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ON THE CATEGORY OF THE SET OF CUT POINTS OF CONTINUA OF A CERTAIN TYPE

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Main results: every n-dimensional continuum C may be imbedded in an n-dimensional continuum C^* such that the set of cut points of C^* as well as its complement are dense in C^*; if the set of endpoints of a continuum is dense, then its complement is of the first category.

I want to report about the question of the category of the set of cut points of a continuum C in the case that the set of cut points of C and the set of non-cut points of C are both dense on C. The class of such continua is very large for it has been found that an arbitrary continuum C in n-dimensional Euclidean space can be embedded in a continuum C^* of dimension n (lying in an Euclidean space of at most $2n + 1$ dimensions) which has the above property.

The answer to the question about the category of the set of all cut points of a continuum C depends on the particular character of non-cut points. This leads to the natural dichotomic classification of non-cut points into two kinds (the first of which are endpoints). If the set of non-cut points of such continua contains a dense (in C) sub-set of points of the first kind, then the set of cut points is of the first category; if, on the other hand, the set of non-cut points contains points of the second kind, then we can say nothing about the category of the set of the cut points, i. e. it may be as well one of the first category as of the second category. This is easily seen from the example given below.

Incidentally, these considerations suggest a new, more natural definition of the notion of simple link which was introduced by R. L. Moore and made use of by G. T. Whyburn. In this way we obtain a more natural definition of cyclic element in the sense of G. T. Whyburn.

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The point x is called cut point of the continuum C if the set $C - x$ is not connected. If the points a and b lie in two disjoint parts of $C - x$

then we say that the point x cuts the continuum C between a and b. The set of all cut points of the continuum C is denoted by $R(C)$. The set of all non-cut points of the continuum C (i. e. points z for which $C - z$ is connected) will be denoted by $N(C)$. Hence

$$R(C) + N(C) = C.$$

If $x \subset R(C)$, then there exists a decomposition of the continuum C into two sets[1])

$$C = A_x + B_x \tag{1}$$

where A_x and B_x are continua (containing more than one point) and where

$$A_x \,.\, B_x = x.$$

We shall say that the decomposition (1) corresponds to the cut point x.

Let a be a fixed non-cut point of the continuum C; for every point $x \subset R(C)$ the point a belongs to one and only one of the two sets A_x and B_x; we will always designate by A_x that of the two sets which contains the point a. We now consider the set

$$P(a) = \prod_x A_x, \tag{2}$$

where x runs over the set $R(C)$. We note that if more than one decomposition (1) corresponds to a point x,[2]) then we must take in (2) all possible A_x, i. e. from every possible decomposition (1).

The set $P(a)$ is always non-empty since $P(a) \supset a$.

Theorem I. *If* $P(a) - a \neq 0$, *then the set* $P(a)$ *is a continuum consisting of more than one point.*

Proof. Let b be an arbitrary point of the set $P(a) - a$. The continuum C contains a continuum K which contains a and b and which is irreducible with respect to this property (i. e. there exists no continuum K' which contains both a and b and which is contained in K and is not equal to K). We shall now show that

$$K \subset A_x \text{ for every } x \subset R(C). \tag{3}$$

If K had a common point z ($z \neq x$) with B_x, then $K \,.\, A_x$ would again be a continuum which contains both a and b and which is different from K (for $K \,.\, A_x$ contains no longer the point z). But this is impossible since K is irreducible between a and b. Thus (3) is proved and then by (2) we have $K \subset P(a)$. Hence, the set $P(a)$ has the property that for every point b it contains a continuum which connects a and b. This means that the set $P(a)$ is connected and since by (2) it is closed, it is actually a continuum.

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1) See *C. Zarankiewicz: Sur les points de division...*, Fund. Math., *IX*, 136.

2) This is the case when x is a branchpoint.

R. L. MOORE introduced the notion of a set which is called „*simple link*“; it was used by G. T. WHYBURN who defines[3]) it in the following way: if p is neither a cut point nor an end point of a connected set M, $p \subset M$, the set consisting of p together with all points of M conjugate[4]) to p will be called a simple link of M.

