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ON COUNTABLE GENERALISED σ-ALGEBRAS[1]), WITH A NEW PROOF OF GÖDEL'S COMPLETENESS THEOREM[2]).

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We give a generalisation of the notion of a countably additive Boolean algebra (alias σ-algebra) in the following sense:

The countably infinite joins and meets can be performed on members of certain s. c. marked (multiple) sequences; these marked sequences form a family defined by the conditions (1) to (6).

In the case of a countable generalised σ-algebra with a countable family of marked sequences, using a theorem of LOOMIS we give an isomorphical representation of the algebra in a countably additive field of sets.

Because the LINDENBAUM algebra of the lower predicate calculus furnishes a typical example of the mentioned generalised σ-algebra, we immediately get (by the set-representation) a new proof of the known fundamental GÖDEL theorem (stating the completeness of the lower predicate calculus) as based on LOOMIS' theorem.

INTRODUCTION

Let B be an algebra. Let Φ be a family of multiple sequences of elements of B. (We shall write $\{a_{n_1, n_2, \ldots, n_k}\}_{n_1, n_2, \ldots, n_k=1}^{\infty}$, $\{b_{m_1, m_2, \ldots, m_l}\}_{m_1, m_2, \ldots, m_l=1}^{\infty}, \ldots \in \Phi$).

We say that Φ is a family of *marked sequences in B* if the following is true:

(1) (The rule of complement).

If $\{a_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k=1}^{\infty} \in \Phi$ then $\{a'_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k=1}^{\infty} \in \Phi$ ($'$ always denotes the complement in the algebra.)

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[1]) For basic notions of Boolean algebras (in the sequel, the attribute Boolean will be omitted) see e. g. G. BIRKHOFF [1] (The numbers in brackets refer to the list at the end of the paper).

[2]) The subject of this paper has been exposed by the author (within a series of communications on the algebra of lower predicate calculus) at the seminarium of prof. MOSTOWSKI in Warsaw, during April 1950. — I take the opportunity to express my gratitude to prof. MOSTOWSKI for many stimulative and helpful criticisms. Another part of the mentioned communications will appear in Fundamenta Mathematicae 1951. An elaborated algebraic theory of the predicate calculus (of mathematical logic) with applications is planned as a monograph.

(2) (The rule of joins and meets.)

a) If $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$, $\{b_{m_1,\ldots,m_l}\}_{m_1,\ldots,m_l=1}^{\infty} \in \Phi$ then $\{a_{n_1,\ldots,n_k} \cup b_{m_1,\ldots,m_l}\}_{n_1,\ldots,n_k,m_1,\ldots,m_l=1}^{\infty} = \{c_{r_1,\ldots,r_{k+l}}\}_{r_1,\ldots,r_{k+l}=1}^{\infty} \in \Phi$.

b) The same shall hold with $\cap$ instead of $\cup$. (Notice that two sequences are equal if and only if the corresponding members are equal.)

(3) (The rule of identification of choosen indices (forming "diagonal" sequences)).

If $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$ and if $1 \leqq i_1 < i_2 < \ldots < i_s \leqq k$ are chosen integers, then putting $n = n_{i_1} = n_{i_2} = \ldots = n_{i_s} = 1, 2, \ldots$ we get a further sequence

$$\{a_{n_1,\ldots,n_{i-1},n,n_{i_1+1},\ldots,n_{i_s-1},n,n_{i_s+1},\ldots,n_k}\}_{n,n_1,\ldots,n_k=1}^{\infty} = \\ = \{b_{m_1,\ldots,m_{k-s+1}}\}_{m_1,\ldots,m_{k-s+1}=1}^{\infty} \in \Phi.$$

(4) (The rule of fixation of indices (forming "cylindric" sequences)).

If $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$, if $1 \leqq i_1 < i_2 < \ldots < i_s \leqq k$ are chosen integers and if $n_{i_1}, n_{i_2}, \ldots, n_{i_s}$ are preassigned positive integers, then the sequence

$$\{a_{n_1,\ldots,n_{i_1},\ldots,n_{i_s},\ldots,n_k}\}_{n_1,\ldots,n_{i_1-1},n_{i_1+1},\ldots,n_{i_s-1},n_{i_s+1},\ldots,n_k=1}^{\infty} = \\ = \{c_{r_1,r_2,\ldots,r_{k-s}}\}_{r_1,r_2,\ldots,r_{k-s}=1}^{\infty} \in \Phi$$

also belongs to Φ.

