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ON SOME ORDERED CONTINUA OF POWER $2^{\aleph_0}$ CONTAINING A DENSE SUBSET OF POWER $\aleph_1$.

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Let P be a lexicographically ordered point-set whose elements x are transfinite sequences of zeros and ones $x = x_0x_1 \ldots x_\lambda \ldots$ $(\lambda < \omega_1)$. After identifying the neighbour sequences we get an ordered continuum Q. In the paper six different systems $\mathfrak{P}_k$, $k = 1, 2, \ldots, 6$ are constructed the elements of which are disjoint closed intervals and one-point-sets of Q. Every $\mathfrak{P}_k$ is an ordered continuum of power $2^{\aleph_0}$. Some properties of the continua $\mathfrak{P}_k$ are studied, for instance the separability, the character of points, the homogeneity, the similarity and so on. At the end the necessary and sufficient condition is given for the ordered continuum satisfying the Souslin property to be a linear continuum.

In the present paper the method of identification of points in certain intervals on a given ordered continuum is used to construct new ordered continua (Theorem 1). The basic ordered continuum is the continuum Q the elements of which are transfinite sequences of zeros and ones $x_0x_1 \ldots \ldots x_\lambda \ldots$ $(\lambda < \omega_1)$, which is lexicographically ordered and in which some identifications of pairs of sequences have been made. After certain further identifications of points in Q we get ordered continua $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$, $\mathfrak{P}_4$, $\mathfrak{P}_5$, $\mathfrak{P}_6$ of power $2^{\aleph_0}$. containing no dense countable subset and such that the least power of a subset which is dense in them is $\aleph_1$ (Theorems 2 and 3). The continuum $\mathfrak{P}_1$ contains points with characters c_{00} and c_{01}, the continuum $\mathfrak{P}_3$ contains points with characters c_{00}, c_{01}, c_{10} and c_{11}, $\mathfrak{P}_4$ contains points with characters c_{00}, c_{01} and c_{11} and the continuum $\mathfrak{P}_6$ contains points with characters c_{00} and c_{11}. The continuum $\mathfrak{P}_2$ is similar to $\mathfrak{P}_1$ reversed and $\mathfrak{P}_5$ is similar to $\mathfrak{P}_4$ reversed. Every continuum $\mathfrak{P}_k$ is similar to a subset of $\mathfrak{P}_3$. All continua $\mathfrak{P}_k$ $(k = 1, 2, \ldots, 6)$ show a certain degree of homogeneity, every continuum $\mathfrak{P}_k$ being quasi-homogeneous in the sense that in every interval on $\mathfrak{P}_k$ there is a subinterval similar to $\mathfrak{P}_k$ (Theorem 4). Every continuum $\mathfrak{P}_k$ possesses the following property π: Any disjoint uncountable system of intervals on $\mathfrak{P}_k$ contains an uncountable subsystem of intervals whose left end-points form an increasing or a decreasing sequence of points in $\mathfrak{P}_k$ (Theorem 7).

In this paper a necessary and sufficient condition is given for an ordered continuum C possessing the Souslin property to be a linear set; it is the similarity of C to an ordered subset of $\mathfrak{P}_k$ (Theorem 8).

At the end some new types of ordered continua of power $2^{\aleph_0}$ are constructed and a list is given of 15 possible kinds of ordered continua of power $2^{\aleph_0}$ with a dense subset of power $\aleph_1$. Some of them (6) are proved to be non-existent whereas the existence of some others (6) is secured by the continua $\mathfrak{P}_k$ ($k = 1, 2, \ldots, 6$). As to 3 remaining continua some problems are put forward concerning their existence and properties.

I.

Let P be an ordered continuum. Let $\{x^\nu\}_0^{\omega_\varrho}$ be an ordinary or transfinite sequence of points $x^\nu \in P$, $\nu < \omega_\varrho$, ω_ϱ being the least ordinal of power $\aleph_\varrho$. We define the convergence in this manner: the point $x \in P$ is a *left* (*right*) *limit* of $\{x^\nu\}$ if either the one-point-set (x) or every closed interval[1]) $\langle y, x\rangle$ ($\langle x, y\rangle$) contains almost all points x^ν i. e. all points x^ν, $\nu \geqq \nu_0$ where ν_0 is a suitable ordinal $< \omega_\varrho$. The left convergence will be denoted by $x^\nu \rightarrow x$ and the right convergence by $x \leftarrow x^\nu$. According to F. Hausdorff[2]) the *character* of a point $x \in P$ is $c_{\varrho\sigma}$ if ϱ and σ are the least ordinals for which there exists an increasing sequence $\{x^\nu\}_0^{\omega_\varrho}$ and a decreasing sequence $\{y^\nu\}_0^{\omega_\sigma}$ such that $x^\nu \rightarrow x \leftarrow y^\nu$. Further we define the character of an interval $\langle x, y\rangle$ or (x, y) as $c_{\varrho\sigma}$ provided that there is a character $c_{\varrho\sigma'}$ of the end-point x and a character $c_{\varrho'\sigma}$ of the end-point y in P.

Definition. Let P be an ordered point-set and let $\mathfrak{S}$ be a disjoint system of intervals and one-point-sets such that $\bigcup \mathfrak{S} \subset P$. Let us define $X \prec Y$ for any two elements X, Y of $\mathfrak{S}$ if $x < y$ for all points $x \in X$ and all $y \in Y$, the sign $<$ indicating the order-relation in P. It is easy to prove that $\prec$ is an order-relation in the system $\mathfrak{S}$. Without inconvenience we shall use the symbol $<$ instead of $\prec$.

We say that we have thus got the ordered system $\mathfrak{S}$ by identifying certain points of the ordered point-set P. The intervals $X \subset P$, $X \in \mathfrak{S}$ will be called interval-points and the other elements $(x) \in \mathfrak{S}$ common points of $\mathfrak{S}$.

Theorem 1. *Let P be an ordered continuum. Let $\mathfrak{P}$ be a disjoint system of one-point-sets and closed intervals of P containing more than one element and such that $\bigcup \mathfrak{P} = P$. Then $\mathfrak{P}$ is an ordered continuum.*

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[1]) The point-set containing no more than one element will not be counted among intervals. Closed intervals will be denoted by $\langle\ \rangle$ and open intervals by $(\ )$. Every ordered continuum is supposed to have two different end-points.

[2]) *F. Hausdorff, Grundzüge der Mengenlehre*, Leipzig (1914), p. 143. In ordered continua the character $c_{\varrho\sigma}$ is defined for all points except both end-points.

Proof. The system $\mathfrak{P}$ contains at least two elements. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two non-void subsystems of $\mathfrak{P}$ such that $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{P}$ and $\mathfrak{A} < \mathfrak{B}$. The couple $(\mathfrak{A}, \mathfrak{B})$ cannot be a saltus in $\mathfrak{P}$. Indeed, there are points of P lying between two disjoint closed intervals or between a closed interval and a point outside it or between two different points of P. Neither can $(\mathfrak{A}, \mathfrak{B})$ be a gap in $\mathfrak{P}$; otherwise $(\bigcup \mathfrak{A}, \bigcup \mathfrak{B})$ would be a gap in the continuum P. Evidently, $\mathfrak{P}$ contains the first and the last element.

In the sequel we shall always use the symbol $\mathfrak{P}$ for systems ordered in the above mentioned way.

Lemma 1. *Let P be an ordered continuum. Let $\mathfrak{P}$ be a disjoint system of one-point-sets and closed intervals of P such that $\bigcup \mathfrak{P} = P$. Then the character of a common point $(x) \in \mathfrak{P}$ in $\mathfrak{P}$ is the same as of the corresponding point $x \in P$ in P and the character of an interval-point $y \in \mathfrak{P}$ in $\mathfrak{P}$ is the same as that one of the interval $y \subset P$ in P.*

Proof. Let $z \in \mathfrak{P}$ be any element in $\mathfrak{P}$ with the character $c_{\varrho\sigma}$. Let $x \in P$ and $y \in P$ be the left and right end-point of $z \subset P$ if z is an interval-point and let $x = y \in P$ correspond to the point z, if it is a common point. Because the point z has a character $c_{\varrho\sigma}$ in $\mathfrak{P}$ both points x and y are different from the end-points of the continuum P. Therefore there exist the characters of the points x and y in P. Let them be $c_{\varrho'\tau'}$ and $c_{\tau\sigma'}$ (in the case when $x = y$ we have $\varrho' = \tau$ and $\tau' = \sigma'$). Let $\{x^\nu\}_0^{\omega_{\varrho'}}$ be an increasing sequence of points in P such that $x^\nu \to x$. As $\bigcup \mathfrak{P} = P$, there is, for any x^ν, an element $u^\nu \in \mathfrak{P}$ such that $x^\nu \in u^\nu$. Because $\mathfrak{P}$ is a disjoint system of one-point-sets and intervals which is, according to Theorem 1, an ordered continuum we get a non-decreasing sequence $u^\nu \to z$ in $\mathfrak{P}$. Therefore $\varrho \leqq \varrho'$. On the other hand, if $\{z^\nu\}_0^{\omega_\varrho}$ is an increasing sequence of elements in $\mathfrak{P}$ such that $z^\nu \to z$ we have $y^\nu \to x$ in P where $y^\nu \in z^\nu$. Therefore $\varrho' \leqq \varrho$, so that $\varrho = \varrho'$. It is easy to prove in a similar way that $\sigma = \sigma'$.