For continua the set $P(a)$ happens to be identical with the simple link. For we have

Theorem 2. *A necessary and sufficient condition that a set M which contains more than one point ($M \subset C$) be a simple link for C is the existence of a non-cut point z in C such that $M = P(z)$.*

Proof. Suppose the set M which contains more than one point is a simple link of C. Then the set M contains a point z belonging to $N(C)$.[5]) Let $x \subset M$. There are no points which cut C between x and z, i. e. in every decomposition (1) both points x and z always belong to the same summand. Hence, $x \subset P(z)$. This implies that $M \subset P(z)$.

Now, let $y \subset P(z)$; this means that in the decomposition (1) we have $y \subset A_x$ for every $x \subset R(C)$; hence there exists no point which cuts C between y and z. Then, by the definition of simple link, we have $y \subset M$; consequently $P(z) \subset M$. The two above inclusions imply that $P(z) = M$.

Suppose now that the set $P(a)$ contains more than one point; then $P(a) - a$ is not empty and, by Theorem 1, $P(a)$ is a continuum. We will show that $P(a)$ is a simple link for C. Let M_a be a simple link for C containing the point a. For every point $t \subset P(a) - a$ there exists no point x which cuts C between t and a; i. e. both points t and a belong to M_a; hence $t \subset M_a$ and consequently $P(a) \subset M_a$.

Let w be a point of M_a. By the definition of simple link there exists no point which cuts C between w and a; i. e. in every decomposition (1) both points w and a always belong to the same summand A_x. Hence $w \subset P(a)$ and furthermore $M_a \subset P(a)$. The two above inclusions imply $M_a = P(a)$.

As the set $P(a)$ is identical with the simple link for continua we can formulate the following theorems on the basis of the results of G. T. WHYBURN:[6])

Theorem 3. *The necessary and sufficient conditions that a point x of a continuum C be an endpoint are the following:*

1. *the point x is not a cut point of C,*
2. $P(x) = x$.

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[3]) *G. T. Whyburn*: *Analytic Topology*, Coll. Publ. of Amer. Math. Soc., 1942, 64.

[4]) Two point a and b of a connected set M will be said to be conjugate provided no point separates a and b in M.

[5]) *G. T. Whyburn*, loc. cit., 65.

[6]) *G. T. Whyburn*, loc. cit., p. 64.

This theorem can be considered as a new definition of an endpoint in the sense of K. MENGER and G. T. WHYBURN. On the other hand, it gives a dichotomic classification of the non-cut points of any continuum C.

A point x which is a non-cut point of an arbitrary continuum C will be called of the first or second kind, according as $P(x) = x$ or $P(x) \neq x$.

The points of the first kind are endpoints. On the basis of further results of G. T. WHYBURN,[7]) we can formulate:

Theorem 4. *The set $P(x)$ contains an at most denumerable set of points which are cut points of the continuum C.*

Theorem 5. *Two sets $P(x)$ and $P(y)$ are either disjoint, identical or have at most one common point which must be a cut point of C.*

From these theorems it follows immediately:

Theorem 6. *An isolated non-cut point of the continuum C must be of the first kind, i. e. it is an endpoint.*

Proof. As a matter of fact let us suppose that a is an isolated non-cut point of C and that $P(a)$ is a continuum; then by Theorem 4 there will exist an uncountable set of points which do not cut C and which lie in every neighborhood of a. This contradicts the hypothesis that a is an isolated non-cut point. Hence, $P(a) = a$. Therefore — by Theorem 3 — the point a is an endpoint.

An interesting class of continua are those on which the sets $R(C)$ and $N(C)$ are both dense. Examples of such continua have been given by T. WAŻEWSKI, K. MENGER, and myself.[8])

The class of such continua is quite large; for we have following

Theorem 7. *For every n-dimensional continuum C there exists a n-dimensional continuum C^** (lying in an Euclidean space of at most $2n+1$ dimensions) *such that:*

1. $C \subset C^*$,
2. *both $R(C^*)$ and $N(C^*)$ are dense on C^*.*

Proof. A given continuum C may be embedded in an at most $2n+1$ dimensional Euclidean space,[9]) in which every point of the continuum is accessible by a simple arc. In the continuum C every point accessible by a simple arc is also accessible by the dendrite of WAŻEWSKI or by one of MENGER (i. e. if p is a point of the continuum C lying in a Euclidean space R_{2n+1}, then there exists a dendrite of WAŻEWSKI W such that both $W - p \subset R_{2n+1} - C$ and $W . C = p$). Let us now choose in C a denu-

[8]) *T. Ważewski*: *Sur les courbes de Jordan...*, Annales de la Société Polonaise de Math., **2** (1923), 49. — *K. Menger*: *Grundzüge einer Theorie der Kurven*, Math. Ann., **96**, 285. — *C. Zarankiewicz*, loc. cit., 158.

