that it is possible to introduce a norm in X in the following manner

$$|x| = \max |x(t)| .$$

It is easy to see that the space X is complete in this norm.

To every $x \in X$ there exists a countable $S(x) \subset P$ such that $x(t) = 0$ for every t which does not belong to $S(x)$.

To every $p \in P$ we introduce a point $e_p \in X$, the unit vector corresponding to p, as follows

$$e_p(p) = 1\,, \quad e_p(t) = 0$$

for every $t \in P$, $t \neq p$.

The set of all unit vectors will be denoted by M.

Let y be a linear functional on X. For every $t \in P$, let us put $y(t) = e_t \,.\, y$. Let $t_1, \ldots, t_n$ be arbitrary points of P distinct from each other. Let $\varepsilon_i = \operatorname{sign} y(t_i)$. Let $x = \Sigma \varepsilon_i e_{t_i}$, so that $|x| \leq 1$. We have then

$$\Sigma |y(t_i)| = \Sigma \varepsilon_i\, y(t_i) = (\Sigma \varepsilon_i e_{t_i})\, y = xy \leq |y|\,.$$

It follows that $\sum\limits_{0 \leq t \leq 1} |y(t)| \leq |y|$, so that there exists a sequence $t_n \in P$ such that $y(t) = 0$ for every t which does not belong to this sequence. We are going to show that, for every $x \in X$, we have

$$xy = \sum_{0 \leq t \leq 1} x(t)\, y(t)\,.$$

For every natural m, let $x_m = \sum\limits_{i \leq m} x(t_i)\, e_{t_i}$. Let $\varepsilon > 0$. Let us choose m such that, for $i > m$ we should have $|x(t_i)| < \varepsilon$, and at the same time that $\sum\limits_{i > m} |y(t_i)| < \varepsilon$. We have then $|x - x_m| \leq \varepsilon$ so that

$$\begin{aligned} xy = x_m y + (x - x_m)\, y &= \sum_{i \leq m} x(t_i)\, y(t_i) + \Theta\, \varepsilon |y| = \\ &= \sum_{0 \leq t \leq 1} x(t)\, y(t) + \Theta\, \varepsilon |x| + \Theta\, \varepsilon |y| \end{aligned}$$

which proves our assertion.

Let e_n be an arbitrary sequence of unit vectors. If there does not exists a unit vector e such that $e_n = e$ for infinitely many n, a subsequence e_n' may be found such that the corresponding points t_n are all distinct from each other. Let y be an arbitrary linear functional on X. We are going to show that $e_n' y \to 0$. This, however, is clear since $e_n' y = y(t_n)$ and the series $\Sigma |y(t_n)|$ converges. It follows that the set $M \cup (0)$ is countably compact in the weak topology. According to a general theorem, the closed convex envelope B of $M \cup (0)$ is weakly compact.

We intend to show now that the closed convex envelope of M coincides with B. To see that, it is sufiicient to show that the point 0 belongs to the weak closure of M. Let $y_1, \ldots, y_n$ be arbitrary linear functionals on X, let ε be an arbitrary positive number. We are going to show that there exists a point $e_p \in M$ such that $|e_p y_i| < \varepsilon$ for every $i = 1, 2, \ldots, n$. Clearly it is sufficient to take p outside the set $S(y_1) \cup \ldots \cup S(y_n)$ and we have $e_p y_i = 0$.

We are going to show now that the set B coincides with the set $\tilde{B}$ of all $x \in X$ which fulfil the following two conditions:

1° $x(t) \geqq 0$ for every $t \in P$,

2° $\sum_{0 \leqq t \leqq 1} x(t) \leqq 1$.