(5) (The rule of trivial sequences)

If $a_{n_1,n_2,\ldots,n_k} = a \in B$ for any $n_1, n_2, \ldots, n_k = 1, 2, \ldots,$ then $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$.

(6) (The rule of L. U. B. and G. L. B.)

To each $\{a_{n_1,n_2,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$ there exists in B the L.U.B., i.e. the (at most) countable join, and the G. L. B., i. e. the (at most) countable meet (in the sense of the lattice ordering, of course) of the members $a_{n_1,\ldots,n_k}$ of the sequence in question.[3])

When b and c are the mentioned L. U. B. and the G. L. B. respectively, then we write

$$\bigcup_{n_1=1}^{\infty} \bigcup_{n_2=1}^{\infty} \cdots \bigcup_{n_k=1}^{\infty} a_{n_1,n_2,\ldots,n_k} = b$$

$$\bigcap_{n_1=1}^{\infty} \bigcap_{n_2=1}^{\infty} \cdots \bigcap_{n_k=1}^{\infty} a_{n_1,n_2,\ldots,n_k} = c$$

[3]) It is to be noticed that since the arranging of the members in a marked sequence is irrelevant for their L. U. B. and G. L. B. we simply could speak about certain "marked" at most countable subsets of the algebra. But formally, this conception would bring, in fact, more complications than simplifications — not speaking about the suitability of the application to logic.

or more shortly

$$\bigcup_{n_1, \ldots, n_k=1}^{\infty} a_{n_1, \ldots, n_k} = b, \quad \bigcap_{n_1, \ldots, n_k=1}^{\infty} a_{n_1, \ldots, n_k} = c.$$

(7) (The rule of partial L. U. B. and G. L. B.)

a) Let $\{a_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k=1}^{\infty}$ be a k-tuple sequence belonging to Φ. Let j with $1 \leqq j \leqq k$ be a fixed integer.

Then the joins

$$\bigcup_{n_j=1}^{\infty} a_{n_1, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k} = b_{n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k}$$

performed (by (6)) after any fixation (by (4)) of the $k-1$ indices $n_1, \ldots, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k$ are mambers of a $k-1$ tuple sequence

$$\{b_{n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k}\}_{n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k=1}^{\infty}$$

belonging also to Φ.

b) The same shall hold with $\cap$ instead of $\cup$.

We say that the algebra B then becomes *a generalised σ-algebra* with *respect to the family Φ of marked sequences*; more shortly, *B is a $\Phi\sigma$-algebra.*

The concept of a $\Phi\sigma$-algebra can obviously be taken for a generalisation of the concept of σ-algebras (i. e. countably additive algebras) where Φ is the family of all multiple sequences. In the trivial case (of any algebra), Φ contains trivial marked sequences (sub (5)) only. In order to present a less trivial example, consider the σ-field of Borel subsets of any topological space. Let Φ be the family of multiple sequences of Borel subsets, formed by completing the set of all multiple sequences of open subsets as to fulfil the above rules (1) to (6). Then the algebra of Borel subsets $G_{\delta\sigma\delta} \ldots$ and $F_{\sigma\delta\sigma} \ldots$ with a finite combination of the suffixes σ, δ is a $\Phi\sigma$-algebra though not a σ-algebra. (If the space in question is perfectly normal, i. e. each closed subset is a G_δ, then the $\Phi\sigma$-algebra in question consists of $G_{\delta, \sigma, \ldots, \delta(\sigma)}$ subsets).

A natural question is whether a $\Phi\sigma$-algebra can be isomorphically represented by a field of sets, i. e. immerged in a suitable σ-field of subsets of a set. Of course, this is not possible in general because it is well known that the σ-algebra of Lebesgue measurable subsets of the interval (0,1) taken modulo subsets of measure zero cannot be isomorphically represented by a σ-field of sets.

In the present note, we shall answer the above question in the positive for any countable $\Phi\sigma$-algebra with a countable family Φ of fundamental sequences. This quite special case is considered for the purpose of mathematical logic. Since the so called LINDENBAUM algebra of the lower predicate calculus furnishes a typical example of such a $\Phi\sigma$-algebra, we almost immediately get a new (algebraical) proof of the well known completeness theorem for lower predicate calculus, due to GÖDEL [2].

Notions and symbols.