II.

Let P be a set whose elements x are transfinite sequences of zeros and ones:

$$x = [x_\lambda] = x_0 x_1 \dots x_\lambda \dots \quad (\lambda < \omega_1)$$

where $x_\lambda = 0$ or $= 1$ for $\lambda < \omega_1$. Let us define $[x_\lambda] < [y_\lambda]$ if there is an index δ such that $x_\lambda = y_\lambda$ for $\lambda < \delta$ whereas $x_\delta = 0 < y_\delta = 1$. Hence, P is a lexicographically ordered set with saltus and without gaps[3]) containing $2^{\aleph_1}$ points. The saltus will be avoided by the following identifications: $[x_\lambda] = [y_\lambda]$ if and only if 1. $x_\lambda = y_\lambda$ for all $\lambda < \omega_1$ or 2. there exists the least ordinal δ such that $x_\lambda = y_\lambda$ for $\lambda < \delta$ and $x_\delta + y_\delta = 1$,

[3]) Cf. *W. Sierpiński*: *Sur une propriété des ensembles ordonnés*, Fund. Math. **36** (1949); Lemma I, p. 57.

whereas $x_\lambda \neq x_\delta$ and $y_\lambda \neq y_\delta$ for $\lambda > \delta$. In such a way we get and ordered continuum Q (without any saltus and gaps) containing $2^{\aleph_1}$ points.

After the identification there is no more a one-to-one correspondence between points $x \in Q$ and the symbols $[x_\lambda]$. Therefore we shall distinguish between the points x and the symbols $[x_\lambda]$, which will be called (dyadic) *developments* of points x. There are points in Q with one development only and points with two different developments. The end-points of Q will be denoted by a and b. The corresponding developments are 000... and 111....

The point $x \in Q$ will be called a point of first, second or third kind if there exists a development $[x_\lambda]$ of x in which $x_\lambda = 1$ or $x_\lambda = 0$ only for a countable number of λ's, i. e. if there exists the least ordinal $0(x) < \omega_1$ such that $x_\lambda = 0$ for $\lambda \geqq 0(x)$ or the least ordinal $1(x) < \omega_1$ such that $x_\lambda = 1$ for $\lambda \geqq 1(x)$. The point $x \in Q$ will be called a point of *first kind* if there exists a limit[4]) ordinal $0(x)$, of *second kind* if there exists a limit ordinal $1(x)$, of *third kind* if there exists an isolated $0(x)$. In the last case there exists a $1(x)$ and the equality holds $0(x) = 1(x)$. It is easy to see that there exist two different developments of a point $x \in Q$ if and only if x is a point of the third kind.

Let $[x_\lambda]$ be a development of a point $x \in Q$. We shall use ordinal superscripts μ only in sequences $\{{}^\mu x\}_\alpha^\beta$ and $\{x^\mu\}_\gamma^\delta$ of points ${}^\mu x \in Q$, $\alpha \leqq \mu < \beta$ and $x^\mu \in Q$, $\gamma \leqq \mu < \delta$ with developments $[{}^\mu x_\lambda]$, $[x^\mu_\lambda]$ where ${}^\mu x_\lambda = x_\lambda$ except ${}^\mu x_\mu = 0$ and $x^\mu_\lambda = x_\lambda$ except $x^\mu_\mu = 1$.

Let $[x_\lambda]$, $[y_\lambda]$ be developments of the points $x \in Q$ of the first kind, $y \in Q$ of the second kind and let $[z_\lambda]$ and $[t_\lambda]$ be two different developments of the point $z \in Q$ of the third kind, where $z_\lambda = 0$ only for a countable number of λ's. Let us consider the following sequences:

$$\{{}^\mu x\}_0^{0(x)},\ \{x^\mu\}_{0(x)}^{\omega_1},\ \{{}^\mu y\}_{1(y)}^{\omega_1},\ \{y^\mu\}_0^{1(y)},\ \{{}^\mu z\}_{1(z)}^{\omega_1},\ \{t^\mu\}_{0(z)}^{\omega_1}.$$

The third and the fifth are increasing, the second and the sixth are decreasing uncountable sequences. As x is a point of the first kind we have $x \neq a$. Let $\langle x', x\rangle \subset Q$ be any closed interval; denote the development of x' by $[x'_\lambda]$. Then there exists the least ordinal δ such that $x'_\delta = 0 < x_\delta = 1$. Evidently $\delta < 0(x)$ so that ${}^\mu x \in \langle x', x\rangle$ for $\delta + 1 \leqq \mu < 0(x)$. Since $0(x) \geqq \omega$ there are $x_{\lambda'} = 1$ for infinitely many $\lambda' \to 0(x)$. Therefore it is possible to choose in the sequence $\{{}^\mu x\}_0^{0(x)}$ an ordinary increasing countable subsequence of points ${}^{\lambda'}x$ left converging to the point x. Consequently the character of the point x in Q is c_{01}. Similarly we can conclude that the character of the point $y(z)$ in Q is c_{10} (c_{11}).

Now, let $[u_\lambda]$ be a development of a point $u \in Q$, $a \neq u \neq b$, which is not a point of any of the three kinds. Then ${}^\mu u \to u \leftarrow u^\mu$ where $\{{}^\mu u\}_0^{\omega_1}$ is

[4]) The ordinal number 0 is not a limit ordinal. Therefore the endpoints a and b of Q fail to be points of any of three kinds.

a sequence containing an increasing uncountable subsequence and $\{u^\mu\}_0^{\omega_1}$ contains a decreasing subsequence of uncountably many points. Evidently $a \leftarrow \{a^\mu\}_0^{\omega_1}$ and $\{^\mu b\}_0^{\omega_1} \rightarrow b$. Therefore we get

Lemma 2. *The character of every point in Q of the first (second, third) kind is c_{01} (c_{10}, c_{11}). The character of every other point in Q except the end-points a and b is c_{11}. There is in Q a decreasing [increasing] sequence of power $\aleph_1$ right [left] converging to the end-point a [b].*

Definition. Let $\alpha > 0$ be a countable ordinal. Let $i_0 i_1 \dots i_\lambda \dots$ $(\lambda < \alpha)$ be a countable sequence of $i_\lambda = 0$ or $= 1$. All points $x \in Q$ with developments $[x_\lambda]$ where $x_\lambda = i_\lambda$ for $\lambda < \alpha$ form a closed interval $I \subset Q$ with the end-point-developments $i_0 i_1 \dots i_\lambda \dots 000 \dots$ and $i_0 i_1 \dots i_\lambda \dots$ $\dots 111 \dots$. I will be called the *interval of order* α and it will be denoted by $I i_0 i_1 \dots i_\lambda \dots (\lambda < \alpha)$. The whole continuum Q will be called the interval of order 0.

Lemma 3. *Two intervals $I i_0 i_1 \dots i_\lambda \dots$ $(\lambda < \varepsilon)$ and $I j_0 j_1 \dots j_\lambda \dots$ $(\lambda < \eta)$ of limit orders ε and η have a point in common if and only if one of the two following conditions is fulfilled:*

1° $i_\lambda = j_\lambda$ *for all* $\lambda < \min(\varepsilon, \eta)$,

2° *there exists the least ordinal* $\delta < \min(\varepsilon, \eta)$ *such that* $i_\lambda = j_\lambda$ *for* $\lambda < \delta$ *and* $i_\delta + j_\delta = 1$, *whereas* $i_\lambda \neq i_\delta$ *for* $\delta + 1 \leqq \lambda < \varepsilon$ *and* $j_\lambda \neq j_\delta$ *for* $\delta + 1 \leqq \lambda < \eta$.

Proof, If the condition 1° is fulfilled the common part of both intervals is an interval. In the second case 2° the point $x \in Q$ with two different developments $[x_\lambda]$ and $[y_\lambda]$ — where $x_\lambda = y_\lambda = i_\lambda$ for $\lambda < \delta$, $x_\delta = i_\delta$, $y_\delta = j_\delta$ (so that $x_\delta + y_\delta = 1$), whereas $x_\lambda \neq x_\delta$ and $y_\lambda \neq y_\delta$ for all $\lambda > \delta$ — is the common point of both intervals.