The set $\tilde{B}$ is clearly convex. Let us show that $\tilde{B}$ is closed. Suppose that x_0 does not belong to $\tilde{B}$. There are two cases possible. (1) There exists a point $t \in P$ such that $x_0(t) < 0$. If y is the linear functional on X defined by $y(t) = 1$, $y(s) = 0$ for $s \neq t$, we have then $\tilde{B}y \geqq 0$ and $x_0 y < 0$, so that x_0 cannot belong to the closure of $\tilde{B}$. (2) We have $x_0(t) \geqq 0$ for every t but $\Sigma x(t) > 1$ (this, of course, is meant to include the case when the series $\Sigma x(t)$ is divergent). We arrange the points t where $x(t) \neq 0$ in a sequence t_n. There exists a natural m such that $\sigma = \sum_{i \leq m} x_0(t_i) > 1$. Let f_k be unit functionals corresponding to the points t_k. Let x fulfil the inequalities $|(x - x_0) f_k| < \frac{\sigma - 1}{2m}$ for $k = 1, 2, \ldots, m$. We have then

$$\sum_{k \leq m} x(t_k) \geqq \sum_{k \leq m} x_0(t_k) - m \frac{\sigma - 1}{2m} = \sigma - \frac{\sigma - 1}{2} = \frac{1 + \sigma}{2} > 1$$

so that x cannot belong to $\tilde{B}$. The set $\tilde{B}$ is therefore closed.

It follows that $B \subset \tilde{B}$. Suppose there exists a point $x_0 \in \tilde{B}$ and a linear functional y on X such that

$$\sup My < x_0 y$$

It follows that $\max_{0 \leqq t \leqq 1} y(t) < x_0 y$. Since $x_0(t) \geqq 0$, we have $x_0(t)\, y(t) \leqq x_0(t)$ . . $\max_{0 \leqq t \leqq 1} y(t)$, so that

$$\sup y(t) < \Sigma x_0(t)\, y(t) \leqq \sup y(t)\, \Sigma x_0(t) \leqq \sup y(t)$$

which is a contradiction which proves that $B = \tilde{B}$.

Let us denote by B_1 the set of all $x \in B$ which fulfil the relation $\sum_{0 \leqq t \leqq 1} x(t) = 1$. Let $b \in B$ and let $y_1, y_2, \ldots$ be an arbitrary sequence of linear functionals on X. Let $t^* \in P$ be chosen so that t^* does not belong to the set $\bigcup_j S(y_j) \cup S(b)$. Let $b_1 \in X$ be defined by the relations $b_1(t) = b(t)$ for every $t \neq t^*$ and $b_1(t^*) = 1 - \sum_{0 \leqq t \leqq 1} b(t)$. Clearly $b_1 \in B_1$ and $b_1 y_j = b y_j$ for every natural j.

Clearly the set B_1 is convex. Let r be a linear form defined on $L(X)$ which lies in the weak closure of B_1. The weak closure of B_1 coincides, however, with B. To see that, we note that $M \subset B_1 \subset B$ and B is the closed convex envelope of M.

It follows that $r \in B$ so that, for every sequence $y_1, y_2, \ldots$, there exists a point $b_1 \in B_1$ which fulfills

$$r y_j = b_1 y_j$$

for every j so that B_1 fulfills condition (C).

We conclude with a remark concerning pseudocompact sets. If $B \subset X$ is pseudocompact in the weak topology, it is easy to see that B is bounded and possesses property (C). We see thus that the theorem (2,1) of [6] is contained in the present result.

§ 2.

Let us examine now more closely what conclusions may be drawn from the formally (and, as may be easily seen, essentially) weaker assumption (H). Let B be a set of property (H) and $r \in R$ belong to the weak closure of B. Let $y_1, y_2, \ldots$ be a sequence of linear functionals on X. It is easy to see that only two cases are possible:

1° There exists a $b \in B$ which belongs to every $H(y_j; ry_j)$,

2° There exists a natural n such that $B \cap H(y_1; ry_1) \cap \ldots \cap H(y_n; ry_n) = 0$. Unfortunately enough, this second disagreable eventuality may occur even with the most innocent sets; this is a source of additional complications which are, however, of a purely formal character.

In the present section we intend to show that a result analogous to theorem **(1,2)** may be proved even under the weaker assumptions of boundedness and property (H). We must limit ourselves, however, to the case of a convex set B.