In the sequel, we take the joins $\bigcup\limits_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k}$ and the meets $\bigcap\limits_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k}$ with $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$ for the only "algebraically" defined infinite operations of the $\Phi\sigma$-algebra in question. (If nothing other is said, then any multiple sequence (as $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty}$, $\{b_{m_1,\ldots,m_l}\}_{m_1,\ldots,m_l=1}^{\infty}, \ldots$) belongs to the considered family Φ of marked sequences.) $+$, Σ always denote set sums, . , Π set products (intersections), $\emptyset$ the void set, 0 the unit and 1 the zero (of an algebra).

We need an appropriate generalisation of the known basic notions of σ-homo(iso)morphic mapping, of a σ-(prime)ideal and of the corresponding quotient algebra.

We *define*:

A mapping φ of a $\Phi\sigma$-algebra A in a $\Psi\sigma$-algebra B is *σ-homomorphic* (in the generalised sense, this phrase often being omitted) if $\{\varphi(a_{n_1,\ldots,n_k})\}_{n_1,\ldots,n_k=1}^{\infty} \in \Psi$ whenever $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$ — and if

$$\varphi(a') = (\varphi(a))', \quad \bigcup_{n_1,\ldots,n_k=1}^{\infty} \varphi(a_{n_1,\ldots,n_k}) = \varphi(\bigcup_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k}), \quad \bigcap_{n_1,\ldots,n_k=1}^{\infty} \varphi(a_{n_1,\ldots,n_k}) =$$

$$= \varphi(\bigcap_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k}).$$ — φ is called a proper *σ-homomorphic mapping* of A *onto* B if φ is a σ-homomorphic mapping of A onto the whole B and, moreover, if to any $\{b_{m_1,\ldots,m_l}\}_{m_1,\ldots,m_l=1}^{\infty} \in \Phi$ there exists a suitable $\{a_{m_1,\ldots,m_l}\}_{m_1,\ldots,m_l=1}^{\infty} \in \Phi$ so that $\varphi(a_{m_1,\ldots,m_l}) = b_{m_1,\ldots,m_l}$. In this case, B is said to be a (generalised) *σ-homomorphic image of A*. If such a φ is a one-one correspondence, then A is said to be *σ-isomorphic with B*.

A nonvoid subset $J \neq A$ of elements of a $\Phi\sigma$-algebra A is said to be a *$\Phi\sigma$-ideal* if

1) $\bigcap\limits_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k} \in J$, whenever $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$, $a_{n_1,\ldots,n_k} \in J$ for each $n_1, \ldots, n_k = 1, 2, \ldots$

2) $a \in J$, whenever $b \subseteq a$, $b \in J$.

A $\Phi\sigma$-ideal P is said to be *prime*, if $\bigcup\limits_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k} \in P$, $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi$ implies $a_{\bar{n}_1,\ldots,\bar{n}_k} \in P$ for suitable positive integers

$\overline{n}_1, \overline{n}_2, \ldots, \overline{n}_k$. (Some authors speak about dual ideals. M. H. Stone sometimes uses the term α-ideal. For the purpose of mathematical logic, the used kind of ideals seems to be more convenient.) — Note that in general, any of the considered assertions possesses a corresponding dual one, the dualisation being left to the reader.

The *$\Phi^*\sigma$-quotient algebra A/J* of the $\Phi\sigma$-algebra A:

Let A be a $\Phi\sigma$-algebra, J a $\Phi\sigma$-ideal in A. Then by the *$\Phi^*\sigma$-quotient algebra A/J* we shall understand the generalised σ-algebra of the classes $X = [x]$, $Y = [y]$, ... (with $x, y, \ldots \in A$ and $[x_1] = [x_2]$, i. e. $x_1 \equiv x_2$ (mod$J$), whenever $x_1 \cap c = x_2 \cap c$ with a suitable $c \in J$) *with respect to the family Φ^* of marked sequences* $\{X_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k = 1}^{\infty} = \{[x_{n_1, \ldots, n_k}]\}_{n_1, \ldots, n_k = 1}^{\infty} \in \Phi^*$ (where $\{x_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k = 1}^{\infty} \in \Phi$). — To see that A/J really becomes $\Phi^*\sigma$-algebra, it is sufficient to point out two facts:

First, $x_1 \equiv x_2$ (mod$J$), if and only if $c^* = (x_1 \cup x_2') \cap (x_1' \cup x_2) \in J$, c^* having the property $x_1 \cap c^* = x_2 \cap c^* = x_1 \cap x_2$.

