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A CONTRIBUTION TO THE METRIC THEORY OF DIOPHANTINE APPROXIMATIONS

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The sets of numbers the Lebesgue measure of which is zero can be investigated by means of the Hausdorff measure. In this way Jarník investigated the null sets of the numbers x for which there are infinitely many pairs of integers p, q such that the inequality

$$\left|x - \frac{p}{q}\right| < \varphi(q) \tag{1}$$

is fulfilled. ($\varphi(q)$ is a given positive function.) Jarník proved his results in the case of the general simultaneous approximation. In this paper the null sets of the numbers x are investigated for which there is only a finite number of pairs of integers p, q such that the inequality (1) holds.

§ 1. Discussion of the problem.

Let us denote by $g(q)$ a given positive function defined for positive values of q. An irrational number x is said to admit the approximation $g(q)$, if there exist infinitely many pairs of integers p, q ($q > 0$) in such a way that the inequality

1,01. $$\left|x - \frac{p}{q}\right| < \frac{1}{q^2 \cdot g(q)} \tag{1,01}$$

holds.

By means of continuous fractions it is easy to show that to any given function g an irrational number x can be found which admits the approximation g.

The question arises now, ,,how many" such numbers x esixt to a given function g. Khintchine (Хинчин) [1][1]) showed that the Lebesgue measure of the set of the numbers x belonging to the interval (0, 1) and admitting the approximation g is zero, if $\int^{\infty} \frac{dx}{x \cdot g(x)}$ converges and that

[1]) Cf. the references at the end of the paper.

this measure is unity, if $\int\limits^{\infty} \frac{\mathrm{d}x}{x \,.\, g(x)}$ diverges. (He supposed that the function $g(q)$ increases steadily to infinity as q increases to infinity.)

Jarník [2], [3] showed that it is possible to investigate the set of the numbers x belonging to the interval $(0, 1)$ and admitting the approximation g by means of the Hausdorff measure, if $\int\limits^{\infty} \frac{\mathrm{d}x}{x \,.\, g(x)}$ converges. If $\int\limits^{\infty} \frac{\mathrm{d}x}{x \,.\, g(x)}$ diverges then the Lebesgue measure of the set of the numbers x belonging to the interval $(0, 1)$ and not admitting the approximation g is zero. Consequently this set can be investigated by means of Hausdorff measure and this investigation is contained in this paper.

1,02. Let $f(d)$ be a continuous positive function defined for $0 < d < D$ and let us suppose that $f(d) \to \infty$ steadily with $d \to 0$ and that $d \,.\, f(d) \to 0$ steadily with $d \to 0$.

Let Q be a given set of real numbers. The Hausdorff measure f of the set Q is defined as follows:

Let the set Q be covered by a finite or enumerable aggregate of intervals, the lengths of which are denoted by $d_1, d_2, \ldots$; these lengths fulfil the inequalities $d_1 < D$, $d_2 < D, \ldots$ Now we define the number σ by the relation:

$$\sigma = \sum_{i=1}^{\infty} d_i \,.\, f(d_i).$$

(We write $\sigma = \infty$, if the sum on the right side diverges.)

Now we choose another number ϱ, $0 < \varrho < D$, and find the number σ for each aggregate of intervals covering Q and fulfilling the inequalities: $d_1 < \varrho$, $d_2 < \varrho, \ldots$ The lower bound of these numbers σ will be denoted by L_ϱ. L_ϱ apparently does not decrease as ϱ tends to zero and

$$\lim L_\varrho = Hmf(Q) \text{ for } \varrho \to 0$$

the limit of the function L_ϱ, as ϱ tends to zero, is called the outer Hausdorff measure f of the set Q.

It is all the same, whether we used for this definition open, closed, half-closed intervals or all together and following theorems are apparently true:

1,03. If $Q_1 \supset Q_2$, then $Hmf(Q_1) \geqq Hmf(Q_2)$.

1,04. Let the functions $f_5(d)$ and $f_6(d)$ fulfil the conditions (1,02) and let us suppose that

$$f_5(d) > cf_6(d), \; c > 0, \; 0 < d < D.$$

If $Hmf_6(Q) = \infty$, then $Hmf_5(Q) = \infty$; if $Hmf_5(Q) = 0$, then $Hmf_6(Q) = 0$.

1,05. Let the functions $f_5(d)$ and $f_6(d)$ fulfil the conditions (1,02) and let us suppose that

$$\frac{f_5(d)}{f_6(d)} \to \infty \text{ as } d \to 0.$$

If $Hmf_6(Q) > 0$, then $Hmf_5(Q) = \infty$, if $Hmf_5(Q) < \infty$, then $Hmf_6(Q) = 0$.

1,06. Let T_n be sets of real numbers, $T = \sum_{n=1}^{\infty} T_n$, $Hmf(T_n) = 0$, $n = 1, 2, \ldots$

Then $Hmf(T) = 0$.

The theorem (1,05) enables us to establish the notion of Hausdorff dimension. Let $f_s(d)$ be an aggregate of functions which fulfil the conditions 1,02 and let s be an element of a densily ordered set S. Let for any pair $s_1, s_2, s_1 < s_2$ the following condition take place:

$$\frac{f_{s_2}(d)}{f_{s_1}(d)} \to 0 \text{ with } d \to 0$$

and let the given set Q and two elements $s', s'', s' < s''$ fulfil the equations $Hmf_{s'}(Q) = \infty$, $Hmf_{s''}(Q) = 0$. If there exist such an element s^*, that for any element s, $s > s^*$ the relation $Hmf_s(Q) = 0$ holds and for any element s, $s < s^*$ the relation $Hmf_s(Q) = \infty$ holds, then such an element s^* will be called the dimension of the set Q and we write

$$\dim Q = s^*.$$

Such an element exists at most one and its existence is warranted, if we suppose that the set S is ordered without gaps. From (1,03) it follows easily

1,07. If $Q_1 \supset Q_2$ and if $\dim Q_1$ and $\dim Q_2$ exist, then $\dim Q_1 \geqq \dim Q_2$.

The usefulness of the notion of the dimension is illustrated by two theorems due to V. Jarník:

1,08. Let α be a real number, $\alpha > 2$. Let us denote by P_α the set of the numbers x from the interval $(0, 1)$ which admit the approximation $q^{\alpha-2}$. If we put $f_s(d) = d^{s-1}$, then

$$\dim P_\alpha = \frac{2}{\alpha} \text{ (Jarník [2]).}$$

1,09. Let α be an integer, $\alpha > 8$. Let M_α be the set if all numbers x, $0 < x < 1$, having a development in a regular continuous fraction the partial denominators of which are at most α. If we put again $f_s(d) = d^{s-1}$, then

$$1 - \frac{4}{\alpha \,.\, \lg 2} \leqq \dim M_\alpha \leqq 1 - \frac{1}{8\alpha \,.\, \lg \alpha} \quad \text{(Jarník [4]).}$$

1,10. From now we mean by $g(q)$ a function fulfilling the following conditions:

α) $g(q)$ is defined and continuous for $q \geqq w$

β) $g(q) \geqq 4\sqrt{2}$

γ) $\displaystyle\int^{\infty} \frac{\mathrm{d}x}{x \,.\, g(x)}$ diverges

δ) $\dfrac{g(q \,.\, h(q))}{g(q)} \to 1$ as $q \to \infty$, if $h(q)$ is any function such that $1 \leqq h(q) \leqq g(q) + 2$.

1,11. Let us denote by Q_g the set of the numbers x from the interval $(0, 1)$ which do not admit the approximation g.

1,12. Let us put

$$f_1(d) = \exp\left\{\tfrac{2}{3} \int_w^{\frac{1}{\sqrt{d}}} \frac{\mathrm{d}x}{x \,.\, g(x)}\right\}, \qquad f_2(d) = \exp\left\{2 \int_w^{\frac{1}{\sqrt{d}}} \frac{\mathrm{d}x}{x \,.\, g(x)}\right\}. \quad (1,12)$$

The main result of the present paper is the

Theorem I. *If* $g(q) > 10^3$, *then* $Hmf_1(Q_g) = 0$ *and* $Hmf_2(Q_g) = \infty$.

According to this theorem and to the theorem 1,04 we obtain a certain survey about the question, what is the value of the Hausdorff measure of a given set Q_g for different functions f. If we put special functions for $g(q)$ and $f_s(d)$, we may find $\dim Q_g$ by means of this theorem.

§ 2. Deduction of two inequalities.

In regular continuous fractions we have an extraordinary powerful means for investigating the question whether a given number x does or does not admit the approximation g.

It is known that to any irrational number x from the interval $(0, 1)$ a sequence of integers $a_1, a_2, \ldots$ can be uniquely found in such a way that the following relation takes place:

2,01.
$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} \quad (2,01)$$

Conversely to any such a sequence an irrational number x is uniquely defined by means of the relation (2,01).

