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Czechoslovak Mathematical Journal, Vol. 3 (1953), No. 3, 291–296

Persistent URL: http://dml.cz/dmlcz/100089

Чехословацкий математический журнал, т. 3 (78) 1953

ON THE CONVERGENCE IN σ-ALGEBRAS OF POINT-SETS.

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(Received Juny 1, 1953.)

In this article the topological convergence of point-sets in σ-algebras is studied. Necessary and sufficient conditions are given for a topological convergence to be a metrical one. An example is constructed showing that the topological convergence possessing Hedrick's property need not be identical with the metrical convergence.

In this paper the symbol X is used to denote an abstract space. Let $\mathbf{A}$ be a σ-algebra of point-sets, i. e. a class of sets in X which contains X and which is closed under the formation of countable unions and differences. The elements of $\mathbf{A}$ are called events. Two σ-algebras $\mathbf{A}$ and $\mathbf{A}_1$ are said to be *isomorphic* if there exists a one-to-one transformation T of $\mathbf{A}$ onto $\mathbf{A}_1$ such that $T(\bigcup A_\iota) = \bigcup TA_\iota$ and $T(A - B) = T(A) - T(B)$. Let $\{A_n\}$ be a sequence of events $A_n \in \mathbf{A}$. According to *F. Hausdorff*[2]) we define $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ and $\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$; the sequence $\{A_n\}$ converges *topologically* to an event $A \in \mathbf{A}$ — in symbols $A_n \xrightarrow{t} A$ or $\lim A_n \stackrel{t}{=} A$ — if $A = \limsup A_n = \liminf A_n$. We say that two convergences $\xrightarrow{1}$ and $\xrightarrow{2}$ are *identical*, if $A_n \xrightarrow{1} A$ implies and is implied by $A_n \xrightarrow{2} A$. The topological convergence will be said to possess *Hedrick's* property[3]) if $\lim_n A_{mn} \stackrel{t}{=} A_m$ and $\lim_m A_m \stackrel{t}{=} A$ implies that there is a sequence $\{A_{m_k n_k}\}_{k=1}^{\infty}$, $m_1 \leq m_2 \leq \ldots$, converging to A.

The scope of this paper is the study of the relationship between topological and metrical convergences. Necessary and sufficient conditions are given for both convergences to be identical.[4]) One of these conditions concerns the

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[1]) On December 1952 I had the occasion to announce the idea of the convergence of random events in a conference held in Wrocław. Using another method *E. Marczewski* succeeded in proving some results contained in this paper (cf. his article: Remarks on the convergence of measurable sets and measurable functions, which will be published in *Colloquium mathematicum*).

[2]) *F. Hausdorff:* Grundzüge der Mengenlehre, 1914, p. 21.

[3]) *M. Fréchet:* Les espaces abstraits, 1928, p. 211.

[4]) Cf. *D. Maharam:* An algebraic characterisation of measure algebras, *Ann. of Math.* *48* (1947), Theorem 1.

notion of *probability*; by this we understand a non negative and countably additive set-function P defined on $\mathbf{A}$ and such that $P(X) = 1$. An example is constructed showing that the topological convergence possessing *Hedrick*'s property need not be identical with the metrical convergence.

Let $\mathbf{A}$ be a σ-algebra. The following relations are easy to be proved:

1. If $A_n \in \mathbf{A}$, $B_n \in \mathbf{A}$, $n = 1, 2, \dots$ and $A_n \xrightarrow{t} A$, $B_n \xrightarrow{t} B$, then $A \in \mathbf{A}$, $B \in \mathbf{A}$ and $A'_n \xrightarrow{t} A'$, $A_n \cup B_n \to A \cup B$, $A_n \cap B_n \xrightarrow{t} A \cap B$, $A_n - B_n \xrightarrow{t} A - B$, $A_n \div B_n \xrightarrow{t} A \div B$.

2. If $A_n \in \mathbf{A}$, $n = 1, 2, \dots$ then $A_n \xrightarrow{t} A$ if and only if $A_n \div A \xrightarrow{t} 0$.[5])
Here A' denotes the complement $X - A$ of A and $\div$ denotes the *symmetric difference:* $A \div B = (A - B) \cup (B - A)$.

Let $\to$ denote any convergence satisfying both well known *Fréchet*'s axioms[6]) of convergence. We shall say that the sequence $\{x_n\}$ of elements converges a posteriori[7]) to the element x, in symbols $x_n \twoheadrightarrow x$, if in every subsequence $\{x_{n_k}\}$ there is a subsequence $\{x_{n_{k_i}}\}$ converging to x. The topological convergence a posteriori will be denoted by $\overset{t}{\twoheadrightarrow}$.