Second, $x_{n_1, \ldots, n_k} \equiv y_{n_1, \ldots, n_k}$ (mod$J$) means that

$c^*_{n_1, \ldots, n_k} = (x_{n_1, \ldots, n_k} \cup y'_{n_1, \ldots, n_k}) \cap (x'_{n_1, \ldots, n_k} \cup y_{n_1, \ldots, n_k}) \in J$ what implies $\{c^*_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k = 1}^{\infty} \in \Phi$ and therefore $c = \bigcap_{m_1, \ldots, m_k = 1}^{\infty} c^*_{m_1, \ldots, m_k} \in J$.

Hence $x_{n_1, \ldots, n_k} \cap c = y_{n_1, \ldots, n_k} \cap c$ for each $n_1, \ldots, n_k = 1, 2, \ldots$ so that $\left(\bigcap_{n_1, \ldots, n_k = 1}^{\infty} x_{n_1, \ldots, n_k}\right) \cap c = \left(\bigcap_{n_1, \ldots, n_k = 1}^{\infty} y_{n_1, \ldots, n_k}\right) \cap c$ and $\bigcup_{n_1, \ldots, n_k = 1}^{\infty} (x_{n_1, \ldots, n_k} \cap c) = \left(\bigcup_{n_1, \ldots, n_k = 1}^{\infty} x_{n_1, \ldots, n_k}\right) \cap c = \bigcup_{n_1, \ldots, n_k = 1}^{\infty} (y_{n_1, \ldots, n_k} \cap c) = \left(\bigcup_{n_1, \ldots, n_k = 1}^{\infty} y_{n_1, \ldots, n_k}\right) \cap c$ by a generalised associative and resp. distributive law (see (II) below).

The *$\Phi\sigma$-subalgebra of a $\Psi\sigma$-algebra*:

A $\Phi\sigma$-algebra $\tilde{A}$ is said to be a *$\Phi\sigma$-subalgebra of the $\Psi\sigma$ algebra A* if

(1) $\tilde{A} \subseteq A$

(2) $\Phi \subseteq \Psi$

(3) The operations in the sense of $\tilde{A}$ have the same effect as the operations in the sense of the whole A.

(When especially A is a σ-algebra then the condition (2) trivially follows from (1), (3).)

Known theorems.

Let us collect the essentially known lemmata and theorems needed in the sequel. (We use the already introduced symbols.)

(I) (De Morgan identities.)

$$\bigcup_{n_1,\ldots,n_k=1}^{\infty} a'_{n_1,\ldots,n_k} = (\bigcap_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k})'$$

and dually.

(II) (A generalised distributive law.)

$$(\bigcup_{n_1,\ldots,n_k=1}^{\infty} a_{n_1,\ldots,n_k}) \cap (\bigcup_{m_1,\ldots,m_l=1}^{\infty} b_{m_1,\ldots,m_l}) = \bigcup_{n_1,\ldots,n_k,m_1,\ldots,m_l}^{\infty} (a_{n_1,\ldots,n_k} \cap b_{m_1,\ldots,m_l})$$

and dually. (Proof with the help of De Morgan identities, see G. BIRKHOFF [1]).

(III) (Immerging of a $\Phi\sigma$-algebra in a σ-algebra.)

Any $\Phi\sigma$-algebra A can be immerged in a σ-algebra $\bar{A}$ under preserving the $\Phi\sigma$-operations. (A is then a $\Phi\sigma$-subalgebra of the σ-algebra $\bar{A}$). (Proof by completing A into $\bar{A}$ with the help of Mc NEILLE [7] cuts.)

(IV) (The first lemma on generalised σ-isomorphism).

Let the $\Psi\sigma$-algebra B be a (generalised) σ-homomorphic image of the $\Phi\sigma$-algebra A, under the proper homomorphic mapping φ of A onto B. — Then the $\Phi^*\sigma$-quotient algebra A/J, where $J = \varphi^{-1}(1)$ is a $\Phi\sigma$-ideal in A, is σ-isomorphic (in the generalised sense) with B. (Proof by an obvious generalisation of the well known argument of Abstract Algebra on account of (II) and of the justification of the definition of the $\Phi^*\sigma$-quotient algebras.)

(V) (The second lemma on generalised σ-isomorphism.)

Let A be a σ-algebra, $\tilde{A}$ a $\Phi\sigma$-subalgebra of A, J a σ-ideal in A. Take $(\tilde{A}, J) = \sum_{x \in \tilde{A}} [x] \supseteq \tilde{A}$ for the $\Phi\sigma$-algebra (with the same fundamental sequences as in $\tilde{A}$).