The number a_i is called the i-th partial denominator, the numbers p_i, q_i defined by the equation

2,02.
$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_i}}} = \frac{p_i}{q_i} \quad p_i, q_i \text{ relatively prime integers, } q_i \text{ positive, } i > 0 \tag{2,02}$$

are called the i-th approximate numerator and denominator respectively. We put for completeness:

$$p_{-1} = 1,\ q_{-1} = 0,\ p_o = 0,\ q_o = 1.$$

We shall need the following formulae:

2,03.
$$p_{i+1} = a_{i+1} \cdot p_i + p_{i-1},\quad q_{i+1} = a_{i+1} \cdot q_i + q_{i-1},\ (i \geqq 0) \tag{2,03}$$

2,04.
$$p_{i+1} \cdot q_i - p_i \cdot q_{i+1} = (-1)^i,\ (i \geqq -1) \tag{2,04}$$

2,05.
$$\frac{1}{(a_{i+1} + 2) \cdot q_i^2} < \left| x - \frac{p_i}{q_i} \right| < \frac{1}{a_{i+1} \cdot q_i^2},\ (i \geqq 0). \tag{2,05}$$

We shall also need the theorem

2,06. If $\left| x - \frac{r}{s} \right| < \frac{1}{2s^2}$, x irrational, r, s integers, then $\frac{r}{s} = \frac{p_i}{q_i}$ for a suitable i

and

2,07.
$$q_i \geqq i. \tag{2,07}$$

Let g be a given function satisfying (1,10) and let us define the sets S_P, T^r_m and T as follows:

Let us denote by $\overline{w}$ the least integer greater than w and D. Let P be an integer greater than $\overline{w}$.

2,08. The number x belongs to the set S_P if and only if its partial denominators fulfil these relations:

$$a_i = 1 \qquad \text{for } i = 1, 2, \ldots, P;$$
$$a_{i+1} \leqq g(q_i) - 2 \ \text{ for } i = P, P+1, \ldots$$

2,09. Let m and r be positive integers, $m \geqq \overline{w}$. The number x belongs to the set T^r_m if and only if its partial denominators fulfil the relations

$$a_i \leqq r \qquad \text{for } i = 1, 2, \ldots, m,$$
$$a_{i+1} \leqq g(q_i) \text{ for } i = m, m+1, \ldots;$$

2,10. $$T = \sum_{r=1}^{\infty} \sum_{m=\overline{w}}^{\infty} T^r_m. \tag{2,10}$$

Now we shall prove the theorem

2,11. $$S_P \subset Q_g \subset T.$$

Proof: If $x \in S_P$, then for $i \geqq P$

$$\frac{1}{g(q_i) \cdot q_i^2} \leqq \frac{1}{(a_{i+1} + 2)\, q_i^2} < \left| x - \frac{p_i}{q_i} \right|$$

(cf. (2,05)) and according to (2,06) x does not admit the approximation g.

If $x \in Q_g$, then there exists such an index m that $a_{i+1} < g(q_i)$ for $i \geqq m$ and we can find an index r in such a way that $a_i \leqq r$ for $i = 1, 2, \ldots, m$ and consequently $x \in T^r_m$.

According to (1,03) the following inequalities hold

$$Hmf(S_P) \leqq Hmf(Q_g) \leqq Hmf(T).$$

In order to prove the Theorem I it is sufficient to prove:

2,12. There is such a P that

$$Hmf_2(S_P) = \infty$$

and for any $m, r, m = \overline{w}, \overline{w} + 1, \ldots, r = 1, 2, \ldots$ it is

$$Hmf_1(T^r_m) = 0. \qquad (1,03,\ 1,06)$$

All irrational numbers from the interval $(0, 1)$ having the first i partial denominators $(a_1, a_2, \ldots, a_i)$ in common, can be written in the form

$$x = \frac{\varepsilon p_i + p_{i-1}}{\varepsilon q_i + q_{i-1}}, \quad \varepsilon \text{ irrational, } \varepsilon > 1$$

and these are just all irrational numbers of an interval, which we shall call the interval of the i-th order and denote by I^i or in detail $I^i_{a_1, \ldots, a_i}$.

2,13. $$I^i_{a_1, a_2, \ldots, a_i} = \left[\frac{p_i + p_{i-1}}{q_i + q_{i-1}}, \frac{p_i}{q_i}\right] \text{ (closed interval).} \tag{2,13}$$

Apparently

2,14. $$I^{i+1}_{a_1, \ldots, a_i, k} \subset I^i_{a_1, \ldots, a_i}. \tag{2,14}$$

2,15. If j, k are different positive integers, then the intervals $I^{i+1}_{a_1, \ldots, a_i, k}$ and $I^{i+1}_{a_1, \ldots, a_i, j}$ have at most one point in common.

The length of an interval I will be denoted by $|I|$. Then it holds:

2,16. $$|I^{i+1}_{a_1, \ldots, a_{i+1}}| = \frac{1}{q_{i+1}(q_{i+1} + q_i)} \quad i \geqq 0. \tag{2,16}$$

Let $g^+(q)$ be the greatest integer less or equal to $g(q)$.

Let us choose the integers m, r, $m \geqq \overline{w}$, $r \geqq 1$ and define the sets V_i, $i = m, m+1, \dots$ which cover T_m. We put

$$V_m = \sum_{a_1, \dots, a_m} I^m_{a_1, \dots, a_m}$$

where the sum runs over all possible combinations of the indices $a_1, \dots$, a_m, where $a_1 = 1, 2, \dots, r, \dots$; $a_m = 1, 2, \dots, r$.

If V_i is defined as the sum of a finite number of intervals I^i $i \geqq m$, then let us define V_{i+1} as follows:

Instead of any interval $I^i_{a_1, \dots, a_i}$ which is used in the definition of V_i we put the sum

$$\sum_{k=1}^{g^+(q_i)} I^{i+1}_{a_1, \dots, a_i, k}$$

and so we can write

$$V_{i+1} = \sum_{a_1, \dots, a_i} \sum_{k=1}^{g^+(q_i)} I^{i+1}_{a_1, \dots, a_i, k}$$

where the sum runs over all combinations of the indices $a_1, \dots, a_i$ which occur in the definition of V_i.

According to the definitions of the sets V_i and T^r_m it is clear that

2,17. $$V_i \supset V_{i+1} \supset T^r_m. \tag{2,17}$$

Now we state the Theorem II, which together with Theorem III is the main aim of this paragraph.

Theorem II. *Suppose that there exists an integer $P \geqq \overline{w}$ in such a way that for any q and t satisfying the conditions $q \geqq P$, $0 < t < 1$, the following inequality is fulfilled*

2,18. $$\frac{f\left(\dfrac{1}{q^2(1+t)}\right)}{1+t} \geqq \sum_{k=1}^{g^+(q)} \frac{f\left(\dfrac{1}{q^2(k+t)(k+t+1)}\right)}{(k+t)(k+1+t)}. \tag{2,18}$$

Then

$$Hmf(T^r_m) < \infty,\ m = \overline{w}, \overline{w}+1, \dots, r = 1, 2, \dots$$

Proof: We divide the inequality (2,18) by q^2, put $q = q_i$, $t = \dfrac{q_{i-1}}{q_i}$ and then for $i \geqq P$ we have

$$\frac{f\left(\dfrac{1}{q_i(q_i + q_{i-1})}\right)}{q_i(q_i + q_{i-1})} \geqq \sum_{k=1}^{g^+(q_i)} \frac{f\left(\dfrac{1}{(kq_i + q_{i-1})((k+1)q_i + q_{i-1})}\right)}{(kq_i + q_{i-1})((k+1)q_i + q_{i-1})}$$

and according to (2,16) and (2,03) the following inequality holds for any interval I^i, $i \geqq P$

$$|I^i_{a_1,\ldots,a_i}| \cdot f(|I^i_{a_1,\ldots,a_i}|) \geqq \sum_{k=1}^{g^+(q_i)} |I^{i+1}_{a_1,\ldots,a_i,k}| \cdot f(|I^{i+1}_{a_1,\ldots,a_i,k}|).$$

Let us put

$$\sigma_i = \Sigma |I^j| \,.\, f(|I^j|)$$

where the sum runs over all intervals of the i-th order, which are subsets of V_i.

According to the last inequality we apparently get:

$$\sigma_P \geqq \sigma_{P+1} \geqq \sigma_{P+2} \geqq \cdots$$

and the proof of the Theorem II is finished.

Now we shall state the Theorem III, which is in a certain sense an opposite one to the Theorem II.