As a matter of fact, the main idea of the proof resting the same as in theorem **(1,2)**, we must — for purely technical reasons — follow the example of Floyd and Klee in introducing a kind of a ,,nonsupport point". The construction of a such a point necessitates the introduction of the additional assumption of convexity.

We begin with the construction of a point which is — in a certain sense — an inner point of B with respect to every linear combination of linear functionals belonging to a given countable set.

(2,1) *Let B be a bounded convex subset of X which possesses property (H). Let $y_1, y_2, \ldots$ be a sequence of points of Y. Then there exists a point $b_0 \in B$ with the following property: if y is a functional of the form $y = \omega_1 y_1 + \ldots + \omega_n y_n$ and $b_0 y$ lies on the boundary of By, then By is a one point set.*

Proof. We begin with a remark concerning notation. If K is a (nonvoid) bounded convex subset of the real line, then there exist two numbers $\varkappa_1 \leqq \varkappa_2$ so that the closure of K coincides with the set of all numbers $\varkappa_1 \leqq \xi \leqq \varkappa_2$. We shall denote by $m(K)$ the number $\frac{1}{2}(\varkappa_1 + \varkappa_2)$.

For the proof of our lemma, we may clearly assume that all $y_n \neq 0$. Let $B_0 = B$.

Let $H_1 = H(y_1; m(B_0 y_1))$ and $B_1 = B_0 \cap H_1$ so that $B_1 \neq 0$. Suppose now that we have already defined hypervarieties $H_1, \ldots, H_n$ such that the set $B_n = B_0 \cap H_1 \cap \ldots \cap H_n$ is nonvoid. To obtain the next step, we take

$H_{n+1} = H(y_{n+1}; m(B_n y_{n+1}))$ and put $B_{n+1} = B_n \cap H_{n+1}$. The set B_n being convex, we have $B_{n+1} \neq 0$. Since B has property (H), we conclude that there exists a point $b_0 \in B$ which belongs to every H_n. We are going to show that b_0 has the desired property.

First of all, we are going to prove the following proposition.

Let n be a natural number such that the set $B_{n-1}y_n$ contains exactly one point β which lies on the boundary of By_n. Then By_n is a one-point set.

The proof goes by induction. For $n = 1$ there is nothing to prove. Now let $n > 1$ and suppose the proposition proved for less than n functionals. Let β be a real number such that $B_{n-1}y_n = \beta$ and such that β lies on the boundary of By_n. In the first part of the proof, we are going to show that, in this case, the set $B_{n-2}y_n$ is a one-point set as well and $B_{n-2}y_n = \beta$. To see that, take an arbitrary point $b \in B_{n-2}$. We are to show that $by_n = \beta$. This is obvious if $b \in B_{n-1}$ so that it is sufficient to consider the case $b \in B_{n-2}$, b non $\in B_{n-1}$. Since

$$B_{n-1} = B_{n-2} \cap H(y_{n-1}; m(B_{n-2}y_{n-1}))$$

it follows that $by_{n-1} \neq m(B_{n-2}y_{n-1})$; we may clearly suppose that $by_{n-1} > m(B_{n-2}y_{n-1})$. Then there exists a point $b' \in B_{n-2}$ such that $b'y_{n-1} < m(B_{n-2}y_{n-1})$. A number $0 < \lambda < 1$ can be found such that the point $\tilde{b} = \lambda b + (1 - \lambda)\, b'$ fulfills the relation $\tilde{b}y_{n-1} = m(B_{n-2}y_{n-1})$. The set B_{n-2} being convex we have $\tilde{b} \in B_{n-2}$. According to the above relation we have $\tilde{b} \in B_{n-1}$. It follows that $\tilde{b}y_n = \beta$. We have now

$$\tilde{b}y_n = \lambda b y_n + (1 - \lambda)\, b'y_n = \beta\,.$$

Both by_n and $b'y_n$ belong to the set By_n and the point β lies on the boundary of By_n. Clearly this equation is impossible unless $by_n = b'y_n = \beta$. Since b was an arbitrary point of B_{n-2} we have $B_{n-2}y_n = \beta$.