Now, let us suppose that both intervals have at least one point x in common. Two cases are possible: either $i_\lambda = j_\lambda$ for all $\lambda < \min(\varepsilon, \eta)$ and then the condition 1° is fulfilled, or there exists the least ordinal $\delta < \min(\varepsilon, \eta)$ such that $i_\delta \neq j_\delta$. The common point x must have two different developments $i_0 i_1 \dots i_\delta 111 \dots$ and $j_0 j_1 \dots j_\delta 000 \dots$ in the case when $i_\delta = 0$, or $i_0 i_1 \dots i_\delta 000 \dots$ and $j_0 j_1 \dots j_\delta 111 \dots$ in the case when $i_\delta = 1$. Therefore $i_\delta + j_\delta = 1$ whereas $i_\lambda \neq i_\delta$ for $\delta < \lambda < \varepsilon$ and $j_\lambda \neq j_\delta$ for $\delta < \lambda < \eta$.

Let $x \in Q$. We say that x has *property* (a) or (b) or ..., (g) if there exists a dovelopment $[x_\lambda]$ of x and the least ordinal α such that the corresponding property holds:

(a) $x_\lambda = 1$ for an infinite number of indices $\lambda < \alpha$,

(b) $x_\lambda = 0$ for an infinite number of indices $\lambda < \alpha$,

(c) there exist two ordinary increasing sequences of indices $\lambda_n \rightarrow \alpha$, $\mu_n \rightarrow \alpha$ such that $x_{\lambda_n} = 0$ and $x_{\mu_n} = 1$,

(d) there exists the least ordinal β such that $x_\lambda = 1$ for $\beta \leqq \lambda < \beta + \omega = \alpha$, β being a limit ordinal or 0,

(e) there exists the least ordinal β such that $x_\lambda = 0$ for $\beta \leq \lambda < \beta + \omega = \alpha$, β being a limit ordinal or 0,

(f) there exists the least ordinal γ such that $x_\lambda = 1$ for $\gamma \leq \lambda < \gamma + \omega = \alpha$, γ being an isolated ordinal > 0,

(g) there exists the least ordinal γ such that $x_\lambda = 0$ for $\gamma \leq \lambda < \gamma + \omega = \alpha$, γ being an isolated ordinal > 0.

If a point $x \in Q$ has one of the properties (a), ..., (g), it is easily seen that α is a *limit ordinal*. One point can have more than one property, for example every point with property (d) has also property (a). Every point $x \in Q$ must have at least one of the properties (c), (d), (e), (f), (g).

Let $[x_\lambda]$ be a development of a point $x \in Q$ and let α be the least ordinal such that some of the properties (a), ..., (g) holds. Then every point $z \in Q$ with the development $[z_\lambda]$, $z_\lambda = x_\lambda$ for $\lambda < \alpha$, has the same property with the same least ordinal α. All points like these form an interval $Ix_0x_1 \dots x_\lambda \dots (\lambda < \alpha)$ of order α and we say that this interval has the property in question.

Now we are going to denote by $\mathfrak{S}_k$ ($k = 1, 2, ..., 6$) the system of intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \subset Q$ of all (least) orders α such that

$\mathfrak{S}_1$: all intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ have property (a),

$\mathfrak{S}_2$: all intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ have property (b),

$\mathfrak{S}_3$: all intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ have property (c),

$\mathfrak{S}_4$: every interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ has property (c) or (d) but no interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha')$ where $\alpha' < \alpha$ has either property (c) or (d),

$\mathfrak{S}_5$: every interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ has property (c) or (e) but no interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha')$ where $\alpha' < \alpha$ has property (c) or (e),

$\mathfrak{S}_6$: every interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ has property (c) or (d) or (e) but no interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha')$ where $\alpha' < \alpha$ has any of these three properties.

Now, let us prove that *every system* $\mathfrak{S}_k$ ($k = 1, 2, \dots, 6$) *is a disjoint system of closed intervals on* Q. In fact, let $Ii_0i_1 \dots i_\lambda \dots (\lambda < \varepsilon)$ and $Ij_0j_1 \dots j_\lambda \dots (\lambda < \eta)$ be two different intervals of (limit) orders ε and η belonging to a certain system $\mathfrak{S}_k$. With respect to the minimality of ε and η and because ε and η are limit orders the condition 1° of lemma 3 could not be satisfied but in the case $\varepsilon = \eta$; but then $Ii_0i_1 \dots i_\lambda \dots (\lambda < \varepsilon) = Ij_0j_1 \dots j_\lambda \dots (\lambda < \eta)$ which would be a contradiction. Therefore the condition 1° cannot be satisfied at all.

Assume that the condition 2° of lemma 3 is satisfied. Then $i_\lambda = j_\lambda$ for $\lambda < \delta$ where δ is a suitable ordinal $< \min(\varepsilon, \eta)$ and $i_\delta + j_\delta = 1$, whereas $i_\lambda \neq i_\delta$ for infinitely many λ viz. if $\delta + 1 \leq \lambda < \varepsilon$ and $j_\lambda \neq j_\delta$ for infinitely many λ viz. if $\delta + 1 \leq \lambda < \eta$. Therefore $\varepsilon = \eta = \delta + \omega$ and one of the two intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \varepsilon)$ and $Ij_0j_1 \dots j_\lambda \dots (\lambda < \eta)$

has the property (f) and the other has the property (g). Thus we have proved that both intervals can only belong to the system $\mathfrak{S}_1$ or $\mathfrak{S}_2$. As $i_\lambda = j_\lambda$ for $\lambda < \delta$ and with respect to the minimality of orders ε and η, it is either $Ii_0i_1 \dots i_\lambda \dots (\lambda < \varepsilon) \in \mathfrak{S}_1$ and then $j_\lambda = 1$ for a finite number of $\lambda < \eta$ so that $Ij_0j_1 \dots j_\lambda \dots (\lambda < \eta)$ non $\in \mathfrak{S}_1$ or it is $Ii_0i_1 \dots i_\lambda \dots (\lambda < \varepsilon) \in$ $\in \mathfrak{S}_2$ and then $j_\lambda = 0$ for a finite number of $\lambda < \eta$ so that $Ij_0j_1 \dots j_\lambda \dots$ $(\lambda < \eta)$ non $\in \mathfrak{S}_2$. Therefore the possibilities $k = 1$ or $k = 2$ cannot occur.

From this it follows that none of the conditions 1° and 2° of lemma 3 can be satisfied. By the lemma quoted both intervals $Ii_0i_1 \dots i_\lambda \dots$ $(\lambda < \varepsilon)$ and $Ij_0j_1 \dots j_\lambda \dots (\lambda < \eta)$ have no point in common.

Let us denote by $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ the system whose elements are closed intervals $y = Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \in \mathfrak{S}_k$ and one-point-sets $(x) \subset Q - \bigcup \mathfrak{S}_k$ where $\bigcup \mathfrak{S}_k$ denotes the set of all points $t \in Q$ belonging to one interval $I \in \mathfrak{S}_k$ at least. By the development of an interval-point $y = Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ we shall mean any development $[x_\lambda]$ of any point $x \in Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \subset Q$. We shall use the symbol $I^*_{j_0j_1\dots j_\lambda\dots}$ $(\lambda < \alpha)$ to denote the set of all elements $z \in \mathfrak{P}_k$ (interval-points and common points) with developments $[z_\lambda]$ where $z_\lambda = j_\lambda$ for $\lambda < \alpha$.

We shall denote by $A^k_{\varrho\sigma} \subset \mathfrak{P}_k$; $\varrho, \sigma = 0, 1$; $k = 1, 2, \dots, 6$, the set of all points $z \in \mathfrak{P}_k$ with the character $c_{\varrho\sigma}$ in $\mathfrak{P}_k$.

Theorem 2. *The system* $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ *is an ordered continuum. None of its intervals contains any countable dense subset. There exists the following disjoint partition of* $\mathfrak{P}_k$:

$$\begin{array}{llllll}
\mathfrak{P}_1 = A^1_{00} & \cup\, A^1_{01} & & & & \cup\, E^1 \\
\mathfrak{P}_2 = A^2_{00} & & \cup\, A^2_{10} & & & \cup\, E^2 \\
\mathfrak{P}_3 = A^3_{00} & \cup\, A^3_{01} & \cup\, A^3_{10} & \cup & A^3_{11} & \cup\, E^3 \\
\mathfrak{P}_4 = A^4_{00} & \cup\, A^4_{01} & & & A^4_{11} & \cup\, E^4 \\
\mathfrak{P}_5 = A^5_{00} & & \cup\, A^5_{10} & \cup & A^5_{11} & \cup\, E^5 \\
\mathfrak{P}_6 = A^6_{00} & & & & A^6_{11} & \cup\, E^6
\end{array}$$

into non-void subsets which are dense in $\mathfrak{P}_k$ *except the two-point set* E^k *containing both end-points of* $\mathfrak{P}_k$.

Proof. I. The system $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ is a disjoint system of one-point-sets and closed intervals on the continuum Q containing more than one element such that $\bigcup \mathfrak{P}_k = Q$. According to Theorem 1 the system $\mathfrak{P}_k$ is an ordered continuum.