Theorem III. *Let us suppose that*

2,19. $$\frac{f\left(\dfrac{1}{q^2 \,.\, g^2(q)}\right)}{f\left(\dfrac{1}{q^2}\right)} < K,\ K > 0 \ \textit{for}\ q \geqq \overline{w} \tag{2,19}$$

and let P be such an integer satisfying $P \geqq \overline{w}$ that for any q and t satisfying $q \geqq P$ and $0 < t < 1$ the following inequality holds

2,20. $$\frac{f\left(\dfrac{1}{q^2(1+t)}\right)}{1+t} \leqq \sum_{k=1}^{g^+(q)-2} \frac{f\left(\dfrac{1}{q^2(k+t)\,(k+1+t)}\right)}{(k+t)\,(k+1+t)}. \tag{2,20}$$

Then

$$Hmf(S_P) > 0.$$

Proof: Similarly as in the preceding case we define the sets U_i, $i \geqq P$ which cover the set S_P.

$$U_P = I^P_{1,\ldots,1}.$$

If U_i is defined as a sum of a finite number of intervals I^i we define U_{i+1} in the following manner:

Instead of any interval $I^i_{a_1,\ldots a_i}$ which is used in the definition of U_i we put the sum

$$\sum_{k=1}^{g^+(q_i)-2} I^{i+1}_{a_1,\ldots,a_i,k}$$

and so we can write

$$U_{i+1} = \sum_{a_1, \ldots, a_i} \sum_{k=1}^{g^+(q_i)-2} I^{i+1}_{a_1, \ldots, a_i, k}$$

where the sum runs over all combinations of the indices $a_1, \ldots, a_i$ occurring in the definition of U_i.

It is apparently true that U_i is a closed set, $U_i \supset U_{i+1} \supset S_P$, and if we put $U = \prod_{i=P}^{\infty} U_i$, then U is a closed set, $U; \supset S_P$ U is a perfect set, as any interval I^i contains at least two mutually disjoint, intervals I^{i+2} and $|I^i| < \frac{1}{i^2}$.

Now we prove an auxiliary theorem:

2,21. $$U = S_P \text{ (and therefore } Hmf(U) = Hmf(S_P)).$$

Proof: Let x be an irrational number, $x \in U$. Then $x \in S_P$ for, according to the definition of the sets U_i, its partial denominators fulfil the conditions (2,08). But U cannot contain any rational number. For to each rational number y from the interval (0, 1) there exists an interval I^i in such a way that y is an end point of this interval. Then y does not belong to U_{i+2}.

Now we return to the proof of the Theorem III.

If Z is a finite or enumerable aggregate of intervals, we put

(*) $$\sigma_Z = \sum_i d_i f(d_i)$$

where d_i is the length of the i-th interval from the aggregate Z and in (*) the sum runs over all intervals from the system Z.

Let Z be an arbitrarily chosen finite or enumerable aggregate of open intervals which covers U.

According to the known Borel theorem we may select a finite aggregate Y of open intervals from the aggregate Z which covers U too. From the aggregate Y we eliminate such intervales which have no point in common with U. Instead of each interval I from the aggregate Y which has at least one point in common with the set U we take a closed interval L, the endpoints of which α and β are defined as follows:

$$\alpha = \inf x, \quad x \in I \cap U;$$
$$\beta = \sup x, \quad x \in I \cap U.$$

(Note: $I \cap U$ contains infinitely many points, as U is a perfect set and I is an open interval.) As U is a closed set, we have $\alpha \in U$ and $\beta \in U$.

Let us denote by X this aggregate of closed intervals L. Consequently X is a finite aggregate of closed intervals the endpoints of which belong to U and which covers U. Then it holds: $\sigma_X \leqq \sigma_Y \leqq \sigma_Z$.

In order to prove that $Hmf(S_P) > 0$ we shall use the following auxiliary theorem:

2,22. *To any interval L from the aggregate X there is a finite number of intervals* $I^i(I_1^{i_1}, I_2^{i_2}, \ldots, I_p^{i_p}; i_k \geqq P)$ *satisfying the following two conditions:*

1. $L \cap U \subset \sum_{k=1}^{p} I_k^{i_k}$,

2. $|L| \cdot f(|L|) \geqq \frac{1}{64K} \sum_{k=1}^{p} |I_k^{i_k}| \cdot f(|I_k^{i_k}|)$.

This auxiliary theorem will be proved at the end of this paragraph. But first of all we shall use it to prove the relation:

$$Hmf(U) > 0.$$

According to the auxiliary theorem we take instead of the aggregate X the finite aggregate of intervals I^i mentioned in (22,2), if needed we omit some intervals from the new aggregate and denote by W the aggregate of intervals I^i which fulfils the following conditions:

1. W covers U;

2. If $I_1^{i_1} \in W$, $I_2^{i_2} \in W$, $I_1^{i_1} \neq I_2^{i_2}$ then the inclusion $I_1^{i_1} \subset I_2^{i_2}$ cannot be true.

Evidently we have [by (22,2)]

$$\frac{1}{64K} \sigma_W < \sigma_X.$$

If all numbers i_k in (2,22) are equal to P, then

$$\sigma_W = \sigma_{U_P} \ (W = U_P).$$

If the equation $i_k = P$ is not true for all numbers i_k from (2,22), we rewrite the inequality (2,20), similarly as in the proof of the Theorem II.

$$2{,}23. \quad |I^i_{a_i, \ldots, a_i}| \cdot f(|I^i_{a_1, \ldots, a_i}|) \leqq \sum_{k=1}^{g^+(q_i)-2} |I^{i+1}_{a_1, \ldots, a_i, k}| \, f(|I^{i+1}_{a_1, \ldots, a_i, k}|). \quad (2{,}23)$$

Let be $I^{i+1}_{a_1, \ldots, a_i, n}$ an interval of the highest order belonging to W. Then all the intervals $I^{i+1}_{a_1, \ldots, a_i, k}$, $k = 1, 2, \ldots, g^+(q_i) - 2$ belong to W as every interval $I^{i+1}_{a_1, \ldots, a_i, k}$ contains infinitely many points of U and according to the second condition fulfilled by the aggregate W the intervals $I^i_{a_1, \ldots, a_i}, I^{i-1}_{a_1, \ldots, a_{i-1}}, \ldots, I^P_{a_1, \ldots, a_P}$ do not belong to W (cf. (2,14)), From the aggregate W we get the aggregate W_1, if we take instead of the intervals $I^{i+1}_{a_1, \ldots, a_i, k}$, $k = 1, 2, \ldots, g^+(q_i) - 2$ the interval $I^i_{a_1, \ldots, a_i}$. Then we have

$$\sigma_{W_1} \leqq \sigma_W.$$

If $W_1 \neq U_P$, we use the same method for the aggregate W_1 as we already used in the case of the system W. After a finite number of u steps we get $W_u = U_P$, consequently $\sigma_{U_P} \leqq \sigma_W$ and

2,24.
$$\frac{1}{64K}\, \sigma_{U_P} \leqq \sigma_X. \tag{2,24}$$

As on the left side of the inequality (2,24) is a positive constant, the relation $Hmf(U) = 0$ cannot be true.

Now we shall prove the auxiliary theorem (2,22).

Proof: By L let us denote an interval from the system X. There is just one interval $I^i_{a_1, \ldots, a_i}$ that fulfils the following conditions

α) $I^i_{a_1, \ldots, a_i} \supset L$;

β) The inclusion $I^{i+1}_{a_1, \ldots, a_i, k} \supset L$ is not true for any k, $k = 1, 2, \ldots$

Apparently $i \geqq P$.

We shall distinguish three cases:

1. There is such an interval $I^{i+1}_{a_1, \ldots, a_i, k}$ that $L \supset I^{i+1}_{a_1, \ldots, a_i, k}$.

Let us denote by

u the least integer for that $L \supset I^{i+1}_{a_1, \ldots, a_i, u}$

v the greatest integer for that $L \supset I^{i+1}_{a_1, \ldots, a_i, v}$

u' the least integer for that $L \cap I^{i+1}_{a_1, \ldots, a_i, u'} \neq \emptyset$

v' the greatest integer for that $L \cap I^{i+1}_{a_1, \ldots, a_i, v'} \neq \emptyset$

The intervals $I^{i+1}_{a_1, \ldots, a_i, k}$, $k = u', u' + 1, \ldots, v'$ apparently fulfil the condition 1. of the theorem (2,22). Now we show that they fulfil the condition 2. too.

$$\sum_{k = u'}^{v'} |I^{i+1}_{a_1, \ldots, a_i, k}| = \left| \frac{u' \cdot p_i + p_{i-1}}{u' \cdot q_i + q_{i-1}} - \frac{(v' + 1)\, p_i + p_{i-1}}{(v' + 1)\, q_i + q_{i-1}} \right| =$$

$$= \frac{v' + 1 - u'}{(u' \cdot q_i + q_{i-1})\, ((v' + 1)\, q_i + q_{i-1})},$$

$$|L| \geqq \sum_{k = u}^{v} |I^{i+1}_{a_1, \ldots, a_i, k}| = \frac{v + 1 - u}{(u \cdot q_i + q_{i-1})\, ((v + 1)\, q_i + q_{i-1})},$$

2,251
$$\frac{\sum_{k=u'}^{v'} |I^{i+1}_{a_1,\dots,a_i,k}|}{|L|} \leqq \frac{v'+1-u'}{v+1-u} \cdot$$
$$\cdot \frac{(uq_i+q_{i-1})((v+1)q_i+q_{i-1})}{(u'q_i+q_{i-1})((v'+1)q_i+q_{i-1})} \leqq 6, \tag{2,251}$$