Then $\tilde{A} \,.\, J$ is a $\Phi\sigma$-ideal in $\tilde{A}$ and we have the generalised σ-isomorphism (in the sense of the above definitions)

$$(\tilde{A}, J)/J \cong \tilde{A}/\tilde{A} \,.\, J$$

given by the one-one mapping $\varphi([x]) = \langle x \rangle \in \tilde{A}/\tilde{A}J$ of the $\Phi^*\sigma$-quotient-algebra $(\tilde{A}, J)/J$ into the $\Phi^*\sigma$-quotient-algebra $\tilde{A}/\tilde{A} \,.\, J$ (if $\langle x \rangle$ denotes the class of elements of $\tilde{A}$ congruent to $x \in \tilde{A}$ modulo $\tilde{A} \,.\, J$.)

(Proof by the well known argument using (IV) (above) to the generalised proper homomorphic mapping $x \rightarrow [x]$ of $\tilde{A}$ onto $(\tilde{A}, J)/J$.)

(VI) (Theorem of LOOMIS [6]).[4])

Any σ-algebra A can be σ-isomorphically represented as a quotient σ-algebra $\mathfrak{F}/\mathfrak{J}$ of a suitable σ-field $\mathfrak{F}$ of sets taken modulo a σ-ideal $\mathfrak{J}$, $A \cong \mathfrak{F}/\mathfrak{J}$.

Let us return to the set representation of countable generalised σ-algebras with a countable family of marked sequences.

Theorem I. *Let A be a countable $\Phi\sigma$-algebra with a countable family Φ of marked sequences. — Then there exists a set field $\mathfrak{a}$ and a family Ψ of marked sequences of sets in $\mathfrak{a}$ so that the $\Psi\sigma$-algebra $\mathfrak{a}$ is σ-isomorphic* (in the generalised sense) *with the $\Phi\sigma$-algebra A.*

Proof. Let $\bar{A}$ be the σ-algebra with the given $\Phi\sigma$-subalgebra A, according to the above lemma (III).

Let $\bar{A} \cong \mathfrak{F}/\mathfrak{J}$ be a σ-isomorphic representation of $\bar{A}$ by the quotient σ-algebra of the σ-field $\mathfrak{F}$ (of subsets $X, Y, \ldots$ of a suitable set F) divided by the σ-ideal $\mathfrak{J}$, according to the lemma (VI) of LOOMIS. Let $\varphi(x) = [X] \in \mathfrak{F}/\mathfrak{J}$ (with $x \in \bar{A}$, $X \in \mathfrak{F}$) be the representing one-one mapping of $\bar{A}$ onto $\mathfrak{F}/\mathfrak{J}$.

Then $\varphi(A) = \tilde{\mathfrak{F}}/\mathfrak{J}$ is a countable $\tilde{\Phi}\sigma$-subalgebra of the σ-algebra $\mathfrak{F}/\mathfrak{J}$, $\tilde{\Phi}$ consisting of the sequences $\{\varphi(x_{n_1, \ldots, n_k})\}_{n_1, \ldots, n_k=1}^{\infty}$ with $\{x_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k=1}^{\infty} \in \Phi$. Choose one representative set $X \in \tilde{\mathfrak{F}}$ in each of the countably many classes $[X] \in \tilde{\mathfrak{F}}/\mathfrak{J}$ and form the (of course countable) set field generated (in $\tilde{\mathfrak{F}}$) by these representants. Further, consider the smallest countable family Φ_1 of sequences (of members of the already formed set field) fulfilling (1) to (5) of the introduction and containing any $\{X_{m_1, \ldots, m_l}\}_{m_1, \ldots, m_l=1}^{\infty}$ with $\{[X_{m_1, \ldots, m_l}]\}_{m_1, \ldots, m_l=1}^{\infty} \in \tilde{\Phi}$ (i.e. with $\varphi(x_{m_1, \ldots, m_l}) = [X_{m_1, \ldots, m_l}]$ $(m_1, \ldots, m_l = 1,2, \ldots)$ whenever $\{x_{m_1, \ldots, m_l}\}_{m_1, \ldots, m_l=1}^{\infty} \in \Phi$ in A). Finally, adjoin the corresponding set sums $\sum_{m_1, \ldots, m_l=1}^{\infty} X_{m_1, \ldots, m_l}$ and set products $\prod_{m_1, \ldots, m_l=1}^{\infty} X_{m_1, \ldots, m_l}$ to the already formed set field. In this way, we get a countable $\Phi_1\sigma$-algebra, i. e. set field, say $\mathfrak{F}^*$, with the countable family Φ_1 of marked sequences. Obviously $\mathfrak{F}^* \subseteq \tilde{\mathfrak{F}}$ (because $\tilde{\mathfrak{F}}/\mathfrak{J}$ is a $\tilde{\Phi}\sigma$-algebra) and using the symbols of the above lemma (V) we can write $\tilde{\mathfrak{F}} = \sum_{Y \in \mathfrak{F}^*} [Y] = (\mathfrak{F}^*, \mathfrak{J})$.