II. Let $I = Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \subset Q$ be an interval with property (c) or (d) or (e) containing no end-point a, b. Then the left end-point z of I is a point of the first kind with $0(x) = \alpha$ or $0(x) = \beta = \alpha - \omega$ and the right end-point y of I is a point of the second kind with $1(y) = \alpha$ or $1(y) =$ $= \beta = \alpha - \omega$. According to lemma 2 the character of I in Q is c_{00}. In the analogous way, using the same lemmas, it is easy to show that the character of every interval I with property (f) in Q is c_{01} and that the character of every interval I with property (g) in Q is c_{10}.

If there is an interval-point in $\mathfrak{P}_k$ with property (c) or (d) or (e) then, evidently, there exists no common point in $\mathfrak{P}_k$ with the same property. Since in each system $\mathfrak{S}_k$ ($k = 1, 2, \ldots, 6$) there are intervals with property (c), there are interval-points with property (c) in each continuum $\mathfrak{P}_k$ so that no common point (x) of $\mathfrak{P}_k$ has the property (c). Therefore $0(x)$ or $1(x)$ exists for every common point $(x) \in \mathfrak{P}_k$ and $x \in Q$ is a point of the first or second or third kind in Q with character c_{01} or c_{10} or c_{11} or it is an end-point in Q.

III. Let $p < q$ be any two different points of $\mathfrak{P}_k$ ($k = 1, 3, 4, 6$) with developments $[p_\lambda]$ and $[q_\lambda]$ such that $p_\lambda = q_\lambda$ for $\lambda < \delta$ whereas $p_\delta = 0 < q_\delta = 1$. If $p(q)$ is an interval-point in $\mathfrak{P}_k$, let $[p_\lambda]([q_\lambda])$ be the development of the right (left) end-point of the interval $p \subset Q(q \subset Q)$. As $p \neq q$ there is the least ordinal $\gamma > \delta$ such that either $p_\gamma = 0$ or $q_\gamma = 1$. Therefore there is an interval $J = Il_0l_1 \ldots l_\lambda \ldots (\lambda < \gamma + 3) \subset Q$ such that $p < J < q$. In fact, let us put in the first case ($p_\gamma = 0$): $e_\lambda = p_\lambda$ for $\lambda < \gamma$ and $e_\gamma = e_{\gamma+2} = 1$, $e_{\gamma+1} = 0$ and in the second case ($q_\gamma = 1$): $e_\lambda = q_\lambda$ for $\lambda < \gamma$, $e_\gamma = e_{\gamma+1} = 0$, $e_{\gamma+2} = 1$.

Let us consider three intervals in Q:

$$U = Iu_0u_1 \ldots u_\lambda \ldots (\lambda < \gamma + \omega), \quad V = Iv_0v_1 \ldots v_\lambda \ldots (\lambda < \gamma + \omega 2)$$
$$W = Iw_0w_1 \ldots w_\lambda \ldots \ (\lambda < \gamma + \omega)$$

where $u_\lambda = v_\lambda = w_\lambda = e_\lambda$ for $\lambda < \gamma + 3$ whereas

$u_\lambda = 0$ for even λ and $u_\lambda = 1$ for odd λ such that $\gamma + 3 \leqq \lambda < \gamma + \omega$,
$v_\lambda = 0$ for $\gamma + 3 \leqq \lambda < \gamma + \omega$ and $v_\lambda = 1$ for $\gamma + \omega \leqq \lambda < \gamma + \omega 2$
$w_\lambda = 1$ for $\gamma + 3 \leqq \lambda < \gamma + \omega$.

Then we have $p < U < q$, $p < V < q$ and $p < W < q$.

IV. From property (a) it follows that a point $x \in \mathfrak{P}_1$ is a common point in $\mathfrak{P}_1$ if and only if *each* of its developments $[x_\lambda]$ contains at the most a finite number of $x_\lambda = 1$. There is only one point $x \in \mathfrak{P}_1$ like this viz. the point $x = (a)$ where $a \in Q$ is the first point with the development $000\ldots$ All other points in $\mathfrak{P}_1$ are the interval-points. It is easy to see that every interval-point in $\mathfrak{P}_1$ has one and only one of the properties: (c), (d), (f).

Now, let us choose any two points $p < q$ in $\mathfrak{P}_1$ with developments defined in section III. With respect to property (a) we have either $p_\gamma = 0$ for the least ordinal $\gamma > \delta$ and $\gamma < \delta + \omega$ or $q_\gamma = 1$ for the least ordinal $\gamma > \delta$; therefore, because $p \neq q$, we get in both cases $e_\lambda = 1$ only for a finite number of $\lambda < \gamma + 3$. Consequently U, V and W are interval-points in $\mathfrak{P}_1$ from which the first has property (c), the second (d) and the third has property (f). According to section II and to lemma 1 the first two elements have character c_{00} and the last one has character c_{01} in $\mathfrak{P}_1$.

Therefore A^1_{00} and A^1_{01} are dense subsets in $\mathfrak{P}_1$ and the following disjoint partition holds $\mathfrak{P}_1 = A^1_{00} \cup A^1_{01} \cup E^1$, where the set E^1 consists of the two end-elements of $\mathfrak{P}_1$.

V. The interval $Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \subset Q$ is an interval-point in $\mathfrak{P}_3$ if and only if it has property (c). Let the points p and q of $\mathfrak{P}_3$ be the same as in section III. As $p < q$ there is the least ordinal $\gamma > \delta$ such that either $p_\gamma = 0$ or $q_\gamma = 1$. Every interval $Ie_0e_1 \dots e_\lambda \dots (\lambda < \alpha') \subset Q$ where $\alpha' < \gamma + 3$ fails to have property (c). For $\alpha' \leq \delta$ it is evident, because otherwise $p = q$, and for $\alpha' > \delta$ it follows from the property of the ordinal γ. Thus the interval U is an interval-point in $\mathfrak{P}_3$ whose character — according to section II and to lemma 1 — is c_{00}. The left (right) end-point v of the interval $V \subset Q$ corresponds to a common point $(v) \in \mathfrak{P}_3$; v is a point of the first (second) kind in Q with $0(v) = \gamma + \omega 2$ $(1(v) = \gamma + \omega)$ whose character in Q — by lemma 2 — is c_{01} (c_{10}). According to lemma 1 the point (v) has the same character in $\mathfrak{P}_3$. The right end-point $w \in Q$ of the interval $W \subset Q$ corresponds to a common point $(w) \in \mathfrak{P}_3$; it is also a point of the third kind in Q with $0(w) = 1(w) = \gamma + 2$ and with character c_{11} in Q and with the same character in $\mathfrak{P}_3$. Therefore $\mathfrak{P}_3 = A^3_{00} \cup A^3_{01} \cup A^3_{10} \cup A^3_{11} \cup E^3$ where $A^3_{\rho\sigma}$ are non-void disjoint subsets which are dense in $\mathfrak{P}_3$ and E^3 is a set containing only the two end-elements of $\mathfrak{P}_3$.

VI. There are two possible kinds of interval-points in $\mathfrak{P}_4$ with property (c) or (d). They have — as stated in section II — the same character c_{00} except the last interval-point $b^* = I111 \dots (\lambda < \omega)$ of $\mathfrak{P}_4$. Further there are two possible kinds of common points in $\mathfrak{P}_4$ with property (e) or (f) and (g) (at the same time). They can have only characters c_{01} or c_{11}. Consequently there is no element in $\mathfrak{P}_4$ with character c_{10}.

Let $p < q$ be any two points of $\mathfrak{P}_4$ mentioned in section III. Then no interval $Ie_0e_1 \dots e_\lambda \dots (\lambda < \alpha') \subset Q$, where $\alpha' < \gamma + 3$, can have any of the properties (c) and (d). As $p \neq q$ this is clear for $\alpha' \leq \delta$. For $\delta < \alpha' < \gamma + 3$ the assertion follows from the fact that either $e_\delta = 0$ and $e_\lambda = 1$ for $\delta + 1 \leq \lambda \leq \gamma$ or $e_\delta = 1$ and $e_\lambda = 0$ for $\delta + 1 \leq \lambda \leq \gamma$.

From this it may be deduced that U and V are interval-points of $\mathfrak{P}_4$ with properties (c) and (d). The left (right) end-point $w \in Q$ of the interval $W \subset Q$ corresponds to a common point $(w) \in \mathfrak{P}_4$ with property (e) ((f) and (g)). Here we have $0(w) = \gamma + \omega$ $(0(w) = 1(w) = \gamma + 2)$. The character of the point (w) in $\mathfrak{P}_4$ is c_{01} (c_{11}). Consequently $A^4_{10} = 0$ and $A^4_{00}, A^4_{01}, A^4_{11}$ are dense subsets in $\mathfrak{P}_4$.

VII. In $\mathfrak{P}_6$ there may exist three kinds of interval-points with properties (c) or (d) or (e) which all — except the first interval-point $a^* \in \mathfrak{P}_6$ and the last interval-point $b^* \in \mathfrak{P}_6$ — have the same character c_{00} and may be only one kind of common points with property (f) and (g) at the same time whose character is c_{11}. There are no elements in $\mathfrak{P}_6$ with character c_{01} or c_{10}.