2,252
$$\frac{f(|I^{i+1}_{a_1,\dots,a_i,v'}|)}{f(|L|)} \leqq \frac{f\left(\frac{1}{(v'\cdot q_i+q_{i-1})((v'+1)q_i+q_{i-1})}\right)}{f\left(\frac{1}{q_i(q_i+q_{i-1})}\right)} \leqq$$
$$\leqq \frac{f\left(\frac{1}{g^2(q_i)\cdot q_i^2)}\right)}{f\left(\frac{1}{q_i^2}\right)} < K, \tag{2,252}$$

as
$$|L| \leqq |I^i_{a_1,\dots,a_i}|,\ v'+2 \leqq g^+(q_i),\ 0 < q_{i-1} < q_i.$$

Hence

2,253
$$\frac{1}{9K}\sum_{k=u'}^{v'} |I^{i+1}_{a_1,\dots,a_i,k}| \cdot f(|I^{i+1}_{a_1,\dots,a_i,k}|) \leqq$$
$$\leqq \frac{f(|I^{i+1}_{a_1,\dots,a_i,v'}|)}{9K}\sum_{k=u'}^{v'} |I^{i+1}_{a_1,\dots,a_i,k}|. \tag{2,253}$$

In the other two cases the inclusion $L \supset I^{i+1}_{a_1,\dots,a_i,k}$ is false for any integer k. Apparently there is an integer n in such a way that
$$L \cap I^{i+1}_{a_1,\dots,a_i,n} \neq \emptyset \neq I^{i+1}_{a_1,\dots,a_i,n+1} \cap L$$
and
$$L \cap I^{i+1}_{a_1,\dots,a_i,k} = \emptyset \text{ for } n \neq k \neq n+1.$$

2. Let the following inequality take place:
$$|L| \geqq |I^{i+2}_{a_1,\dots,a_i,n+1,1}| = \frac{1}{((n+2)q_i+q_{i-1})((2n+3)q_i+2q_{i-1})}.$$

Then we may replace the interval L by the intervals $I^{i+1}_{a_1,\dots,a_i,k}$, $k = n, n+1$.
$$\frac{\sum_{k=n}^{n+1} |I^{i+1}_{a_1,\dots,a_i,k}|}{|L|} < \frac{2|I^{i+1}_{a_1,\dots,a_i,n}|}{|I^{i+2}_{a_1,\dots,a_i,n+1,1}|} =$$
$$= \frac{2((n+2)q_i+q_{i-1})((2n+3)q_i+2q_{i-1})}{(nq_i+q_{i-1})((n+1)q_i+q_{i-1})} < \frac{4(n+3)^2}{n^2} \leqq 64,$$

$$|L| < 2 \,.\, |I^{i+1}_{a_1, \dots, a_i, n}| = \frac{2}{(nq_i + q_{i-1})\,((n+1)\,q_i + q_{i-1})},$$

$$\frac{f(|I^{i+1}_{a_1, \dots, a_i, n+1}|)}{f(|L|)} \leqq \frac{f\left(\dfrac{1}{((n+1)\,q_i + q_{i-1})\,((n+2)\,q_i + q_{i-1})}\right)}{f\left(\dfrac{2}{(nq_i + q_{i-1})\,((n+1)\,q_i + q_{i-1}))}\right)} \leqq$$

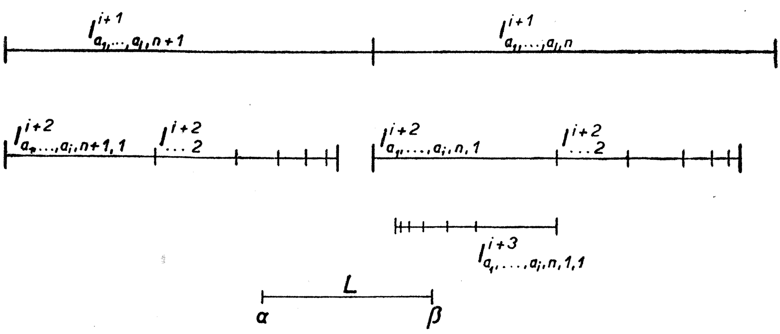
$$\leqq \frac{f\left(\dfrac{1}{2\left(1+\dfrac{3}{n}\right)^2} \cdot \dfrac{2}{(nq_i)^2}\right)}{f\left(\dfrac{2}{(nq_i)^2}\right)} < K$$

and therefore

2,26.
$$\frac{1}{64K} \sum_{k=n}^{n+1} |I^{i+1}_{a_1, \dots, a_i, k}| \,.\, f(|I^{i+1}_{a_1, \dots, a_i, k}|) \leqq$$
$$\leqq \frac{1}{64K} f(|I^{i+1}_{a_1, \dots, a_i, n+1}|) \sum_{k=n}^{n+1} |I^{i+1}_{a_1, \dots, a_i, k}| < |L| \,.\, f(|L|). \tag{2,26}$$

Let us turn to the last case characterised by the relation:

3. $|L| < |I^{i+2}_{a_1, \dots, a_i, n+1, 1}|$.



Let us suppose that n is even. Then according to (2,03), (2,04) and (2,13) it follows that (see the figure)

$I^{i+1}_{a_1, \dots, a_i, n+1}$ preceds $I^{i+1}_{a_1, \dots, a_i, n}$

$I^{i+2}_{a_1, \dots, a_i, l, k}$ preceds $I^{i+2}_{a_1, \dots, a_i, l, k+1}$

$I^{i+3}_{a_1, \dots, a_i, n, 1, k+1}$ preceds $I^{i+3}_{a_1, \dots, a_i, n, 1, k}$.

According to our assumptions $\alpha \in I^{i+1}_{a_1, \ldots, a_i, n+1}$ and $\beta \in I^{i+2}_{a_1, \ldots, a_i, n, 1}$. and there is a u and a v so that $\alpha \in I^{i+2}_{a_1, \ldots, a_i, n+1, u}$, $\beta \in I^{i+3}_{a_1, \ldots, a_i, n, 1, v}$.

Now we show that all the intervals $I^{i+2}_{a_1, \ldots, a_i, n+1, k}$, $k = u, u+1, \ldots$ $\ldots, g^+(q_{i+1}) - 2$ and $I^{i+3}_{a_1, \ldots, a_i, n, 1, k}$, $k = v, v+1, \ldots, g^+(q_{i+2}) - 2$ which have at least one point in common with the interval $[\alpha, \beta]$ fulfil the conditions of the theorem (2,22).

$$\frac{\sum\limits_{k=u}^{g^+(q_{i+1})-2} |I^{i+}_{a_1, \ldots, a_i\, n+1, k}|}{|L|} \leq \frac{\sum\limits_{k=u}^{\infty} |I^{i+2}_{a_1, \ldots, a_i, n+1, k}|}{\sum\limits_{k=u+1}^{\infty} |I^{i+1}_{a_1, \ldots, a_i, n+1,}} =$$

$$= \frac{q_{i+1}((u+1)\, q_{i+1} + q_i)}{q_{i+1}(uq_{i+1} + q_i)} < 3.$$

Similarly

$$\frac{\sum\limits_{k=v}^{g^+(q_{i+2})-2} |I^{i+3}_{a_1, \ldots, a_i, n, 1, k}|}{|L|} < 3.$$

The shortest from these intervals is either $I^{i+3}_{a_1, \ldots, a_i, n, 1, g^+(q_{i+2})-2}$ or $I^{i+2}_{a_1, \ldots, a_i, n+1, g^+(q_{i+1})-2}$. Let us denote it by I^*.

$$\frac{f(|I^{i+3}_{a_1, \ldots, a_i, n, 1, g^+(q_{i+2})-2}|)}{f(|L|)} \leq \frac{f(|I^{i+3}_{a_1, \ldots, a_i, n, 1, g^+(q_{i+2})-2}|)}{f(|I^{i+2}_{a_1, \ldots, a_i, n, 1}|)} =$$

$$= \frac{f\left(\dfrac{1}{((g^+(q_{i+2}) - 2)\, q_{i+2} + q_{i+1})\, ((g^+(q_{i+2}) - 1)\, q_{i+2} + q_{i+1})}\right)}{f\left(\dfrac{1}{q_{i+2}(q_{i+2} + q_{i+1})}\right)} \leq$$

$$\leq \frac{f\left(\dfrac{1}{g^2(q_{i+2}) \cdot q^2_{i+2})}\right)}{f\left(\dfrac{1}{q^2_{i+2}}\right)} < K,$$

and similarly

$$\frac{f(|I^{i+2}_{a_1, \ldots, a_i, n+1, g^+(q_{i+1})-2}|)}{f(|L|)} \leq \frac{f(|I^{i+2}_{a_1, \ldots, a_i, n+1, g^+(q_{i+1})-2}|)}{f(|I^{i+1}_{a_1, \ldots, a_i, n+1}|)} < K.$$

Consequently

$$\frac{f(|I^*|)}{f(|L|)} < K$$

and

$$2{,}27. \qquad \frac{1}{64K}\left\{\sum_{k=u}^{g^+(q_{i+1})-2} |I^{i+2}_{a_1,\dots,a\ \ ,n+1,k}| \cdot f(|I^{i+2}_{a_1,\dots,a_i,\,n+1,k}|) + \right.$$

$$\left. + \sum_{k=v}^{g^+(q_{i+2})-2} |I^{i+3}_{a_1,\dots,a_i,\,n,1,k}| \cdot f(I^{i+3}_{a_1,\dots,a_i,\,n,1,k}|)\right\} \leqq \qquad (2{,}27)$$

$$\leqq \frac{1}{64K} f(I^*)\{\sum_k |I^{i+2}_{a_1,\dots,a_i\,n+1,k}| + \sum_k |I^{i+3}_{a_1,\dots,a_i,\,n,1,k}|\} \leqq |L| \cdot f(|L|).$$

The auxiliary theorem (2,22) is true according to (2,251), (2,252), (2,253), (2,26) and (2,27).