Now, taking Φ_1 as the family of marked sequences of $\tilde{\mathfrak{F}} = (\mathfrak{F}^*, \mathfrak{J})$ we easily see that the derived family of marked sequences of the quotient algebra $\tilde{\mathfrak{F}}/\mathfrak{J}$ (in the sense of the above definition of

[4]) Restated in BIRKHOFF [1]. A simple proof can be found in SIKORSKI [11]. For another proof and a topological strengthening of this LOOMIS theorem, see RIEGER [10].

a generalised quotient σ-algebra) equals the original family $\tilde{\Phi}$ ot marked sequences of $\mathfrak{F}/\mathfrak{J}$, as corresponding to Φ of A. Hence using the lemma (V) we get the generalised σ-isomorphisms

$$A \cong (\mathfrak{F}^*, \mathfrak{J})/\mathfrak{J} \cong \mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J} \tag{*}$$

given by the one-one mappings $\varphi(X) = [x] \leftrightarrow \langle X \rangle \in \mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J}$ if $X \in \mathfrak{F}^*$, $x \in A$.

Since $\mathfrak{F}^* . \mathfrak{J}$ is a countable subideal of the σ-ideal $\mathfrak{J}$ (of $\mathfrak{F}$) hence to each class $\langle X \rangle \in \mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J}$ we can form the set $\tilde{X} = \prod_{Y \in \langle X \rangle} Y \in [X]$.

Let us prove that also $\tilde{X} = Y . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z$ with an arbitrary $Y \in \langle X \rangle$. — First, we have $Y_1 . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z = Y_2 . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z$ whenever $Y_1 \equiv Y_2 \pmod{\mathfrak{F}^* . \mathfrak{J}}$ (with $Y_1, Y_2 \in \mathfrak{F}^*$) because $Y_1 . Z = Y_2 . Z$ with a suitable $Z \in \mathfrak{F}^* . \mathfrak{J}$.

Therefore $\tilde{X} = \prod_{Y \in \langle X \rangle} Y \supseteq \prod_{Y \in \langle X \rangle} Y . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z = Y_1 . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z$ with an arbitrary $Y_1 \in \langle X \rangle \in \mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J}$. On the other side, since $Y_1 . Z \in \langle X \rangle$ with arbitrary $Y_1 \in \langle X \rangle$, $Z \in \mathfrak{F}^* . \mathfrak{J}$ we get $\tilde{X} = \prod_{Y \in \langle X \rangle} Y \subseteq \prod_{Z \in \mathfrak{F}^*.} Y_1 . Z =$
$= Y_1 . \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z.$

Now, we have to show that the sets $\tilde{X}$ form a set field $\mathfrak{a}$ (of certain subsets of the set $\prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z$) and, moreover, that $\mathfrak{a}$ is the desired $\Psi\sigma$-algebra (of sets) σ-isomorphic (in the generalised sense) with the $\Phi^*\sigma$-algebra $\mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J}$ under the one-one correspondence $\langle X \rangle \leftrightarrow \tilde{X}$ with $X \in \mathfrak{F}^*$. ($\mathfrak{a}$ also is a representative $\Psi\sigma$-set field of the $\tilde{\Phi}\sigma$-quotient algebra $\mathfrak{F}/\mathfrak{J}$ (but, in general not of $\mathfrak{F}^*/\mathfrak{F}^* . \mathfrak{J}$) though this fact will not be used in the sequel.) — Thereby, the proof will be completed on the ground of the above generalised σ-isomorphisms (*).