Let $p < q$ be any two different points of $\mathfrak{P}_6$. Using the same argumentation as in the previous section VI we can prove that there is no interval $Ie_0e_1 \dots e_\lambda \dots (\lambda < \alpha') \subset Q$ where $\alpha' < \gamma + 3$ having any of the three properties (c), (d), (e) so that U and V are interval-points and the right end-point of the interval $W \subset Q$ corresponds to a common point in $\mathfrak{P}_6$. Therefore $\mathfrak{P}_6 = A^6_{00} \cup A^6_{11} \cup E^6$ where A^6_{00} and A^6_{11} are disjoint dense subsets in $\mathfrak{P}_6$ and E^6 is a set of the two end-elements of $\mathfrak{P}_6$.

VIII. Let us attach to every common point $x \in \mathfrak{P}_1$ with the development $[x_\lambda]$ the point $\varphi(x) \in Q$ with the development $[1 - x_\lambda]$ and to every interval-point $y = Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha) \in \mathfrak{P}_1$ the interval $\varphi(y) = Ij_0j_1 \dots \dots j_\lambda \dots (\lambda < \alpha) \subset Q$ where $j_\lambda = 1 - i_\lambda$ for $\lambda < \alpha$. Then $(\varphi(x))$ is a common point in $\mathfrak{P}_2$, $\varphi(y)$ is an interval-point in $\mathfrak{P}_2$ and φ is a mapping of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$. The relation $z_1 < z_2$ between any two elements of $\mathfrak{P}_1$ implies the relation $\varphi(z_1) > \varphi(z_2)$ in $\mathfrak{P}_2$ so that $\mathfrak{P}_2$ is similar to $\mathfrak{P}_1^*$ where * is used to denote the reverse order. In a similar way it is easy to prove that $\mathfrak{P}_5$ is similar to $\mathfrak{P}_4^*$.

To complete the proof it remains to show that in no interval of $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ there is any dense countable subset. This follows from the fact that there are elements of $\mathfrak{P}_k$ with character c_{01} or c_{10} or c_{11} inside every interval (p, q) of $\mathfrak{P}_k$ so that there is an uncountable disjoint system of intervals in (p, q).

Remark. The end-points of the ordered continua $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ are of 4 kinds namely (a), $a^* = I000 \dots (\lambda < \omega)$, (b) and $b^* = I111 \dots (\lambda < \omega)$. From the properties (a) — (g) it follows that $E^1 = \{(a), b^*\}$, $E^2 = \{a^*, (b)\}$, $E^3 = \{(a), (b)\}$, $E^4 = \{(a), b^*\}$, $E^5 = \{a^*, (b)\}$, $E^6 = \{a^*, b^*\}$.

Theorem 3. *The cardinal number of every ordered continuum* $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ *is* $2^{\aleph_0}$. *The cardinal number of* A^k_{00} *is* $2^{\aleph_0}$ *and the cardinal number of* A^k_{01}, A^k_{10} *and* A^k_{11} — *provided these sets are non-void — is* $\aleph_1$.

Proof.[5]) Because the cardinal number of the system of intervals of all countable orders $\alpha < \omega_1$ is $2^{\aleph_0\aleph_1} = 2^{\aleph_0}$, the power of the system $\mathfrak{S}_k$ is

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[5]) The ordered continuum with analogous properties mentioned in Theorems 2 and 3 for $k = 3$ is known under the name *ultracontinuum*. *F. Bernstein* defined in his article „*Untersuchungen aus der Mengenlehre*", Math. Ann. **61** (1905), p. 146 the point-set X_u in this manner: the elements of X_u are the ordinary infinite sequences $x = \alpha_1\alpha_2 \dots \alpha_n \dots$ of ordinals α_n of the first and second ordinal class $0 \leq \alpha_n < \omega_1$. The order of X_u is introduced by the following order-rule: $\alpha_1\alpha_2 \dots \alpha_n \dots < \beta_1\beta_2 \dots \dots \beta_n \dots$ if there exists an index k such that $\alpha_i = \beta_i$ for $i = 1, 2, \dots, k-1$ whereas $\alpha_k < \beta_k$ if k is an odd integer or $\alpha_k > \beta_k$ if k is an even integer. The point-set X_u is no continuum, there are gaps in it. But after filling up the gaps we get an ordered continuum of power $2^{\aleph_0}$ in which there are elements with characters c_{00}, c_{01}, c_{10} and c_{11}. Cf. my article *Zwei Bemerkungen zum Bernsteinschen Ultrakontinuum*, *Čas. pro pěst. mat. a fys.* **68** (1939), p. 147. Probably, Bernstein's ultracontinuum is similar to the continuum $\mathfrak{P}_3$.

$\leqq 2^{\aleph_0}$. On the other hand, every system $\mathfrak{S}_k$ contains intervals with property (c), especially all intervals $Ii_0i_1 \dots i_\lambda \dots (\lambda < \omega) \subset Q$ with infinitely many $i_\lambda = 0$ and infinitely many $i_{\lambda'} = 1$ for $\lambda < \omega$ and $\lambda' < \omega$. From this it follows that the power of the system $\mathfrak{S}_k$ $(k = 1, 2, \dots, 6)$ is $2^{\aleph_0}$. Because every set A^k_{00} consists of interval-points and because every interval-point with property (c) belongs to the set A^k_{00}, the cardinal number of A^k_{00} is also $2^{\aleph_0}$ for every $k = 1, 2, \dots, 6$.

Let $z \in \mathfrak{P}_k - A^k_{00}$, $k = 1, 2, \dots, 6$, be a common point with the development $[z_\lambda]$ *or* an interval-point $z = Iz_0z_1 \dots z_\lambda \dots (\lambda < \alpha)$. Let z be not the last point in $\mathfrak{P}_k$. Then it is possible to attach to z a sequence of ordinals

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \dots < \alpha_\nu < \beta_\nu < \dots$$
$$(0 \leqq \alpha_\nu < \beta_\nu \leqq \omega_1 \ \textit{or}\ 0 \leqq \alpha_\nu < \beta_\nu \leqq \alpha)$$

such that $z_\lambda = 0$ for and only for those λ which satisfy the inequality $\alpha_\nu \leqq \lambda < \beta_\nu$ where $\nu = 0, 1, \dots$. Because there are neither common points in $\mathfrak{P}_k - A^k_{00}$ with property (c) nor interval-points with property (c) the sequence $\alpha_0 < \beta_0 < \dots$ must be finite. The cardinal number of all sequences like these is $\aleph_1$. Evidently, the corresponding sequences of two different elements of $\mathfrak{P}_k - A^k_{00}$ are different. Therefore the cardinal number of the set $\mathfrak{P}_k - A^k_{00}$ cannot exceed the cardinal number $\aleph_1$. On the other hand there are elements $t_\mu \in A^k_{01}$ $(k = 1, 3, 4)$ with corresponding finite sequences $\alpha_0 = 0 < \beta_0 = \omega\mu < \omega(\mu + 1) = \alpha_1 < \beta_1 = \omega_1$ where $0 < \mu < \omega_1$ the cardinal number of which is $\aleph_1$. The same holds true for elements $t_\mu \in A^k_{10}$ $(k = 2, 3, 5)$ with corresponding sequences $\alpha_0 = \omega\mu < \omega(\mu + 1) = \beta_0$ $(0 \leqq \mu < \omega_1)$ and for elements $t_\mu \in A^k_{11}$ $(k = 3, 4, 5, 6)$ with corresponding sequences $\alpha_0 = 0 < \beta_0 = 1 < \alpha_1 = \mu < \mu + 1 = \beta_1$ $(2 \leqq \mu < \omega_1)$. Thus, with respect to Theorem 2, the proof is complete.

An ordered point-set T will be said to be quasi-homogeneous if in every interval on T there is a subinterval which is similar to T.