§ 3. The proof of the Theorem I.

We put

$$f_3(d) = \exp\left\{\frac{2}{2{,}9}\int_w^{\frac{1}{\sqrt{d}}} \frac{dx}{x \cdot g(x)}\right\}$$

$$f_4(d) = \exp\left\{\frac{2{,}04}{1{,}025}\int_w^{\frac{1}{\sqrt{d}}} \frac{dx}{x \cdot g(x)}\right\}$$

and then we prove by means of the Theorems II and III that

3,01. the inequality $Hmf_3(T^r_m) < \infty$ holfs for any positive integers m, r and that

3,02. there is such a positive integer P that the inequality $Hmf_4(S_P) > 0$ takes place.

It follows according to (1,05) that $Hmf_1(T^r_m) = 0$ and that $Hmf_2(S_P) = \infty$ and the Theorem I is now proved (cf. (2,12)).

First of all we make us sure that the functions $f_i(d)$ $(i = 1, 2, 3, 4)$ fulfil the suppositions (1,02).

Apparently the functions $f_i(d)$ increase steadily to infinity as d tends to zero. Now we prove that $d \,.\, f(d) \to 0$ steadily with $d \to 0$. We put $q = d^{-\frac{1}{2}}$, consequently $q \to \infty$.

Suppose that $0 < c < 3$. As $g(x) > 4$, we have

$$\frac{1}{q^2}\exp\left\{c\int\limits_w^q \frac{\mathrm{d}x}{x\,.\,g(x)}\right\} \leqq \frac{1}{q^2}\exp\left\{\tfrac{3}{4}\lg\frac{q}{w}\right\} < \frac{C}{q}$$

and therefore $d\,.\,f_i(d) \to \infty$ with $d \to 0$.

Suppose that $q_1 > q_2 \geqq w$. Then

$$\frac{1}{q_1^2}\exp\left\{c\int\limits_w^{q_1}\frac{\mathrm{d}x}{x\,.\,g(x)}\right\} : \frac{1}{q_2^2}\exp\left\{c\int\limits_w^{q_2}\frac{\mathrm{d}x}{x\,.\,g(x)}\right\} = \left(\frac{q_2}{q_1}\right)^2\exp\left\{c\int\limits_{q_2}^{q_1}\frac{\mathrm{d}x}{x\,.\,g(x)}\right\} \leqq$$

$$\leqq \left(\frac{q_2}{q_1}\right)^2\exp\left\{\tfrac{3}{4}\lg\frac{q_1}{q_2}\right\} = \left(\frac{q_2}{q_1}\right)^{\frac{5}{4}} < 1$$

and therefore $d\,.\,f_i(d)$ are monotonous functions. The proof that the functions $f_i(d)$ fulfil the suppositions (1,02) is finished.

Now we prove (3,01) and (3,02).

We make us sure according to the assumptions made about the function g (1,10) and according to the assumption of the Theorem I ($g(q) > 1000$) that the function f_3 fulfils the inequality (2,18) and and that the function f_4 fulfils the inequalities (2,19) and (2,20).

We choose $P \geqq \overline{w}$ in such a way that the following inequality

3,04. $$\left|\frac{g(y)}{g(q)} - 1\right| < 0{,}01,\ q < y < (g(q) + 2)\,.\,q \qquad (3{,}04)$$

takes place for any $q \geqq P$. Then the following inequalities are true:

3,05. $$\frac{g(y)}{g(q)}\,.\,\frac{1}{1 + \dfrac{2}{g(q)}} > 0{,}99\,.\,0{,}998 > 0{,}98,\ q < y < (g(q) + 2)\,.\,q \quad (3{,}05)$$

3,06. $$\exp\left\{\frac{2}{2{,}9}\int\limits_q^{q.\lambda}\frac{\mathrm{d}x}{x\,.\,g(x)}\right\} < 1 + \frac{2}{2{,}8}\int\limits_q^{q.\lambda}\frac{\mathrm{d}x}{x\,.\,g(x)},\ 1 < \lambda \leqq g(q) + 2. \quad (3{,}06)$$

Proof: We put $z = \dfrac{2}{2{,}9}\displaystyle\int\limits_q^{q.\lambda}\frac{\mathrm{d}x}{x\,.\,g(x)}$ and start from the inequality $\exp z < 1 + z(1 + z)$ for $0 < z < 1$. According to the mean value theorem we have

$$1 + z \leqq 1 + \frac{2}{2{,}9}\,\frac{\lg(g(q) + 2)}{g(y)} \leqq 1 + \frac{2}{2{,}9}\,\frac{g(q)}{g(y)}\,\frac{\lg 1.002}{1.000} < 1{,}03 < \frac{2{,}9}{2{,}8}$$

and therefore (2,06) is true.

(3,07) $$\frac{g(y)}{g(q)-2} < 1{,}02, \quad q \leqq y \leqq (g(q)+2)\,q \tag{3,07}$$

(3,08) $$\frac{2\lg g(q)}{g(q)-2} + \frac{2}{g(q)-3} < 0{,}02 \tag{3,08}$$

Now we show that the following inequality

3,09. $$\frac{\exp\left\{\dfrac{2}{2{,}9}\displaystyle\int_w^{q\sqrt{1+t}} \frac{\mathrm{d}x}{x\,.\,g(x)}\right\}}{1+t} \geqq \sum_{k=1}^{g^+(q)} \frac{\exp\left\{\dfrac{2}{2{,}9}\displaystyle\int_w^{q\sqrt{(k+t)(k+1+t)}} \frac{\mathrm{d}x}{x\,.\,g(x)}\right\}}{(k+t)\,(k+1+t)} \tag{3,09}$$

is true for q, $q \geqq P$ and t, $0 < t < 1$.

We divide the inequality by the expression $\exp\left\{\dfrac{2}{2{,}9}\displaystyle\int_w^{q\sqrt{1+t}} \frac{\mathrm{d}x}{x\,.\,g(x)}\right\}$, then we replace the value $\exp\left\{\dfrac{2}{2{,}9}\displaystyle\int_{q\sqrt{1+t}}^{q\sqrt{(k+t)(k+1+t)}} \frac{\mathrm{d}x}{x\,.\,g(x)}\right\}$ by the following greater terms

$$1 + \frac{2}{2{,}8}\int_{q\sqrt{1+t}}^{q\sqrt{(k+t)(k+1+t)}} \frac{\mathrm{d}x}{x\,.\,g(x)} < 1 + \frac{1}{2{,}8}\cdot\frac{1}{g(y)}\lg\frac{(k+t)\,(k+1+t)}{1+t}$$

where y does not depend on k (if q is given).

Instead of (3,09) it is sufficient to prove

3,10. $$\frac{1}{1+t} \geqq \sum_{k=1}^{g^+(q)} \frac{1}{(k+t)\,(k+1+t)} + \\ + \frac{1}{2{,}8\,.\,g(y)} \sum_{k=1}^{g^+(q)} \frac{\lg\dfrac{(k+t)\,(k+1+t)}{1+t}}{(k+t)\,(k+1+t)}; \tag{3,10}$$

$$\frac{1}{(k+t)\,(k+1+t)} = \frac{1}{k+t} - \frac{1}{k+1+t}$$

and hence

$$\sum_{k=1}^{g^+(q)} \frac{1}{(k+t)(k+1+t)} = \frac{1}{1+t} - \frac{1}{g^+(q)+1+t} \leqq$$

$$\leqq \frac{1}{1+t} - \frac{1}{g(q)+2}.$$

The second sum we estimate similarly

$$\sum_{k=1}^{g^+(q)} \frac{\lg \dfrac{(k+t)(k+1+t)}{1+t}}{(k+t)(k+1+t)} =$$

$$= \sum_{k=1}^{g^+(q)} \lg \frac{(k+t)(k+1+t)}{1+t} \cdot \left\{\frac{1}{k+t} - \frac{1}{k+1+t}\right\} =$$

$$= \frac{\lg(2+t)}{1+t} - \frac{\lg \dfrac{(g^+(q)+t)(g^+(q)+1+t)}{1+t}}{g^+(q)+1+t} + \sum_{k=2}^{g^+(q)} \frac{\lg \dfrac{k+1+t}{k-1+t}}{k+t} <$$

$$< \frac{\lg(2+t)}{1+t} + 2 \sum_{k=2}^{g^+(q)} \frac{1}{(k-1+t)(k+t)} < \frac{\lg(2+t)}{1+t} + \frac{2}{1+t} <$$

$$< \frac{\lg 2 + 2}{1} < 2{,}7$$

and consequently it is sufficient to prove that $\dfrac{1}{g(q)+2} \geqq \dfrac{2{,}7}{2{,}8 \,.\, g(y)}$ or, what is the same, that

$$\frac{g(y)}{g(q)\left(1+\dfrac{2}{g(q)}\right)} \geqq \frac{2{,}7}{2{,}8};$$

but this is true according to (3,05) and so (3,01) follows from Theorem II.