Indeed, first $\langle X_1 \rangle \neq \langle X_2 \rangle$ implies $\tilde{X}_1 \neq \tilde{X}_2$ (assumed $X_1, X_2 \in \mathfrak{F}^*$, of course) by $[X_i] = [\tilde{X}_i]$ $(i = 1, 2)$ and by the above one-one σ-isomorphic mapping $[X] \leftrightarrow \langle X \rangle$(*), using the expression $\tilde{X} = \prod_{Y \in \langle X \rangle} Y$ for $\tilde{X}$.

Second, use the other expression $\tilde{X} = Y \prod_{Z \in \mathfrak{F}^* . \mathfrak{J}} Z$ $(Y \in \langle X \rangle)$ for $\tilde{X}$ and suppose $X_{n_1, \ldots, n_k} \in \mathfrak{F}^*$, $\{X_{n_1, \ldots, n_k}\}_{n_1, \ldots, n_k = 1}^{\infty} \in \Phi_1$ and hence $\sum_{n_1, \ldots, n_k = 1}^{\infty} X_{n_1, \ldots, n_k} \in \mathfrak{F}^*$, $\prod_{n_1, \ldots, n_k = 1}^{\infty} X_{n_1, \ldots, n_k} \in \mathfrak{F}^*$.

Then $\overline{\sum_{n_1,\ldots,n_k=1}^{\infty} X_{n_1,\ldots,n_k}} = (\sum_{n_1,\ldots,n_k=1}^{\infty} X_{n_1,\ldots,n_k})(\prod_{Y \in \mathfrak{F}^*.\mathfrak{J}} Y) =$

$= \sum_{n_1,\ldots,n_k=1}^{\infty} (X_{n_1,\ldots,n_k} \cdot \prod_{Y \in \mathfrak{F}^*.\mathfrak{J}} Y) = \sum_{n_1,\ldots,n_k=1}^{\infty} \tilde{X}_{n_1,\ldots,n_k}$ and analogously

$\overline{\prod_{n_1,\ldots,n_k=1}^{\infty} X_{n_1,\ldots,n_k}} = (\prod_{n_1,\ldots,n_k=1}^{\infty} X_{n_1,\ldots,n_k})(\prod_{Y \in \mathfrak{F}^*.\mathfrak{J}} Y) =$

$= \prod_{n_1,\ldots,n_k=1}^{\infty} (X_{n_1,\ldots,n_k} \cdot \prod_{Y \in \mathfrak{F}^*.\mathfrak{J}} Y) = \prod_{n_1,\ldots,n_k=1}^{\infty} \tilde{X}_{n_1,\ldots,n_k}.$

From these facts we easily conclude that $\mathfrak{a}$ has the desired properties with respect to the family Ψ of marked sequences $\{\tilde{X}_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty}$ whose members $\tilde{X}_{n_1,\ldots,n_k}$ correspond one-by-one (by $\langle X \rangle \leftrightarrow X$) to the members $\{\langle X_{n_1,\ldots,n_k} \rangle\}_{n_1,\ldots,n_k=1}^{\infty} \in \Phi_1^*$ of marked sequences of $\mathfrak{F}^*/\mathfrak{F}^*.\mathfrak{J}$, q. e. d.

Lindenbaum[5]) algebra of the lower predicate calculus.

For the lower predicate calculus, let us make a systematical use of the standard formulation and of the symbols of HILBERT, ACKERMANN [3], P. 53 — [6]), with the original numbering of axioms and rules of inference. The basic notions of the predicate calculus will be assumed as known and the reader may refer to [3].

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ be formulae (Formeln) of the lower predicate calculus. Considering the equivalence relation (between formulae) given by the fact that $\mathfrak{A} \sim \mathfrak{B}$ is an inferred formula (abgeleitete Formel) (i. e. that both $\mathfrak{A} \rightarrow \mathfrak{B}$ and $\mathfrak{B} \rightarrow \mathfrak{A}$ can be inferred from the axioms by the calculus) denote by $|\mathfrak{A}|, |\mathfrak{B}|, |\mathfrak{C}|, \ldots$ the classes of mutually equivalent formulae. — As it is well known and can be easily proved, these classes form a Boolean algebra A under the following definitions:

$$|\mathfrak{A}| \cup |\mathfrak{B}| = |\mathfrak{A} \vee \mathfrak{B}|,$$
$$|\mathfrak{A}| \cap |\mathfrak{B}| = |\mathfrak{A} \mathbin{\&} \mathfrak{B}|,$$

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[5]) The name of the essentialy known concept is chosen in honour of ADOLF LINDENBAUM, a Polish logician and mathematician murdered by the nazis. — LINDENBAUM, TARSKI and MOSTOWSKI iniciated the application of modern algebraic notions to mathematical logic. Comp. e. g. MOSTOWSKI [9].