In the second part of the proof, we are going to use the induction hypothesis. Clearly our proposition may now be applied to the set B and the sequence of functionals $y_1, \ldots, y_{n-2}, y_n$. We obtain at once that By_n is a one-point set so that our proposition is completely proved.

Now we are able to complete the proof of our lemma. Let y be a linear combination $y = \omega_1 y_1 + \ldots + \omega_n y_n$ and suppose that $b_0 y$ lies on the boundary of By. We are going to show that, in this case, the set By is a one-point set.

For the purpose of an induction proof we shall introduce the following classification of functionals. The point y is said to be of height n if it may be expressed as a linear combination of $y_1, \ldots, y_n$ but not of $y_1, \ldots, y_{n-1}$. For functionals of height 1 the lemma is trivial. Now let $n > 1$ and suppose the lemma proved for all functionals of height less than n. Let y be of height n and let $b_0 y$ be a boundary point of By. Since $B_{n-1} \subset B$, $b_0 y$ is a boundary point of $B_{n-1}y$ as well. Note that, for every $j < n$, we have $B_{n-1}y_j = b_0 y_j$. If $y^* = \omega_1 y_1 + \ldots + \omega_{n-1}y_{n-1}$, we have $B_{n-1}y^* = b_0 y^*$. It follows immediately that the point $\omega_n b_0 y_n$ lies on the boundary of $\omega_n B_{n-1} y_n$. Since y is of height n, we have $\omega_n \neq 0$ so that $b_0 y_n$ is a boundary point of $B_{n-1}y_n$. Since $b_0 \in B_n$ we have $b_0 y_n = m(B_{n-1}y_n)$. It follows that $B_{n-1}y_n$ is a one-point set, so that $B_{n-1}y_n = b_0 y_n$. Each of the sets $B_{n-1}y^*$ and $B_{n-1}y_n$ contains exactly one point. It follows that

$$B_{n-1}y = B_{n-1}(y^* + \omega_n y_n) = b_0 y \, .$$

To sum up, we have the following result. The set $B_{n-1}y$ contains exactly one point $b_0 y$ which lies on the boundary of By. Now the proposition proved above may be applied to the sequence of functionals $y_1, \ldots, y_{n-1}, y$. It follows that By is a one-point set which concludes the proof.

(2,2) *Let B be a bounded convex subset of X which possesses property (H). Let r be contained in the weak closure of B. Let $y_1, y_2, \ldots$ be a sequence of points of Y. Let b_0 be the point constructed in the preceding lemma. Then there exists a point $b \in B$ such that*

$$\tfrac{1}{2}(b_0 + r)\, y_j = b y_j$$

is fulfilled for every natural j.

Proof. Let n be an arbitrary natural number. We are going to show that there is a point $b \in B$ such that $\frac{1}{2}(b_0 + r)\, y_j = b y_j$ for $j = 1, 2, \ldots, n$. This is clearly sufficient to prove our lemma. We shall proceed by induction with respect to n.

First of all, let $n = 1$. The point $\frac{1}{2}(b_0 + r)$ lies in the closure of B. If $\frac{1}{2}(b_0 + r)\, y_1$ lies in the interior of By_1, the existence of b is evident. Suppose now that $\frac{1}{2}(b_0 + r)\, y_1$ lies on the boundary of By_1. It follows that for a suitable $\varepsilon (\varepsilon \neq 0)$ we shall have $\varepsilon B y_1 \leqq \varepsilon \, . \, \frac{1}{2}(b_0 + r)\, y_1$.

Let b be an arbitrary point of B. Since $\frac{1}{2}(b + r)$ lies in the closure of B, we shall have also $\varepsilon \, . \, \frac{1}{2}(b + r)\, y_1 \leqq \varepsilon \, . \, \frac{1}{2}(b_0 + r)\, y_1$. Since b was arbitrary we obtain $\varepsilon B y_1 \leqq \varepsilon b_0 y_1$. According to lemma **(2,1)** it follows that By_1 is a one-point set so that $\frac{1}{2}(b_0 + r)\, y_1 = b_0 y_1$.