Now we prove (3,02). First of all there is

$$\frac{\exp\left\{\dfrac{2{,}04}{1{,}025}\displaystyle\int_w^{q \,.\, g(q)} \frac{dx}{x \,.\, g(x)}\right\}}{\exp\left\{\dfrac{2{,}04}{1{,}025}\displaystyle\int_w^{q} \frac{dx}{x \,.\, g(x)}\right\}} = \exp\left\{\frac{2{,}04}{1{,}025}\int_q^{q \,.\, g(q)} \frac{dx}{x \,.\, g(x)}\right\} =$$

$$= \exp\left\{\frac{2{,}04}{1{,}025}\,\frac{1}{g(y)}\lg g(q)\right\},$$

$$q < y < q \cdot g(q)$$

and the last expression is really bounded.

We show that the inequality

3,11.
$$\frac{\exp\left\{\dfrac{2,04}{1,025}\displaystyle\int_w^{q\sqrt{1+t}}\frac{\mathrm{d}x}{x \cdot g(x)}\right\}}{1+t} \leqq \sum_{k=1}^{g^+(q)-2} \frac{\exp\left\{\dfrac{2,04}{1,025}\displaystyle\int_w^{q\sqrt{(k+t)(k+1+t)}}\frac{\mathrm{d}x}{x \cdot g(x)}\right\}}{(k+t)(k+1+t)} \tag{3,11}$$

is satisfied.

Similarly as in the preceding case we divide the inequality by the expression $\exp\left\{\dfrac{2,04}{1,025}\displaystyle\int_w^{q\sqrt{1+t}}\frac{\mathrm{d}x}{x \cdot g(x)}\right\}$ and use the inequality $\exp z > 1 + z$ $(z > 0)$,

$$\int_{q\sqrt{1+t}}^{q\sqrt{(k+t)(k+1+t)}} \frac{\mathrm{d}x}{x \cdot g(x)} \geqq \frac{1}{2 \cdot g(y)} \lg \frac{(k+t)(k+1+t)}{1+t}.$$

Instead of (3,11) it suffices to prove

3,12.
$$\frac{1}{1+t} \leqq \sum_{k=1}^{g^+(q)-2} \frac{1}{(k+t)(k+1+t)} + \frac{1,02}{1,025}\frac{1}{g(y)} \sum_{k=1}^{g^+(q)-2} \frac{\lg\dfrac{(k+t)(k+1+t)}{1+t}}{(k+t)(k+1+t)} \tag{3,12}$$

$$\sum_{k=1}^{g^+(q)-2} \frac{1}{(k+t)(k+1+t)} = \frac{1}{1+t} - \frac{1}{g^+(q)-1+t} \geqq$$

$$\geqq \frac{1}{1+t} - \frac{1}{g(q)-2}.$$

The estimation of the second sum will be rather longer:

3,13.
$$\sum_{k=1}^{g^+(q)-2} \frac{\lg\dfrac{(k+t)(k+1+t)}{1+t}}{(k+t)(k+1+t)} = \tag{3,13}$$

$$= \sum_{k=1}^{g^+(q)-2} \lg \frac{(k+t)(k+1+t)}{1+t}\left(\frac{1}{k+t} - \frac{1}{k+1+t}\right) =$$

$$= \frac{\lg(2+t)}{1+t} - \frac{\lg \dfrac{(g^+(q)-2+t)\,(g^+(q)-1+t)}{1+t}}{g^+(q)-1+t} + $$

$$+ \sum_{k=2}^{g^+(q)-2} \frac{1}{k+t} \lg\frac{k+1+t}{k-1+t}, \tag{3,13}$$

$$\lg\left(1 + \frac{2}{k-1+t}\right) > \frac{2}{k-1+t} - \frac{2}{(k-1+t)^2}$$

and so (3,13) is greater than

3,14.
$$\frac{\lg(2+t)}{1+t} - \frac{2\lg g(q)}{g(q)-2} + 2\sum_{k=2}^{g^+(q)-2} \frac{1}{(k-1+t)\,(k+t)} - $$

$$- 2\sum_{k=2}^{g^+(q)-2} \frac{1}{(k-1+t)^2\,(k+t)} \tag{3,14}$$

$$\sum_{k=2}^{g^+(q)-2} \frac{1}{(k-1+t)^2\,(k+t)} <$$

$$< \frac{1}{(1+t)^2\,(2+t)} + \sum_{k=3}^{g^+(q)-2} \frac{1}{(k-2+t)\,(k-1+t)\,(k+t)}$$

$$\frac{1}{(k-2+t)\,(k-1+t)\,(k+t)} = \frac{1}{2}\frac{1}{k-2+t} - \frac{1}{k-1+t} +$$

$$+ \frac{1}{2}\frac{1}{k+t}$$

and therefore

$$\sum_{k=2}^{g^+(q)-2} \frac{1}{(k-1+t)^2\,(k+t)} < \frac{1}{(1+t)^2\,(2+t)} +$$

$$+ \frac{1}{2}\frac{1}{1+t} - \frac{1}{2}\frac{1}{2+t}$$

and consequently 3,14 is greater than

3,15.
$$\frac{\lg(2+t)}{1+t} - \frac{2\lg g(q)}{g(q)-2} + \frac{2}{1+t} - \frac{2}{g(q)-3} - \frac{2}{(1+t)^2\,(2+t)} -$$

$$- \frac{1}{1+t} + \frac{1}{2+t} \tag{3,15}$$

and it is sufficient to prove, instead of (3,12), the inequality

3,16. $$\frac{g(y)}{g(q)-2} \leqq \frac{1{,}02}{1{,}025}\left\{\frac{\lg(2+t)}{1+t} + \frac{1}{1+t} + \frac{1}{2+t} - \right.$$
$$\left. - \frac{2}{(1+t)^2\,(2+t)} - \frac{2\lg g(q)}{g(q)-2} - \frac{2}{g(q)-3}\right\}. \qquad (3{,}16)$$

According to (3,07) and (3,08) we get

$$\frac{g(y)}{g(q)-2} < 1{,}02, \quad \frac{2\lg g(q)}{g(q)-2} + \frac{2}{g(q)-3} < 0{,}02.$$

Further we have

$$\frac{\lg(2+t)}{1+t} > \frac{\lg 3}{2}, \quad \frac{1}{1+t} + \frac{1}{2+t} - \frac{2}{(1+t)^2\,(2+t)} > \frac{1}{2}$$

and (3,16) would follow from

3,17. $$1{,}02 \leqq \frac{1{,}02}{1{,}025}\left\{\frac{\lg 3 + 1}{2} - 0{,}02\right\}; \qquad (3{,}17)$$

but this inequality is true as $\lg 3 > 1{,}09$ and (3,02) follows from the Theorem III.

§ 4. Discussion of a certain condition concerning $g(q)$.

In order to prove the Theorem I we wanted the following (partial) condition:

$g(q)$ is defined for $q \geqq w$ and continuous,

$\frac{g(q\,.\,h(q))}{g(q)} \to 1$ with $q \to \infty$, $h(q)$ is an arbitrary function fulfilling the inequality $1 \leqq h(q) \leqq g(q) + 2$.

There is a question whether this condition does not restrict the set of admissible functions too strongly. To this question we can give only the answer contained in the theorems (4,03) and (4,04).

We shall say that

4,01. the function $g(x)$ has the property A, if it is defined for $x \geqq w$, if it is positive, increasing and if there is an integer n in such a way that $g(x) < \lg^n x$ and if $\frac{g(x \lg x)}{g(x)} \to 1$ with $x \to \infty$ (w is an arbitrary constant).

We shall say that

4,02. the function $g(x)$ has the property B, if it is defined for $x \geqq w$ (w is an arbitrary constant), if it is continuous, positive, if the function

$$\tau(x) = \frac{g'(x)}{g(x)} x \,.\, \mathrm{lglg}x$$

is defined and continuous for $x \geqq w$ (with the possible exception of points not having any point of accumulation), if there is such an integer n, that

$$\tau(x) < n \frac{\mathrm{lglg}x}{\mathrm{lg}x}$$

and if

$$\tau(x) \to 0 \text{ with } x \to \infty.$$

4,03. If the function g has the property A, then $\frac{g(x\,.\,h(x))}{g(x)} \to 1$ with $x \to \infty$, $1 \leqq h(x) \leqq g(x) + 2$.