The following statement can be easily proved:

In the σ-algebra $\mathbf{A}$ the topological convergence and the topological convergence a posteriori are identical.

As a matter of fact, $A_n \xrightarrow{t} A$ evidently implies $A_n \overset{t}{\twoheadrightarrow} A$. On the other hand, let $A_n \overset{t}{\twoheadrightarrow} A$. Suppose that, on the contrary, this sequence $\{A_n\}$ fails to converge topologically to the event A. Then, according to 2. $\limsup (A_n \div A) \neq 0$. Consequently, there is a point $x_0 \in X$ such that $x_0 \in A_n \div A$ for infinitely many n. Let $\{A_{n_k}\}$ be the sequence of all those events $A_n \in \mathbf{A}$ for which $x_0 \in A_n \div A$. According to our supposition it is possible to choose a subsequence $\{A_{n_{k_i}}\}$ such that $A_{n_{k_i}} \xrightarrow{t} A$, i. e. $A_{n_{k_i}} \div A \xrightarrow{t} 0$. This is contradictory to the fact that $x_0 \in \limsup (A_{n_{k_i}} \div A)$. Therefore $A_n \overset{t}{\twoheadrightarrow} A$ implies $A_n \xrightarrow{t} A$.

Lemma 1. *Let $\mathbf{A}$ be a σ-algebra. Let $\varrho(A, B)$ be a metric in $\mathbf{A}$ such that both convergences in $\mathbf{A}$, the convergence defined by ϱ and the topological one, are identical. Then*

1° *Every subsystem of mutually disjoint events is at most countable.*

2° *Every strictly monotone sequence $\{A_n\}$ of events is at most countable.*

3° *$\mathbf{A}$ is isomorphic with the system of all subsets of an abstract point-set which is at most countable.*

Proof. 1° Let $\{A_\lambda\}_{\lambda < \gamma}$ be a sequence (possibly transfinite) of mutually disjoint events $A_\lambda \in \mathbf{A}$ which are different from one another. Let $\mathbf{A}_n$ be the system of

[5]) Cf. *D. Maharam:* l. c. p. 154, 155.

[6]) *M. Fréchet:* Sur quelques points du calcul fonctionnel. *Rendiconti del circolo Palermo 22* (1906), p. 6.

[7]) *P. S. Urysohn,* Sur les classes (L) de M. Fréchet, *L'Enseignement mathématique 25* (1926), 77—83.

all A_λ in the sequence $\{A_\lambda\}$ for which $\varrho(A_\lambda, 0) \geqq \frac{1}{n}$ holds true, n being a natural integer. If the system $\mathbf{A}_n$ were infinite, it would be possible to choose a sequence of mutually different events $A_{\lambda_k} \in \mathbf{A}_n$ such that $A_{\lambda_k} \xrightarrow{t} 0$ and consequently $A_{\lambda_k} \xrightarrow{\varrho} 0$ which would be a contradiction. Therefore $\mathbf{A}_n$ is a finite system and the system $\bigcup_{n=1}^{\infty} \mathbf{A}_n$ of all events A_λ such that $\varrho(A_\lambda, 0) > 0$ is at most countable. Since the sequence $\{A_\lambda\}$ can contain only one element A_{λ_0} such that $\varrho(A_{\lambda_0}, 0) = 0$, the ordinal γ must be at most countable.

2° follows immediately from 1°.

3° Let $x \in X$ and let $\bar{x}$ denote the common part of all events $A \in \mathbf{A}$ containing x. Then — by 2° — there is a sequence $\{A_n\}$ such that $\bar{x} = \bigcap_{n=1}^{\infty} A_n$. Therefore $0 \neq \bar{x} \in \mathbf{A}$ and, for $x, y \in X$, either $\bar{x} = \bar{y}$ or $\bar{x} \cap \bar{y} = 0$. The sets $\bar{x}$ are the least non-empty events and — according to 1° — the system of all $\bar{x}$ is at most countable. Every event $A \in \mathbf{A}$ is a disjoint union of some least events $\bar{x}$ and the event 0. Therefore $\mathbf{A}$ is isomorphic with the system of all subsets of the set whose elements are $\bar{x}$ which is at most countable.