Theorem 4. *Every continuum* $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ *is quasi-homogeneous.*

Proof. Let J be any closed interval on $\mathfrak{P}_k$ $(k = 1, 3, 4, 6)$ with end-points $p < q$ and let $Ie_0e_1 \dots e_\lambda \dots (\lambda < \gamma + 3) \subset Q$ be an interval of order $\gamma + 3$ defined in section III on the page 000. Then $I^* = I^*e_0e_1 \dots \dots e_\lambda \dots (\lambda < \gamma + \omega)$ where $e_\lambda = 0$ for $\gamma + 3 \leqq \lambda < \gamma + \omega$ is an interval in $\mathfrak{P}_k$ $(k = 1, 3, 4, 6)$ such that $p < I^* < q$ in $\mathfrak{P}_k$. Now, we shall distinguish two cases:

The first case: $k = 1, 3, 4$. The interval I^* fails to have property (a) (for $k = 1$) or (c) (for $k = 3, 4$) or (d) (for $k = 4$). Consequently, if $z = Ii_0i_1 \dots i_\lambda \dots (\lambda < \alpha)$ is any interval-point in $\mathfrak{P}_k$ then $f(z) = Ij_0j_1 \dots \dots j_\lambda \dots (\lambda < \gamma + \omega + \alpha)$ where $j_\lambda = e_\lambda$ for $\lambda < \gamma + \omega$ and $j_\lambda = i_{-(\gamma+\omega)+\lambda}$

for $\gamma + \omega \leq \lambda < \gamma + \omega + \alpha$ is also an interval-point in $\mathfrak{P}_k$ and $f(z) \in I^*$. If $z \in \mathfrak{P}_k$ is a common point with development $[z_\lambda]$ then the point $f(z) = z'$ with development $[z'_\lambda]$ where $z'_\lambda = e_\lambda$ for $\lambda < \gamma + \omega$ and otherwise $z'_\lambda = z_{-(\gamma+\omega)+\lambda}$ for $\lambda \geq \gamma + \omega$ is a common point in $\mathfrak{P}_k$ and $f(z) \in I^*$. Now, if $t = It_0t_1 \dots t_\lambda \dots$ $(\lambda < \beta)$, $\beta > \gamma + \omega$, is an interval-point in I^* or if $[t_\lambda]$ is a development of any common point $t \in I^*$ then $f(z) = t$ where $z = Iz_0z_1 \dots z_\lambda \dots \in \mathfrak{P}_k$, $z_\lambda = t_{\gamma+\omega+\lambda}$, is an interval-point in $\mathfrak{P}_k$ or $[z_\lambda]$, $z_\lambda = t_{\gamma+\omega+\lambda}$, is a development of a common point $z \in \mathfrak{P}_k$. Evidently, the inequality $z_1 < z_2$ implies $f(z_1) < f(z_2)$ so that $z' = f(z)$ is a one-to-one order-preserving correspondence between $\mathfrak{P}_k$ and $I^* \subset J$.

The second case: $k = 6$. Let $p' = Ip'_0p'_1 \dots p'_\lambda \dots$ $(\lambda < \gamma + \omega 3)$ and $q' = Iq'_0q'_1 \dots q'_\lambda \dots$ $(\lambda < \gamma + \omega 2)$ be two interval-points in $\mathfrak{P}_6$ where $p'_\lambda = q'_\lambda = e_\lambda$ for $\lambda < \gamma + \omega$ and $p'_\lambda = 0$, $q'_\lambda = 1$ for $\gamma + \omega \leq \lambda < \gamma + \omega 2$ and $p'_\lambda = 1$ for $\gamma + \omega 2 \leq \lambda < \gamma + \omega 3$. Then $p < p' < q' \leq q$. For $z = a^* \in \mathfrak{P}_6$ we put $f(a^*) = p'$. For $z \neq a^*$, $z = Ii_0i_1 \dots i_\lambda \dots$ $(\lambda < \alpha) \in \mathfrak{P}_6$ we put $f(z) = Ij_0j_\lambda \dots j_\lambda \dots$ $(\lambda < \gamma + \omega + \alpha)$ where $j_\lambda = p'_\lambda$ for $\lambda < \gamma + \omega$ and $j_\lambda = i_{-(\gamma+\omega)+\lambda}$ for $\gamma + \omega \leq \lambda < \gamma + \omega + \alpha$. Then $f(z)$ is an interval-point in $\mathfrak{P}_6$ belonging to the interval $\langle p', q' \rangle$. If $z \in \mathfrak{P}_6$ is a common point with the development $[z_\lambda]$, then $[z'_\lambda]$, where $z'_\lambda = p'_\lambda$ for $0 \leq \lambda < \gamma + \omega$ and otherwise $z'_\lambda = z_{-(\gamma+\omega)+\lambda}$ for $\lambda \geq \gamma + \omega$, is the development of a common point in $\mathfrak{P}_6$ belonging to the interval $\langle p', q' \rangle$. In the same manner as in the first case we can easily prove that $f(z)$ is a one-to-one order-preserving transformation of $\mathfrak{P}_6$ onto $\langle p', q' \rangle$. Therefore $\mathfrak{P}_6$ is similar to the interval $\langle p', q' \rangle \subset J$.

For $k = 2, 5$ Theorem 4 follows from the fact that $\mathfrak{P}_2$ is similar to $\mathfrak{P}_1^*$ and $\mathfrak{P}_5$ is similar to $\mathfrak{P}_4^*$.

Theorem 5. *There exists a subset P_k of $\mathfrak{P}_3$ which is similar to $\mathfrak{P}_k$* $(k = 1, 2, \dots, 6)$.

Proof. Let us attach to every element $z \in \mathfrak{P}_k$ an element $z' \in \mathfrak{P}_3$ in the following manner: If z is an interval-point $z = Ii_0i_1 \dots i_\lambda \dots$ $(\lambda < \alpha)$ in $\mathfrak{P}_k$ with property (c), then $z' = I'i_0i_1 \dots i_\lambda \dots$ $(\lambda < \alpha)$ is an interval-point in $\mathfrak{P}_3$. If $z = Ii_0i_0 \dots i_\lambda \dots$ $(\lambda < \alpha) \in \mathfrak{P}_k$ has not property (c) then let z' denote a point in $\mathfrak{P}_3$ with the development $[z'_\lambda]$ where $z'_\lambda = i_\lambda$ for $\lambda < \alpha$ and $z'_\lambda = 0$ for $\alpha \leq \lambda < \omega_1$. If z is a common point in $\mathfrak{P}_k$ with development $[z_\lambda]$ then it fails to have property (c) so that the corresponding point z' with development $[z'_\lambda]$, $z'_\lambda = z_\lambda$ for $\lambda < \omega_1$ is a common point in $\mathfrak{P}_3$. It is easy to see that all elements z' form a subset $P_k \subset \mathfrak{P}_3$ which is similar to $\mathfrak{P}_k$.

Let $\mathfrak{S}$ be a disjoint system of intervals on $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ of whatever kind satisfying only the condition that for every point $z \in \mathfrak{P}_k$ with character $c_{\varrho 1}$ $(c_{1\sigma})$; $\varrho, \sigma = 0, 1$ and for both end-points of $\mathfrak{P}_k$ there exists an interval $S \in \mathfrak{S}$ such that z is an inner point or a left (right) end-point of S (which may or may not belong to S). We say, that the system $\mathfrak{S}$ possesses *property π^**.

Lemma 4. *Let $\mathfrak{S}$ be a system of intervals on $\mathfrak{P}_3$ possessing property π^*. Let $[x_\lambda]$ be the development of a common point $(x) \in \mathfrak{P}_3$. Then there exists an interval $S \in \mathfrak{S}$ and an interval $I^* x_0 x_1 \dots x_\lambda \dots (\lambda < \alpha) - (x) \subset S$.*

Proof. Since (x) is a common point in $\mathfrak{P}_3$ it is $x \in Q$ and there exists $1(x)$ or $0(x)$. Therefore $\{^\mu x\}_{1(x)}^{\omega_1} \to x$ or $x \leftarrow \{x^\mu\}_{0(x)}^{\omega_1}$ in Q where $\{^\mu x\}_{1(x)}^{\omega_1}$ is an increasing and $\{x^\mu\}_{0(x)}^{\omega_1}$ a decreasing uncountable sequence. From the property π^* it follows the existence of $S \in \mathfrak{S}$ and of an ordinal $\mu' > 1(x)$ or $> 0(x)$ such that $(^{\mu'}x) \in S$ or $(x^{\mu'}) \in S$ and that $((^{\mu'}x), (x)) \subset S$ or $((x), (x^{\mu'})) \subset S$, $(^{\mu'}x)$ and $(x^{\mu'})$ being common points in $\mathfrak{P}_3$. Therefore $I^* x_0 x_1 \dots \dots x_\lambda \dots (\lambda < \mu' + 1) - (x) \subset S$, one of the developments of $^{\mu'}x$ being $[t_\lambda]$ where $t_\lambda = x_\lambda$ for $\lambda < \mu' + 1$ whereas $t_\lambda = 0$ for $\lambda \geqq \mu' + 1$ or analogously $t_\lambda = x_\lambda$ for $\lambda < \mu' + 1$ whereas $t_\lambda = 1$ for $\lambda \geqq \mu' + 1$.

Theorem 6. *Every disjoint system $\mathfrak{S}$ of intervals on $\mathfrak{P}_3$ possessing property π^* is countable.*

Proof. To prove this theorem we shall make use of the following statement[6]):

Let Z be a non-void abstract set. Let the following rule for the construction of subsets $N_\gamma \subset Z$ be prescribed: If all countable subsets N_ν, S_ν, A_ν of Z are constructed for all $\nu < \gamma$ where $N_0 = S_0 = 0 \neq A_0$ and $S_\nu \neq 0$ for $\nu > 0$, we put $N_\gamma = \bigcup_{\nu<\gamma} A_\nu - \bigcup_{\nu<\gamma} S_\nu$ and choose $S_\gamma \neq 0$, $S_\gamma \subset N_\gamma$ supposing $N_\gamma \neq 0$ and then choose the countable set $A_\gamma \subset Z - \bigcup_{\nu<\gamma} A_\nu$. Then there exists a countable ordinal number $\vartheta > 0$ with the property $N_\vartheta = 0$ that is $\bigcup_{\nu<\vartheta} A_\nu = \bigcup_{\nu<\vartheta} S_\nu$.