Proof: Let us put for $x > \mathrm{e}$

$$x_1 = x,\ x_k = x_{k-1} \,\mathrm{lg} x_{k-1},\ k = 2, 3, \ldots, n+2$$

Then we have $x_{n+2} > x \,\mathrm{lg}^n(x) + 2$ for large x and

$$1 \leqq \frac{g(x\,.\,h(x))}{g(x)} < \frac{g(x(\mathrm{lg}^n x + 2))}{g(x)} < \frac{g(x_{n+1})}{g(x)} \to 1.$$

4,04. If the function $g(x)$ has the property B, then $\frac{g(x\,.\,h(x))}{g(x)} \to 1$ for $x \to \infty$, $1 \leqq h(x) \leqq g(x) + 2$.

Proof: We can write

$$g(x) = \exp\left\{\int_w^x \frac{\tau(q)}{q\,.\,\mathrm{lglg}q}\,\mathrm{d}q + C_1\right\},$$

$$g(x) < \exp\left\{\int_w^x \frac{n}{q\,.\,\mathrm{lg}q}\,\mathrm{d}q + C_1\right\} = C_2 \,\mathrm{lg}^n q;$$

$$\frac{g(x\,.\,h(x))}{g(x)} = \exp\left\{\int_x^{x\,.\,h(x)} \frac{\tau(q)}{q\,.\,\mathrm{lglg}q}\,\mathrm{d}q\right\},$$

$$\left|\int_x^{x\,.\,h(x)} \frac{\tau(q)}{q\,.\,\mathrm{lglg}q}\,\mathrm{d}q\right| \leqq \frac{c(x)}{\mathrm{lglg}x} \int_x^{x\,(C_2\,.\,\mathrm{lg}^n x\,+\,2)} \frac{\mathrm{d}q}{q} \leqq n\,.\,c(x)\left(1 + \frac{C_3}{\mathrm{lglg}x}\right),$$

where $c(x) = \sup|\tau(q)|$, $x \leqq q \leqq x\,.\,(C_2 \,\mathrm{lg}^n x + 2)$.

4,05. Let us put $\lg_{(1)}q = \lg q$, $\lg_{(k+1)}q = \lg\lg_{(k)}q$, $k = 1, 2, \ldots$, $g(q) = \lg^{\alpha_1}_{(1)}q \,.\, \lg^{\alpha_2}_{(2)}q \,.\, \ldots \,.\, \lg^{\alpha_n}_{(n)}q$. Among the exponents $\alpha_1, \ldots, \alpha_n$ let α_j be the first one different from zero and let us suppose that $\alpha_j > 0$; then $g(q)$ is an increasing function.

We choose the interval of definition (w, ∞) in such a way that $g(q) > 1000$, $\lg_{(n)}q > 1$. Then $\dfrac{g(q \,.\, h(q))}{g(q)} \to 1$ for $q \to \infty$. Proof can be made by means of 4,03.

It may be true that $\dfrac{g(q \,.\, h(q))}{g(q)} \to 1$ with $q \to \infty$, even of the function g is not monotonous and the following conditions take place:

4,06.
$$\limsup \frac{g(q)}{\lg(q)} = C \text{ with } q \to \infty \tag{4,06}$$

$$\liminf g(q) = C \text{ with } q \to \infty,\ C > 1000.$$

Such a function can be found by means of (4,04).

We put

$$\tau(q) = 2\frac{\lg\lg q}{\lg q} \text{ for } q_{2n-1} \leqq q < q_{2n}$$

$$\tau(q) = -2\frac{\lg\lg q}{\lg q} \text{ for } q_{2n} \leqq q < q_{2n+1},\ n = 1, 2, \ldots$$

and

$$\lg\left\{\frac{1}{C}g(q)\right\} = 2 \,.\int\limits_{q_{2n-1}}^{q} \frac{dx}{x \lg x},\ q_{2n-1} \leqq q < q_{2n};$$

$$\lg\left\{\frac{1}{C}g(q)\right\} = \lg\lg q_{2n} - 2 \,.\int\limits_{q_{2n}}^{q} \frac{dx}{x \lg x},\ q_{2n} \leqq q < q_{2n+1}.$$

The numbers $q_1, q_2, \ldots$ are defined recurrently in such a way that $g(q)$ is a continuous function:

$$q_1 = \exp e,\ 2 \,.\int\limits_{q_{2n-1}}^{q_{2n}} \frac{dx}{x \lg x} = \lg\lg q_{2n},$$

consequently

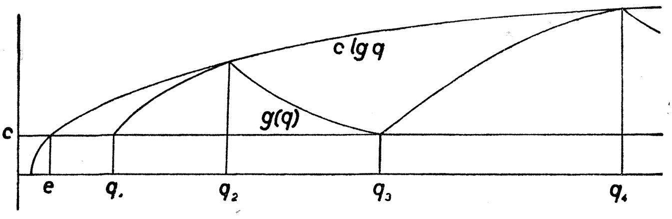
$$q_{2n} = (q_{2n-1})^{\lg q_{2n-1}};$$

$$\lg q_{2n} - 2 \int_{q_{2n}}^{q_{2n+1}} \frac{dx}{x \lg x} = 0,$$

consequently

$$q_{2n+1} = (q_{2n})^{\sqrt{\lg q_{2n}}}.$$

Apparently $q_n \to \infty$ with $n \to \infty$ and the graph of the function is shown in the figure:



Let $g_1(q)$ and $g_2(q)$ be two functions increasing to infinity and let us suppose that

$$\frac{g_2(q)}{g_1(q)} \to \infty \text{ with } q \to \infty.$$

Now we seek conditions under which a function f can be found in such a way that it distinguishes the sets Q_{g_1} and Q_{g_2} according to these respective relations:

$$Hmf(Q_{g_1}) = 0 \text{ and } Hmf(Q_{g_2}) = \infty.$$

This question is answered by

Theorem IV. *Let* $g_1(q) > 10^3$ and $q_2(q) > 10^3$ *be two functions fulfilling the assumptions* (1,10); *further let both functions* g_1 *and* g_2 *have the property* A *or* B *and let us suppose that*

$$\frac{g_2(q)}{g_1(q)} \to \infty \textit{ with } q \to \infty$$

If we put

$$f(d) = \exp\left\{\int_{w}^{\frac{1}{\sqrt{d}}} \frac{dx}{x \, . \, \sqrt{g_1(x) \cdot g_2(x)}}\right\}$$

then

$$Hmf(Q_{g_1}) = 0 \textit{ and } Hmf(Q_{g_2}) = \infty.$$

Proof: If g_1 and g_2 have the property $A(B)$, then $\sqrt{g_1 g_2}$, has the property $A(B)$ too and for q large enough we have

$$2\int_w^q \frac{\mathrm{d}x}{x \cdot g_2} < \int_w^q \frac{\mathrm{d}x}{x \cdot \sqrt{g_1 g_2}} < \frac{2}{3}\int_w^q \frac{\mathrm{d}x}{x \cdot g_1}$$

and the proof can be finished by means of (4,03), (4,04), Theorem I and (1,05).

§ 5. The dimension of some sets.

Theorem V. *Let us put* $g(q) = \lg^\alpha q,\ 0 < \alpha \leqq 1$;

$$f_s(d) = \exp \frac{1}{\lg^s \frac{1}{d}}, \quad -1 < s \leqq 0.$$

Then

$$\dim Q_g = \alpha - 1.$$

More generally, let us consider the following function

$$g(q) = \lg_{(1)}^{\alpha_1} q \cdot \lg_{(2)}^{\alpha_2} q \cdot \ldots \cdot \lg_{(n)}^{\alpha_n} q$$

where $g(q)$ satisfies the following conditions: $\int^\infty \frac{\mathrm{d}x}{x \cdot g(x)}$ diverges and $g(q) \to \infty$ as $q \to \infty$.

The above conditions are satisfied if and only if the following conditions are fulfilled:

1. There is a j such that $\alpha_j \neq 0$, if j is the least index with this property, then $\alpha_j > 0$.

2. For the least k with $\alpha_k \neq 1$ we have $\alpha_k < 1$.

(We may and we shall suppose that there is an $\alpha_k \neq 1$; if not, we replace n by $n + 1$ and we put $\alpha_{n+1} = 0$.)

The functions $f_s(d)$ which will be considered in connection with the function $g(q)$ are all functions of the form

$$f_s(d) = \exp\left\{\frac{1}{\lg_{(1)}^{s_1} \frac{1}{d} \cdot \lg_{(2)}^{s_2} \frac{1}{d} \cdot \ldots \cdot \lg_{(r)}^{s_r} \frac{1}{d}}\right\}, \quad s = (s_1, s_2, \ldots, s_r)$$

for which $f_s(d) \to \infty$, $d f_s(d) \to 0$ as $d \to 0$. It is clear, that these conditions are satisfied if and only if

1. There is a j such that $s_j \neq 0$; if j is the least index with this property, then $s_j < 0$.

2. For the least k with $s_k \neq -1$ we have $s_k > -1$. (If all s_k are -1, we replace r by $r+1$ and put $s_{r+1} = 0$.)