Now let $n > 1$ and suppose the lemma proved for less than n functionals. For every $x \in X$, let $\varphi(x) = (xy_1, \ldots, xy_n) \in E_n$. The point $\varphi(\frac{1}{2}(b_0 + r))$ lies in the closure of $\varphi(B)$. If it is contained in the interior of $\varphi(B)$, the existence of b is evident. Suppose now that $\varphi(\frac{1}{2}(b_0 + r))$ lies on the boundary of $\varphi(B)$. It follows that that there exists a functional f on E_n such that $f(\varphi(B)) \leqq f(\varphi(\frac{1}{2}(b_0 + r))$. Clearly there exist numbers $\omega_1, \ldots, \omega_n$ such that we have $By \leqq \frac{1}{2}(b_0 + r)\, y$ if $y = \omega_1 y_1 + \ldots + \omega_n y_n$. By repeating the argument used in the case $n = 1$ (with y instead of y_1) we obtain that $By = b_0 y = ry$.

Now we shall distinguish two cases according to the height h of the functional y.

If $h < n$, let us repeat the procedure described in the preceding lemma with the sequence $y_1, \ldots, y_{h-1}, y_{h+1}, y_{h-2}, \ldots$ As result of this procedure, we obtain a point b_0' with properties analogous as b_0 but with respect to the modified sequence of functionals. First of all, let us note that $b_0' y_h = b_0 y_h$. In fact, we have

$b'_0y = b_0y$. The functional y may be expressed as $y = \omega_1 y_1 + \dots + \omega_{h-1}y_{h-1} + \omega_h y_h$ with $\omega_h \neq 0$. Since $b'_0 y_1 = b_0 y_j$ for all $j < h$, we have clearly $b'_0 y_h = b_0 y_h$. According to the induction hypothesis, there exists a point $b \in B$ such that $by_j = \frac{1}{2}(b'_0 + r)\, y_j$ for every $j \neq h$, $1 \leqq j \leqq n$. Since By is a one-point set, we have also $by = \frac{1}{2}(b'_0 + r)\, y$ which, in its turn, implies that $by_h = \frac{1}{2}(b'_0 + r)\, y_h$. It follows that $by_j = (b'_0 + r)\, y_j$ for every $1 \leqq j \leqq n$. According to what has been said above, b'_0 may be replaced here by b_0 so that the proof is complete. If $h = n$, we may use the induction hypothesis to find a point $b \in B$ such that $by_j = \frac{1}{2}(b_0 + r)\, y_j$ for every $j < h$. The height of y being n, we have $y = \omega_1 y_1 + \dots + \omega_{n-1}y_{n-1} + \omega_n y_n$ with $\omega_n \neq 0$. The set By being a one-point set, we have $by = \frac{1}{2}(b_0 + r)\, y$. A similar argument as that used above yields $by_n = \frac{1}{2}(b_0 + r)\, y_n$ which completes the proof.

(2,3) Theorem. *Let B be a bounded convex subset of X, which posseses property (H). Let r be contained in the weak closure of B. Then r is almost continuous.*

Proof. With view to the results of [6], it is sufficient to prove that, for every neighbourhood of zero U in X, there exists a sequence $x_n \in X$ such that, if $y \in U^*$ and $x_n y = 0$ for every n, we have $ry = 0$.

The set U^* being weakly compact, a point $y_1 \in U^*$ can be found such that

$$ry_1 = \max ry\,, \quad y \in U^*\,.$$

Let us put $\mu_1 = m(By_1)$, $B'_1 = B \cap H(y_1; \mu_1)$. According to lemma **(2,2)**, we shall have then $B_1 = B \cap H(y_1; \frac{1}{2}(\mu_1 + ry_1)) \neq 0$. Let us choose $b_1 \in B_1$, $b'_1 \in B'_1$.

The set $U^* \cap Z(b_1) \cap Z(b'_1)$ being weakly compact, a point $y_2 \in U^* \cap Z(b_1) \cap Z(b'_1)$ can be found such that

$$ry_2 = \max ry\,, \quad y \in U^* \cap Z(b_1) \cap Z(b'_1)\,.$$

Let us put $\mu_2 = m(B'_1 y_2)$, $B'_2 = B'_1 \cap H(y_2; \mu_2)$. According to lemma **(2,2)** we shall have then $B_2 = B_1 \cap H(y_2; \frac{1}{2}(\mu_2 + ry_2)) \neq 0$. Let us choose $b_2 \in B_2$, $b'_2 \in B'_2$.