Let Z denote the set of all developments of common points in $\mathfrak{P}_3$. Since there are at most two developments of a common point in $\mathfrak{P}_3$ the cardinality of Z — according to Theorem 3 — is $\aleph_1$. Let $N_0 = 0$. Let us choose an infinite countable subset $A_0 = N_1 \subset Z$ of all developments $[x_\lambda]$ of common points $(x) \in \mathfrak{P}_3$ with $0(x) \leqq \tau_0$ or $1(x) \leqq \tau_0$, τ_0 being a countable limit ordinal. Let us choose a development $[s_\lambda^1] \in N_1$ of a common point $(s^1) \in \mathfrak{P}_3$. According to Lemma 4 there is an interval $S^1 \in \mathfrak{S}$ and an interval $I^* s_0^1 s_1^1 \dots s_\lambda^1 \dots (\lambda < \alpha_1) - (s^1) \subset S^1$. Let τ_1 be a limit ordinal $> \tau_0$ and $> \alpha_1$. Let $A_1 \subset Z - A_0$ be a countable subset of all developments $[x_\lambda]$ of common points $(x) \in \mathfrak{P}_3$ with $\tau_0 < 0(x) \leqq \tau_1$ or $\tau_0 < 1(x) \leqq \leqq \tau_1$. Denote $N_2 = A_0 \cup A_1 - [s_\lambda^1]$ choose a development $[s_\lambda^2] \in N_2$ of a common point $(s^2) \in \mathfrak{P}_3$ and go on. After having constructed — for all $\nu < \gamma$ — countable subsets N_ν and chosen disjoint subsets $A_\nu \subset Z$ of developments $[x_\lambda]$ of common points $(x) \in \mathfrak{P}_3$ with $0(x) \leqq \tau_\nu$ or $1(x) \leqq \tau_\nu$ such that $\tau_0 < \tau_1 < \dots < \tau_\nu < \dots$ and after having chosen developments $[s_\lambda^\nu] \in N_\nu$ of common points $(s^\nu) \in \mathfrak{P}_3$ and intervals $S^\nu \in \mathfrak{S}$ such that $I^* s_0^\nu s_1^\nu \dots s_\lambda^\nu \dots (\lambda < \alpha_\nu) - (s^\nu) \subset S$ where $\alpha_\nu < \tau_\nu$ for all $\nu < \gamma$, we put

[6]) See *J. Novák: A paradoxical theorem, Fund. Math.* **37** (1950), p. 77—83.

$N_\gamma = \bigcup_{\nu<\gamma} A_\nu - \bigcup_{\nu<\gamma} [s^\nu_\lambda]$ and choose a development $[s^\gamma_\lambda] \in N_\gamma$ of a common point $(s^\gamma) \in \mathfrak{P}_3$ supposing $N_\gamma \neq 0$. According to Lemma 4 there is an interval $S^\gamma \in \mathfrak{S}$ and an interval $I^* s^\gamma_0 s^\gamma_1 \dots s^\gamma_\lambda \dots (\lambda < \alpha_\gamma) - (s^\gamma) \subset S^\gamma$. We choose a limit ordinal $\tau_\gamma > \alpha_\gamma$ such that $\tau_\nu < \tau_\gamma < \omega_1$ for all $\nu < \gamma$ and then we denote by $A_\gamma \subset Z - \bigcup_{\nu<\gamma} A_\nu$ the set of all developments $[x_\lambda]$ of common points $(x) \in \mathfrak{P}_3$ with $\tau_\nu < 0(x) \leqq \tau_\gamma$ or $\tau_\nu < 1(x) \leqq \tau_\gamma$ for all $\nu < \gamma$. According to the statement quoted there exists a countable ordinal ϑ such that $\bigcup_{\nu<\vartheta} A_\nu = \bigcup_{\nu<\vartheta} [s^\nu_\lambda]$.

Let Θ denote the least ordinal which is $> \tau_\nu$ for all $\nu < \vartheta$. Then both $\bigcup_{\nu<\vartheta} A_\nu$ and $\bigcup_{\nu<\vartheta} [s^\nu_\lambda]$ are the same set viz. the set of all developments $[x_\lambda]$ of common points $(x) \in \mathfrak{P}_3$ with $0(x) < \Theta$ or $1(x) < \Theta$.

Let S be any element of $\mathfrak{S}$. According to Theorem 2 there is a common point $(x) \in \mathfrak{P}_3$, inside S, with development $[x_\lambda]$ and with character c_{10} so that $1(x)$ does exist. If $1(x) < \Theta$, then there is a development $[s^\nu_\lambda] \in N_\nu$ of a point $(s)^\nu \in \mathfrak{P}_3$ such that $x_\lambda = s^\nu_\lambda$ for all $\lambda < \omega_1$. Consequently $I^* s^\nu_0 s^\nu_1 \dots$ $\dots s^\nu_\lambda \dots (\lambda < \alpha_\nu) - (s^\nu) \subset S^\nu$. As the point $(x) = (s^\nu)$ is an inner point of S we see that $(I^* s^\nu_0 s^\nu_1 \dots s^\nu_\lambda \dots (\lambda < \alpha_\nu) - (s^\nu) \cap S \neq 0$; therefore $S^\nu = S$, $\mathfrak{S}$ being a disjoint system of intervals. If $1(x) \geqq \Theta$, then there is the least ordinal $\beta < \Theta$ such that $x_\lambda = 0$ for $\beta \leqq \lambda < \Theta$ (or $x_\lambda = 1$ for $\beta \leqq \lambda < \Theta$), because the point (x) has not property (c). From this it follows the existence of an ordinal $\nu < \vartheta$ and a development $[s^\nu_\lambda] \in N_\nu$ such that $s^\nu_\lambda = x_\lambda$ for $\lambda < \beta$ and $s^\nu_\lambda = 0$ for $\lambda \geqq \beta$ (or $s^\nu_\lambda = 1$ for $\lambda \geqq \beta$). Thus $s^\nu_\lambda = x_\lambda$ for all $\lambda < \Theta$ and, since $\alpha_\nu < \Theta$, we get $(x) \in (I^* s^\nu_0 s^\nu_1 \dots$ $\dots s^\nu_\lambda \dots (\lambda < \alpha_\nu)) \cap S$. We conclude again that $S = S^\nu$.

Because the set of all ordinals $\nu < \vartheta$ is countable, the system $\mathfrak{S}$ must be countable too.

The ordered continuum C will be said to possess *property* π if every disjoint uncountable system of intervals on C contains an uncountable subsystem of intervals whose left end-points form an increasing or a decreasing sequence of points in C.

By means of Theorem 6 we can prove

Theorem 7. *Every continuum* $\mathfrak{P}_k$ $(k = 1, 2, \dots, 6)$ *possesses property* π.

Proof. Suppose that $\mathfrak{P}_3$ does not possess property π. Let $\mathfrak{S}$ be a disjoint uncountable system of intervals on $\mathfrak{P}_3$. Let $\mathfrak{T}$ be a system of all intervals $J \subset \mathfrak{P}_3$ such that 1. $J \cap (\bigcup \mathfrak{S}) = 0$, 2. if $J' \subset \mathfrak{P}_3$ is an interval then $J' \supset J$ and $J' \cap (\bigcup \mathfrak{S}) = 0$ implies that $J = J'$. From properties 1 and 2 it follows that the system $\mathfrak{S} \cup \mathfrak{T}$ is a disjoint system of intervals in $\mathfrak{P}_3$.

Now, let $z \in \mathfrak{P}_3$ be a point with character $c_{\varrho 1}$; $\varrho = 0, 1$, or $z = a^*$ $(c_{1\sigma}$; $\sigma = 0, 1$, or $z = b^*)$ which is not an interior point of any interval of

$\mathfrak{S} \cup \mathfrak{T}$ or a left (right) end-point of any interval of $\mathfrak{S}$. Then from our supposition it follows that there is a point $y \in \mathfrak{P}_3$, $y > z$, $(y < z)$ such that $(z, y) \cap (\bigcup \mathfrak{S}) = 0$ $((y, z) \cap (\bigcup \mathfrak{S}) = 0)$. Therefore we can conclude — by 2 — that z is a left (right) end-point of an interval $J \in \mathfrak{T}$ (which may or may not belong to J). Therefore the system $\mathfrak{S} \cup \mathfrak{T}$ has property π^* and according to Theorem 6 it is countable. This contradicts to the fact that $\mathfrak{S}$ is an uncountable system.