Definition. *The probability function P defined on a σ-algebra $\mathbf{A}$ will be said to possess property (α) if $P(A) = 0$ implies $A = 0$.*

Theorem 1. *Let $\mathbf{A}$ be a σ-algebra of point-sets. The following three conditions are equivalent:*

a) *There exists a metric ϱ in $\mathbf{A}$ such that the metrical and topological convergences are identical.*

b) *There exists a probability function P defined on $\mathbf{A}$ with the property (α).*

c) *$\mathbf{A}$ is isomorphic with the system of all subsets of a set which is at most countable.*

Proof. According to 3° the condition a) implies c). Now, assume that the condition c) holds true. We can suppose that $\mathbf{A}$ is the system of all subsets of a point-set $X = \{x_0, x_1, \ldots, x_n, \ldots\}$, $n < s$, where s is a either finite or the least infinite ordinal ω. In the first case we define: $P((x_n)) = s^{-1}$, in the second case we put: $P((x_n)) = 2^{-n-1}$, (x_n) denoting a one-point-set containing x_n. In both cases, let $P(A) = 0$ if $A = 0$ and $P(A) = \sum_{x_n \in A} P((x_n))$ if $0 \neq A \in \mathbf{A}$. Clearly, $P(A)$ is a probability function on $\mathbf{A}$ possessing the property (α). Therefore c) implies b).

Now we are going to suppose that the condition b) holds true. Let us define for $A, B \in \mathbf{A}$ the function $\varrho(A, B) = P(A \dot{-} B)$. It is well known that ϱ is a metric function[8]) in $\mathbf{A}$. Further suppose that $A_n \xrightarrow{t} A$. By 2. we have $A_n \dot{-} A \xrightarrow{t} 0$.

[8]) Cf. *O. Nikodym:* Sur une généralisation des intégrales de M. J. Radon. *Fund. Math. 15* (1930) p. 137.

From the continuity of the probability function it follows that $P(A_n \div A) \to$ $\to P(0) = 0$ so that $\varrho(A_n, A) \to 0$, i. e. $A_n \xrightarrow{\varrho} A$. Conversely, let $A_n \xrightarrow{\varrho} A$ and suppose that, on the contrary, the sequence $\{A_n\}$ does not converge topologically to A; then $\limsup\, (A_n \div A) \neq 0$. Consequently, there is a point $y \in X$ belonging to infinitely many events $A_n \div A$. Put $C = \bigcap_{y \in A_n \div A} (A_n \div A)$. Then $0 \neq C \subset \limsup\, (A_n \div A)$. Therefore $\varrho(A_n, A) = P(A_n \div A) \geq P(C) > 0$. Thus we can conclude that the sequence $\{A_n\}$ does not converge to A in the metric ϱ; this is a contradiction. We have proved that $A_n \xrightarrow{t} A$ if and only if $A_n \xrightarrow{\varrho} A$. Consequently b) implies a) and Theorem 1 is proved.

Let $\mathbf{B}$ be a σ-algebra which is isomorphic with the system of all subsets of an at most countable abstract set. Let ϱ be any metric in $\mathbf{B}$. We say that ϱ has *property* (β) if the following conditions are satisfied[9])

I $\varrho(0, X) = 1$,

II $\varrho(A, B) = \varrho(A \div B, 0)$,

III $\varrho(A, 0) + \varrho(B, 0) = \varrho(A \cup B, 0) + \varrho(A \cap B, 0)$,

IV *The metrical and the topological convergences in* $\mathbf{B}$ *are identical.*

We denote by $\mathfrak{A}$ the system of all probability functions P defined on $\mathbf{B}$ and possessing the property (α). By $\mathfrak{B}$ will be denoted the system of all metrics ϱ defined on $\mathbf{B}$ and having the property (β).

Theorem 2. *The correspondence* $P = f(\varrho)$, $\varrho \in \mathfrak{B}$, *where* $f(\varrho)$ *is a set-function defined by* $P(A) = \varrho(A, 0)$, $A \in \mathbf{B}$, *is a one-to-one mapping of* $\mathfrak{B}$ *onto* $\mathfrak{A}$. *The inverse mapping* f^{-1} *satisfies the condition* $\varrho(A, B) = P(A \div B)$.

Proof. The additivity of the function $P(A) = \varrho(A, 0)$ follows immediately from III and the σ-additivity from III and IV. The function $\varrho(A, B)$ is a metric such that the metrical convergence and the topological one in $\mathbf{B}$ are identical (cf. the proof of Theorem 1). The proof of the remaining assertions makes no difficulties.

Remark. Let us denote by V the following condition:

V The metric ϱ defined on $\mathbf{B}$ is strongly monotone, i. e. $A \subset B$, $A \neq B$, implies that $\varrho(A, 0) < \varrho(B, 0)$.

Evidently, from III it follows V. On the other hand, if the σ-algebra $\mathbf{B}$ contains more than two elements, then the systems of conditions I—IV and I, II, IV, V are not equivalent. Indeed, if $\varrho \in \mathfrak{B}$ and $\varrho(A, B) = P(A \div B)$ where $P \in \mathfrak{A}$, then $\varrho_1 = \dfrac{2\varrho}{1 + \varrho}$ is a metric fulfilling the conditions I, II, IV and V; if III holds true for ϱ_1, an easy calculation shows that, for any A, $B \in \mathbf{B}$ we have $A \subset B$ or $B \subset A$.