Suppose now that we have already defined the elements y_i, b_i, b'_i, μ_i for $i = 1, 2, \dots, n$ so that the following relations are fulfilled

1° $b_i y_j = \frac{1}{2}(\mu_j + ry_j)$ for $j \leqq i$,

2° $b'_i y_j = \mu_j$ for $j \leqq i$,

3° $b_i y_j = b'_i y_j = 0$ for $i < j$.

First of all, let us take a point y_{n+1} of the set $U^* \cap Z(b_1) \cap Z(b'_1) \cap \dots \cap Z(b_n) \cap Z(b'_n)$ which realizes the maximum of ry on this set. The set $B'_n = B \cap H(y_1; \mu_1) \cap \dots \cap H(y_n; \mu_n)$ is nonempty since $b'_n \in B'_n$. It follows that $B'_{n+1} = B'_n \cap H(y_{n+1}; \mu_{n+1}) \neq 0$, where $\mu_{n+1} = m(B'_n y_{n+1})$. The set

$$B_n = B \cap H(y_1; \tfrac{1}{2}(\mu_1 + ry_1)) \cap \dots \cap H(y_n; \tfrac{1}{2}(\mu_n + ry_n))$$

is nonvoid since $b_n \in B_n$. It follows from lemma **(2,2)** that $B_{n+1} = B_n \cap \cap H(y_{n+1}; \frac{1}{2}(\mu_{n+1} + ry_{n+1})) \neq 0$. It is sufficient now to take an arbitrary $b_{n+1} \in B_{n+1}$ and an arbitrary $b'_{n+1} \in B'_{n+1}$. The induction is thus complete.

The sequence y_j has a limit point $y_0 \in U^*$. According to 3°, we shall have

4° $b_i y_0 = b'_i y_0 = 0$ for every natural i.

For every natural n, we have

$$b'_n \in B \cap H(y_1; \mu_1) \cap \ldots \cap H(y_n; \mu_n) \cap Z(y_0) .$$

It follows that there exists a point $b_0 \in B$ such that

$$b_0 y_0 = 0 , \quad b_0 y_j = \mu_j \quad \text{for} \quad j = 1, 2, \ldots .$$

Let us write now $H_j = H(y_j; \frac{1}{2}(\mu_j + ry_j))$. According to 1° and 4°, for every natural n we have

$$b_n \in B \cap H_1 \cap \ldots \cap H_n \cap Z(y_0) .$$

It follows that there exists a point $b \in B$ such that

$$by_0 = 0, \; by_j = \tfrac{1}{2}(b_0 y_j + ry_j) \quad \text{for } j = 1, 2, \ldots .$$

Clearly ry_j is a non-increasing sequence of nonnegative numbers. Let $\varepsilon = \inf ry_j$, so that $\varepsilon \geqq 0$. We are going to show that $\varepsilon = 0$.

Since $by_j = \frac{1}{2}(b_0 y_j + ry_j)$ for every natural j, we obtain $by_j \geqq \frac{1}{2} b_0 y_j + \frac{1}{2}\varepsilon$ for every natural j, so that

$$by_0 \geqq \tfrac{1}{2} b_0 y_0 + \tfrac{1}{2}\varepsilon .$$

Since $by_0 = b_0 y_0 = 0$, it follows that $\varepsilon = 0$ which concludes the proof.

We shall need the following simple lemma.

(2,4) *For a bounded convex set B condition (C) is equivalent to the following. Let y_j be a given sequence of linear functionals on X and β_j a sequence of real numbers. Suppose that, for every natural m and every sequence $\lambda_1, \ldots, \lambda_m$ of real numbers, we have*

$$\sum_1^m \lambda_j \beta_j \leqq \sup \; B(\sum_1^m \lambda_j y_j) .$$

Then there exists a point $b \in B$ such that $by_j = \beta_j$ for every j.