[9]) *K. Menger*: *Kurventheorie*, Berlin-Leipzig, 1932.

[7]) *G. T. Whyburn*, loc. cit., p. 65.

merable dense set L of points and order them in an infinite sequence $u_1, u_2, \ldots$

We now attach at every point u_k a dendrite on which the endpoints are dense (for example a dendrite of WAŻEWSKI or MENGER), which has only a single endpoint in common with C, and is otherwise disjoint from C. We choose the diameters of the attached dendrites so that they converge sufficiently rapidly to zero; in addition we require that no dendrite has a point in common with one previously attached. Such a construction is obviously possible. We define the continuum C^* as the union of C with all the attached continua. It is evident that the continuum C^* thus defined satisfies the conditions of our theorem.

K. MENGER proved that the set of the endpoints of a curve is a G_δ-set.[10]) It is possible to prove the same not only for curves but for all continua.

If a set P is a G_δ-set and is dense in the continuum C, then $C - P$ is of the first category in C (i. e. the sum of a denumerable number of sets which are nowhere dense in C). Taking into consideration the above fact, we can now formulate

Theorem 8. *If the set of all endpoints of a continuum C is dense in C then the set of all non-endpoints of C is of the first category in C.*

Hence each n-dimensional continuum C in which the set of its endpoints is dense is an example of closed set with the following-seemed paradoxical-property: all points at which this set is locally more than one dimensional constitute a set of the first category.

The question arises what is the category of the set $R(C)$ in the case of continua in which $R(C)$ and $N(C)$ are both dense. It is possible to say that if the set $N(C)$ contains a dense (on C) set of non-cut points of the first kind, then $R(C)$ is by Theorem 8 of the first category. But if the sets $N(C)$ and $R(C)$ are both dense in C and the set $N(C)$ consists of points of the second kind (i. e. of non endpoints), then $R(C)$ need not be of the first category any longer — as the following example shows.

Example. We take the curve $y = \sin\frac{1}{x}$ between $x = -\frac{2}{\pi}$ and $x = +\frac{2}{\pi}$ together with the segment $x = 0, -1 \leqq y \leqq +1$.

Every arc of this curve which lies between two successive extrema is replaced by a closed surrounding strip. The strip shall consist of all points of the plane which lie on both sides of the original arc at a distance so small that the boundaries of two adjacent strips have only one point in common, namely the point where the extremum lies. Two non-adjacent strips shall have no points in common and the thickness of the strips con-

[10]) *K. Menger: Grundzüge einer Theorie der Kurven,* Math. Ann., **96**, 286.

verges to zero as $x \to 0$. The segment $x = 0, \ -1 \leqq y \leqq +1$ remains without change. The first approximation J_1 of the curve consists of the segment $x = 0, \ -1 \leqq y \leqq +1$ and the sum of all the strips just defined. Every extremal point of the original curve will be a cut point of J_1 but no other point of J_1 will be a cut point. In order to obtain the second approximation J_2 we replace every strip of J_1 by a continuum which is similar to J_1 with correspondingly smaller thickness but which lies entirely in the strip and has its „ends" at the extremal points of the strip. Thus we obtain J_2. Carrying on this process to infinity, we obtain the approximations $J_1, J_2, \ldots$

We now set

$$J = \prod_{n=1}^{\infty} J_n.$$

The set J is obviously a continuum irreducible between the points $x = -\dfrac{2}{\pi}$ and $x = +\dfrac{2}{\pi}$. The sets $R(J)$ and $N(J)$ are dense on J. It is easily seen that the set $R(J)$ is no longer of the first category, but of the second category, since it is a dense G_δ-set. But the set $N(J)$ consists only of points of the second kind and contains no endpoints at all.