Thus we have proved Theorem 7 for $k = 3$. Now, let $k = 1, 2, \ldots, 6$. According to Theorem 5 there is a subset $f(\mathfrak{P}_k) = P_k \subset \mathfrak{P}_3$ where f denotes the similarity function. To any interval $(p, q) \in \mathfrak{P}_k$ there corresponds an interval $(f(p), f(q)) \in \mathfrak{P}_3$. Hence because $\mathfrak{P}_3$ possesses property π it is easily seen that $\mathfrak{P}_k$ possesses property π as well.

We say that the ordered continuum C possesses the *Souslin property* if there is no uncoutable disjoint system of intervals in C.

Theorem 8. *Let C be an ordered continuum possessing the Souslin property. Then C contains a countable subset which is dense in C if and only if C is similar to an ordered subset $R \subset \mathfrak{P}_k$ $(k = 1, 2, \ldots, 6)$.*

Proof. Every ordered continuum contains a subset which is similar to the interval $\langle 0, 1 \rangle$ of real numbers. Thus the necessity of the condition is evident.

To prove the sufficiency we can assume — with respect to the theorem 5 — that C is similar to a subset $R \subset \mathfrak{P}_3$. The complementary set $\mathfrak{P}_3 - R$ consists of intervals which form a disjoint system $\mathfrak{S}$ of intervals on $\mathfrak{P}_3$ possessing property π^*. Indeed, if the character of a point $z \in \mathfrak{P}_3$ is $c_{\varrho 1}$ $(c_{1\sigma})$; $\varrho, \sigma = 0, 1$, or if $z = (a)$ $(z = (b))$, then z is not a left (right) limit of any uncoutable decreasing (increasing) sequence of points of R the continuum C having the Souslin property. From this we conclude that the point z is either an inner point or a left (right) end-point ot an interval $S \in \mathfrak{S}$. According to Theorem 6 the system $\mathfrak{S}$ is countable.

Let us denote by R' the set of end-points of all intervals $S \in \mathfrak{S}$ belonging to the set R. The set R' is countable. Let S be any interval of $\mathfrak{S}$ and let $a_1 < a_2$ be its end-points. Then either $a_1 \in R$ or $a_2 \in R$. Otherwise there would be a gap (A, B) in R where A denotes the set of all points x of R such that $x < a_1$ and B the set of all points $x \in R$ such that $x > a_2$. Therefore either $a_1 \in R'$ or $a_2 \in R'$.

Let $b_1 < b_2$ be any two points of R. Then there exists an interval $S \in \mathfrak{S}$ with the end-points $a_1 < a_2$ such that $b_1 \leqq a_1 < a_2 \leqq b_2$. As we have just shown one of the end-points a_1, a_2 belongs to the set R' so that R' is a dense subset of R. Consequently the continuum C contains a countable dense subset.

III.

Let P be an ordered continuum. For $x \in P$ let P_x denote an ordered continuum or a set containing only one point. Then we obtain a new non-void ordered set by lexicographically ordering the set **P** of points (x,y) where $x \in P$ and $y \in P_x$.

Let P be an ordered continuum and let $P' \subset P$ be a subset in it. Put $P_x = P$ for $x \in P'$ and $P_x = (x)$ for $x \in P - P'$. Then we get a special ordered set **P** which will be denoted by $\mathbf{P} = [P; P']$.

We shall prove

Lemma 5. *Let P be an ordered continuum. Then* **P** *is an ordered continuum as well.*

Proof. Let $(\mathbf{A}, \mathbf{B})$ be a section of **P**. Let $A \subset P$ be a subset of all points $x \in P$ such that $(x, y) \in \mathbf{A}$ for at least one $y \in P_x$. Put $B = P - A$. Then (A, B) is a section of P. As P is an ordered continuum there is a point $x_0 \in P$ which is determined by the section (A, B). Let A' denote the set of all points $y \in P_{x_0}$ such that $(x_0, y) \in \mathbf{A}$ and let B' denote the set of all points $y \in P_{x_0}$ such that $(x_0, y) \in \mathbf{B}$. Then $A' \cup B' = P_{x_0}$ and $x' \in A'$, $y' \in B'$ implies $x' < y'$ so that (A', B') is a section of P_{x_0} by which a point $y_0 \in P_{x_0}$ is determined. Evidently (x_0, y_0) is either the last element in **A** whereas **B** does not contain the first element or **A** fails to contain the last element and **B** contains the first element (x_0, y_0).

Let $\mathfrak{P}'_k \subset \mathfrak{P}_k$ $(k = 1, 2, \ldots, 6)$ be a subset of $\mathfrak{P}_k$ of power $\aleph_1$ which does not contain any uncountable decreasing or increasing sequence of points.[7] According to lemma 5 the set $[\mathfrak{P}_k; \mathfrak{P}'_k]$ is an ordered continuum. Its power is $\aleph_1 \cdot 2^{\aleph_0} = 2^{\aleph_0}$ and the least power of the subset which is dense in $[\mathfrak{P}_k; \mathfrak{P}'_k]$ is evidently $\aleph_1$. All points $(x_0, y) \in [\mathfrak{P}_k; \mathfrak{P}'_k]$ vhere x_0 is a point in $\mathfrak{P}'_k$ and $y \in \mathfrak{P}_k$ form an interval I_{x_0}. The system of all intervals I_{x_0}, $x_0 \in \mathfrak{P}'_k$ is disjoint and uncountable. From our supposition it follows that there is no uncountable, decreasing or increasing sequence of left endpoints of intervals belonging to this system. Therefore the continuum $[\mathfrak{P}_k; \mathfrak{P}'_k]$ has not property π and according to Theorem $\mathfrak{T}$ it is not similar to any $\mathfrak{P}_k$ $(k = 1, 2, \ldots, 6)$.

Thus we got new types of ordered continua $[\mathfrak{P}_k; \mathfrak{P}'_k]$ of power $2^{\aleph_0}$ such that the least power of its dense subset is $\aleph_1$.

Now, we are going to use the symbol $P(c_{00}, c_{01}, \ldots)$ to denote the ordered continuum of power $2^{\aleph_0}$ containing a dense subset of the least power $\aleph_1$ and containing points with characters $c_{00}, c_{01}, \ldots$ as given in the parentheses. There may exist at most 15 kinds of continua like these viz.

$$P(c_{01})\ P(c_{10})\ P(c_{11})\ P(c_{01}c_{11})\ P(c_{10}c_{11})\ P(c_{01}c_{10}),$$
$$P(c_{00}c_{01})\ P(c_{00}c_{10})\ \mathrm{A}(c_{00}c_{01}c_{01}c_{11})\ P(c_{00}c_{01}c_{11})\ P(c_{00}c_{10}c_{11})\ P(c_{00}c_{11}),$$
$$P(c_{00}c_{01}c_{10})\ P(c_{00}c_{10}c_{11})\ P(c_{00}).$$

We shall show that the continua in the first row do not exist at all. In

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[7]) The existence of subsets like these follows from Theorem 8 where C denotes the interval $\langle 0, 1 \rangle$.

fact, every countable decreasing and every countable increasing infinite sequence of points in an ordered continuum contains an infinite subsequence which converges to a limit. Therefore in every ordered continuum there is at least one point with character $c_{\varrho 0}$ and at least one point with character $c_{0\sigma}$. Further, according to F. HAUSDORFF[8]) there are points with symetrical character $c_{\varrho\varrho}$ in every ordered continuum. From these reasons the continua in the first row cannot exist.

On the other hand theorems 2 and 3 secure the existence of ordered continua in the second row.

The continuum $P(c_{00}c_{01}c_{10})$ exists. Let us consider, for example, the set $T = \mathfrak{P}_1 \cup \mathfrak{P}_2$ which will be ordered like this: x precedes y if $x \in \mathfrak{P}_1$, $y \in \mathfrak{P}_2$ or if $x \in \mathfrak{P}_i$, $y \in \mathfrak{P}_i$ $(i = 1, 2)$ and $x < y$. After having identified the last element of $\mathfrak{P}_1$ with the first element of $\mathfrak{P}_2$ we get an ordered continuum $P(c_{00}c_{01}c_{10})$.This continuum fails to be quasi-homogeneous. I did not succeed in constructing a quasi-homogeneous ordered continuum $\mathfrak{P}_7 = A^7_{00} \cup A^7_{01} \cup A^7_{10} \cup E^7$ by means of the properties (a) — (g).

The continuum $P(c_{00})$ exists as well. For instance the continuum $\boldsymbol{P} = [\langle 0, 1\rangle; D]$, the set D being a subset of the interval $\langle 0, 1\rangle$ of power $\aleph_1$. This continuum fails to possess the property π. Up to this day it is completely unknown whether there exists a continuum $P(c_{00})$ with property π. The question of the existence of such an ordered continuum is equivalent to the well known problem of Souslin. With respect to Theorem 8 we can only assert that such a continuum, if it exists, is similar to no subset $R \subset \mathfrak{P}_k$ for any $k = 1, 2, \ldots, 6$.

As to the ordered continuum $P(c_{01}c_{10}c_{11})$, I do not know whether it exists. It could exist only under the assumption that $2^{\aleph_0} = \aleph_1$.

[8]) *F. Hausdorff*, l. c., p. 142.