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[9]) According to III the function $\varrho(A, 0)$, $A \in \mathbf{B}$ is a valuation. Cf. *G. Birkhoff:* Lattice theory, 1948, p. 74.

Let us call *H-convergence* any convergence fulfilling the *Hedrick*'s property. It is well known that every metrical convergence is an H-convergence. On the other hand, the example of the σ-algebra of all linear Borel sets shows that the topological convergence need not possess the *Hedrick*'s property. There is a question *whether in* **A** *the topological H-convergence and the metrical one are identical.* The answer to this question is *negative*. As a matter of fact, let **A** be the σ-algebra of all at most countable subsets and their complements in an uncountable abstract space X. Let $\lim_n A_{mn} \stackrel{t}{=} A_m$ and $\lim_m A_m \stackrel{t}{=} A$; $A_{mn}, A_m, A \in \mathbf{A}$. Without any restriction of generality we can assume that all the sets A_{mn} or all their complements A'_{mn} are at most countable. Under this assumption the abstract subspace $Y = \bigcup_{m,n} A_{mn}$ (or $Z = \bigcup_{m,n} A'_{mn}$) is at most countable. According to Theorem 1 the topological convergence in the system of all subsets of Y (or of Z) is a metrical convergence. Therefore it is possible to choose a sequence $\{A_{m_k n_k}\}_{k=1}^{\infty}$ (or $\{A'_{m_k n_k}\}_{k=1}^{\infty}$), $m_1 \leqq m_2 \leqq \ldots$, topologically converging to A (or to A'). By 1. we conclude that $A_{m_k n_k} \xrightarrow{t} A$.

Since the σ-algebra **A** is not isomorphic with any system of all subsets of an at most countable abstract set, the topological H-convergence in **A** is not — according to Theorem 1 — a metrical convergence.

Резюме.

О ТОПОЛОГИЧЕСКОЙ СХОДИМОСТИ В σ-АЛГЕБРАХ МНОЖЕСТВ

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(Поступило в редакцию 1. 6. 1953 г.)

Пусть **A** означает σ-алгебру множеств, то есть непустую σ-аддитивную и комплементативную систему подмножеств абстрактного множества X. Согласно *Ф. Хаусдорфу*, последовательность множеств $\{A_n\}$ топологически сходится к множеству A, если

$$A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n .$$

В работе прежде всего доказывается, что топологическая сходимость в **A** эквивалентна своей апостериорной сходимости; другими словами: если $\{A_n\}$ не сходится топологически к A, то существует выделенная последовательность $\{A_{n_k}\}$ такая, что ни одна выделенная из нее последовательность $\{A_{n_{k_i}}\}$ не сходится топологически к A.

Пусть теперь в σ-алгебре множеств **A** определена метрика ϱ такая, что оба типа сходимости, как топологическая, так и метрическая, эквивалентны

друг другу. Тогда 1) каждая дизъюнктная подсистема системы **A** будет не более чем счетной, 2) каждая строго возрастающая или убывающая последовательность элементов из **A** будет не более чем счетной и 3) система **A** изоморфна системе всех подмножеств не более чем счетного множества.

Пусть P означает функцию вероятности, определенную на σ-алгебре **A**. Мы будем говорить, что P обладает свойством (α), если из $A \in$ **A** и $P(A) = 0$ следует, что $A = 0$. Следующие три условия эквивалентны друг другу:

а) На σ-алгебре множеств **A** существует метрика ϱ такая, что топологическая сходимость и метрическая сходимость эквивалентны одна другой,

б) на **A** существует вероятность P со свойством (α),

в) система **A** изоморфна системе всех подмножеств некоторого не более чем счетного множества.

Мы будем говорить, что топологическая сходимость обладает диагональным свойством, если имеет место следующее: если последовательность $\{A_{mn}\}_{n=1}^{\infty}$ топологически сходится к A_m, а $\{A_m\}_{m=1}^{\infty}$ к элементу A, то существует последовательность $\{A_{m_k n_k}\}_{k=1}^{\infty}$, $m_1 \leqq m_2 \leqq \ldots$, топологически сходящаяся к A. Каждая метрическая сходимость обладает диагональным свойством. Топологическая сходимость может, однако, не иметь этого свойства, как видно на примере σ-алгебры всех борелевских множеств на прямой. В работе решается вопрос, каждая ли топологическая сходимость с диагональным свойством является метрической. Это не всегда так, как показывает пример σ-алгебры **A**, элементами которой служат все не более чем счетные подмножества и их дополнения какого-либо несчетного абстрактного множества X.