This set of indices is ordered by the relation: we put $s'' > s'$ if and only if $\frac{f_{s''}(d)}{f_{s'}(d)} \to 0$ for $d \to 0$. Then we can state the

Theorem VI. $\dim Q_g = s^* = (0, 0, \ldots, 0, \alpha_p - 1, \alpha_{p+1}, \ldots, \alpha_n)$ *if* $\alpha_1 = \alpha_2 = \ldots = \alpha_{p-1} = 1$, $\alpha_p < 1$. *(If* $\alpha_1 < 1$, *then* $s^* = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n$.)

Proof: According to (4,05) we may use the Theorem I and according to this theorem and (1,04) all we have to prove is

$$\frac{\int\limits_w^q \frac{dx}{xg(x)}}{\lg f_s\left(\frac{1}{q^2}\right)} \to \infty \text{ for } q \to \infty \text{ if } s > s^* \text{ and}$$

$$\frac{\int\limits_w \frac{dx}{xg(x)}}{\lg f_s\left(\frac{1}{q^2}\right)} \to 0 \text{ for } q \to \infty \text{ if } s < s^*$$

As it is

$0 < c(s) < \dfrac{\lg f_s\left(\frac{1}{q}\right)}{\lg f_s\left(\frac{1}{q^2}\right)} < 2$ for q large enough, it will be sufficient to prove

5,1. $$\frac{\int\limits_w^q \frac{dx}{xg(x)}}{\lg f_s\left(\frac{1}{q}\right)} \to \infty \text{ for } q \to \infty \text{ if } s > s^* \qquad (5,1)$$

and

5,2. $$\frac{\lg f_s\left(\frac{1}{q}\right)}{\int\limits_w^q \frac{dx}{xg(x)}} \to \infty \quad \text{for } q \to \infty \text{ if } s < s^*. \qquad (5,2)$$

Now we calculate the relations:

5,3. $\frac{\mathrm{d}}{\mathrm{d}q} \lg f_{s^*}\left(\frac{1}{q}\right) = \frac{1}{qg(q)} \psi_1(q)$ where $\lim \psi_1(q) = 1 - \alpha_p, q \to \infty$. (5,3)

5,4. If $s < s^*$, then $\frac{\mathrm{d}}{\mathrm{d}q} \lg f_s\left(\frac{1}{q}\right) = \frac{1}{qg(q)} \cdot \psi_2(q)$ where (5,4)

$$\lim \psi_2(q) = \infty, q \to \infty.$$

5,5. If $s > s^*$, then $\psi_3(q) \cdot \frac{\mathrm{d}}{\mathrm{d}q} \lg f_s\left(\frac{1}{q}\right) = \frac{1}{q \cdot g(q)}$ where (5,5)

$$\lim \psi_3(q) = \infty, \quad q \to \infty.$$

According to (5,3) and (3,04) the functions f_s fulfil the conditions (1,02).

(5,1) and (5,2) can be proved according to (5,4), (5,5) and according to the following lemma:

5,6. Let h_1 and h_2 be positive integrable functions defined for $x \geqq X$. Let us suppose that

$$\frac{h_1(x)}{h_2(x)} \to \infty \text{ with } x \to \infty \text{ and that}$$

$$\int_x^q h_2(x)\,\mathrm{d}x \to \infty \text{ with } q \to \infty.$$

Let C_3 and C_4 be given constants.

Then we have

$$\frac{\int_x^q h_1(x)\,\mathrm{d}x + C_3}{\int_x^q h_2(x)\,\mathrm{d}x + C_4} \to \infty \text{ for } q \to \infty.$$

§ 6. g is a constant.

In the preceding paragraph we investigated special functions $g(q)$ increasing towards infinity. In this paragraph we assume g to be a constant. The greatest integer less or equal to g we denote again by g^+. Now we prove

Theorem VII. *Let us put* $f_s(d) = d^{s-1}$, $0 < s \leqq 1$. *Then we have*

$$Hmf_{s_1}(Q_g) < \infty, \; s_1 > 1 - \frac{1}{4g}, \; g \geqq 10$$

$$Hmf_{s_2}(Q_g) > 0, \; s_2 < 1 - \frac{0{,}99}{g}, \; g \geqq 1000.$$

Proof: Even in this case we can use the Theorems II and III and in the same way like in the preceding text we find that the first part of the Theorem VII will be proved, if we prove the following inequality

6,1.
$$\frac{1}{1+t}\left(\frac{1}{q^2(1+t)}\right)^{-\frac{1}{4g}} \geqq$$
$$\geqq \sum_{k=1}^{g^+} \frac{1}{(k+t)(k+1+t)}\left(\frac{1}{q^2(k+t)(k+1+t)}\right)^{-\frac{1}{4g}}. \tag{6,1}$$

By means of an easy arrangement we get

6,2.
$$\frac{1}{1+t} \geqq \sum_{k=1}^{g^+} \frac{\exp\left\{\frac{1}{4g}\lg\frac{(k+t)(k+1+t)}{1+t}\right\}}{(k+t)(k+1+t)}. \tag{6,2}$$

We use the inequality: $\exp x < 1 + x(1 + x)$ if $0 < x < 1$ and consequently we have

$$\exp\left\{\frac{1}{4g}\lg\frac{(k+t)(k+1+t)}{1+t}\right\} < 1 +$$
$$+ \frac{1}{4g}\lg\frac{(k+t)(k+1+t)}{1+t}\left\{1+\frac{\lg(g^+ + 2)}{2g}\right\}$$

Instead of (6,2) it is sufficient to prove

6,3.
$$\frac{1}{1+t} \geqq \sum_{k=1}^{g^+}\frac{1}{(k+t)(k+1+t)} +$$
$$+ \frac{1}{4g}\left(1+\frac{\lg(g+2)}{2g}\right)\sum_{k=1}^{g^+}\frac{\lg\frac{(k+t)(k+1+t)}{1+t}}{(k+t)(k+1+t)}. \tag{6,3}$$

The sums in (6,3) will be estimated in the same way like in (3,10) and so it suffices to prove

6,4.
$$\frac{1}{g+2} \geqq \frac{1}{4g}\left(1+\frac{\lg(g+2)}{2g}\right).\,2{,}7 \tag{6,4}$$

or what is the same

$$1 \geqq \frac{2{,}7}{4}\left(1+\frac{2}{g}\right)\left(1+\frac{\lg(g+2)}{2g}\right)$$

and this inequality is true for $g \geqq 10$.

The second part of the Theorem VII will be proved, if we prove the inequality:

6,5.
$$\frac{1}{1+t}\left(\frac{1}{q^2(1+t)}\right)^{-\frac{0,99}{q}} \leqq$$
$$\leqq \sum_{k=1}^{g+-2} \frac{1}{(k+t)(k+1+t)}\left(\frac{1}{q^2(k+t)(k+t+1)}\right)^{-\frac{0,99}{g}} \tag{6,5}$$

It is sufficient to prove

6,6.
$$\frac{1}{1+t} \leqq \sum_{k=1}^{g+-2} \frac{\exp\left\{-\frac{0,99}{g}\lg\frac{(k+t)(k+1+t)}{1+t}\right\}}{(k+t)(k+1+t)} \tag{6,6}$$

or even

6,7.
$$\frac{1}{1+t} \leqq \sum_{k=1}^{g+-2} \frac{1}{(k+t)(k+1+t)} +$$
$$+\frac{0,99}{g}\sum_{k=1}^{g+-2} \frac{\lg\frac{(k+t)(k+1+t)}{1+t}}{(k+t)(k+1+t)}. \tag{6,7}$$

The sums in (6,7) will be estimated in the same way as in (3,12):

6,8.
$$\frac{1}{g-2} \leqq \frac{0,99}{g}\left\{\frac{\lg(2+t)+1}{1+t}+\frac{1}{2+t}-\right.$$
$$\left.-\frac{2}{(1+t)^2(2+t)}-\frac{2}{g-3}-\frac{2\lg g}{g-2}\right\} \tag{6,8}$$

$$1 \leqq 0,99\,.\left(1-\frac{2}{g}\right)\left(\frac{\lg 3+1}{2}-\frac{2}{g-3}-\frac{2\lg g}{g-2}\right)$$

$$1 \leqq 0,99\,.\left(1-\frac{2}{1000}\right)(1,04-0,02).$$

The result of the Theorem VII can be stated in the following way:

Theorem VIII. *Let us put* $f_s(d) = d^{s-1}$, $0 < s \leqq 1$, $g \geqq 1.000$. *Then we have*

$$1-\frac{0,99}{g} \leqq \dim Q_g \leqq 1-\frac{1}{4g}.$$

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