Proof. Suppose that the above condition is fulfilled and that $r \in R$ belongs to the weak closure of B. Let y_j be a given sequence of functionals. Put $\beta_j = ry_j$. If m is an arbitrary natural number and $\lambda_1, \ldots, \lambda_m$ given real numbers, we have

$$\sum_1^m \lambda_j \beta_j = r(\sum_1^m \lambda_j y_j) \leqq \sup \; B(\sum_1^m \lambda_j y_j) .$$

It follows that there exists a point $b \in B$ which fulfills $by_j = \beta_j = ry_j$ so that condition (C) is fulfilled as well.

On the other hand, suppose that B possesses property (C). Since B is bounded, $\sup By$ is finite for every $y \in L(X)$. Clearly, the function $p(y) = \sup By$ is subadditive and fulfills $p(\lambda y) = \lambda\, p(y)$ for every $y \in L(X)$ and every $\lambda \geqq 0$. Now let y_j and β_j be two sequences which properties mentioned above. Let Y_0 be the subspace of Y consisting of all linear combinations of the points y_j. It is easy to see that the linear form r defined on Y_0 by the postulates $ry_j = \beta_j$ fulfills the inequality

$$ry \leqq p(y)$$

for every $y \in Y_0$. According to the Hahn-Banach theorem there exists an extension of r to the whole of Y such that the above estimate remains true. We are going to show that r lies in the weak closure of B. In fact, if this were not true, there would be a point $y_0 \in L(X)$ such that $\sup By_0 < ry_0$, which is a contradiction with the estimate $ry_0 \leqq p(y_0)$. This completes the proof.

We summarize the results obtained in the following

(2,5) Theorem. *Let X be a complete convex topological linear space. Let B be a bounded closed convex subset of X. Then the following conditions are equivalent*

1° *B is weakly compact,*

2° *B is weakly pseudocompact,*

3° *B fulfills condition (C),*

4° *B fulfills the condition of lemma* **(2,4)**,

5° *B fulfills condition (H).*

Applied to the unit sphere of a complete normed linear space, this yields a simple characterization of reflexivity.

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Резюме

ДВА ЗАМЕЧАНИЯ К СЛАБОЙ КОМПАКТНОСТИ

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(Поступило в редакцию 11/VI 1955 г.)

По известной теореме [1] замкнутое выпуклое подмножество B полного топологического линейного пространства является слабо компактным тогда и только тогда, если пересечение каждой убывающей последовательности непустых замкнутых выпуклых подмножеств $C_n \subset B$ непустое. В недавно вышедшей работе [3] Е. Е. Флойд и В. Л. Кли получили подобный результат с более слабым предположением, что C_n — замкнутые линейные многообразия. В работе, посвященной исследованию сущности теорем эберлейновского типа, автором был развернут один метод доказательства, являющийся особенно выгодным для обсуждения вопросов, касающихся пересечений линейных многообразий. Цель настоящей статьи — показать, что упомянутый метод можно использовать без существенных изменений для доказательства еще более общей теоремы, заключающей в себе и результат работы [6]. Доказательство здесь значительно проще, однако упрощение носит чисто формальный характер.

Работа разделяется на две части. В первой доказывается теорема, аналогичная теореме работы [6], для множеств, которые удовлетворяют некоторому условию (см. условие (C) выше), касающемуся счетных пересечений гипер-многообразий. Этот результат справедлив и для невыпуклых множеств.

Во второй части доказывается подобный же результат для выпуклых множеств. Мы скажем, что подмножество B выполняет условие (H), если для любой последовательности замкнутых гипер-многообразий H_n в X, для которой $B \cap H_1 \cap \ldots \cap H_n \neq 0$ для любого n, существует точка $b \in B$, лежащая во всех H_n. Далее доказываем следующую теорему:

Пусть B — ограниченное выпуклое множество полного выпуклого топологического линейного пространства X. Пусть B удовлетворяет условию (H). Тогда замыкание множества B слабо компактно.