[6]) It only may be pointed out that we interpret the symbols of [3] (to be used in the sequel) as belonging all to the usual mathematical language and as *denoting* elements of the calculus or certain finite sequences of them, the s. c. formulae, i. e. these signs do not belong to the calculus itself. — This is in accordance with the recent conception of a structural (s. c. syntactical) mathematical theory of a logical calculus the objects of which need not be symbols since they can be e. g. certain geometrical or topological entities or even switching electrical circuits. — Hence e. g. the letter x is a sign of an individual variable, not the variable itself.

(assumed there is no individual variable occuring as free in one and as bounded in the other of the formulae $\mathfrak{A}$, $\mathfrak{B}$; this allways can be satisfied without loss of equivalence, by a suitable change of bounded individual variables according to the rules δ) of [3]),

$$|\mathfrak{A}|' = |\overline{\mathfrak{A}}|, \quad 1 = |\mathfrak{A} \rightarrow \mathfrak{A}|, \quad 0 = |\overline{\mathfrak{A}} \,\&\, \mathfrak{A}|,$$

$|\mathfrak{A}| \subseteq |\mathfrak{B}|$, if and only if $\mathfrak{A} \rightarrow \mathfrak{B}$ is an inferred formula. Now, denote by $x_1, x_2, \ldots$ all the countably many individual variables. Somewhat less known is the fact that $|\exists x \mathfrak{B}(x)| \in A$ and $|(x)\mathfrak{B}(x)| \in A$ (provided $\mathfrak{B}(x)$ containing the individual variable x in a free manner only) are resp. the L. U. B. and the G. L. B. of the elements $|\mathfrak{B}^*(x_n)| \in A$ $(n = 1, 2, \ldots)$, the formulae $\mathfrak{B}^*(x_n)$ being given by substituting the free individual variable x_n for x in $\mathfrak{B}(x)$[7]) — under a preceding suitable change (in the sequel denoted by the asterisk) of the bounded variable x_n when occurring in $\mathfrak{B}(x)$. (The last is always possible without loss of equivalence, on account of the rules δ) (Umbenennungsregeln für die gebundenen Variablen) of [3].)

Indeed, considering the case of the G. L. B. we observe that the axiom e) of [3]: $(x)F(x) \rightarrow F(y)$ (with the help of rules of substitution α_1), α_3) of [3]) says $|(x)\mathfrak{B}(x)| \subseteq |\mathfrak{B}^*(x_n)|$ for each $n = 1, 2, \ldots$ — On the other hand, let us recall the scheme γ_1) of [3]:

If $\mathfrak{A} \rightarrow \mathfrak{B}(x)$ is a derived formula where the individual variable x occurs in $\mathfrak{B}(x)$ freely and does not occur in $\mathfrak{A}$ at all, then $\mathfrak{A} \rightarrow (x)\mathfrak{B}(x)$ is a further derived formula.

Hence if $|\mathfrak{A}| \subseteq |\mathfrak{B}^*(x_n)|$ in A, i. e. $\mathfrak{A} \rightarrow \mathfrak{B}^*(x_n)$ are derived formulae for each $n = 1, 2, \ldots$, then, of course, there is a free variable $x_m = x$ not occurring in $\mathfrak{A}$ at all so that $\mathfrak{A} \rightarrow (x)\mathfrak{B}(x)$ is a derived formula, i. e. $|\mathfrak{A}| \subseteq$ $\subseteq |(x)\mathfrak{B}(x)|$ in A.

The dual case (of the L. U. B.) being analogous with the axiom f) instead of e) and with the scheme γ_2) instead of γ_1) (of [3]) our assertion is proved. — Hence we can and will write

$$\bigcup_{n=1}^{\infty} |\mathfrak{B}^*(x_n)| = |\exists x \mathfrak{B}(x)|$$

and

$$\bigcap_{n=1}^{\infty} |\mathfrak{B}^*(x_n)| = |(x)\mathfrak{B}(x)|$$

for the "algebraically" defined countably infinite joins and meets in the Lindenbaum algebra A.[8])

Now, observe the usual definiton of the notion of a formula. (See [3], P. 53, 54.) We easily see that A becomes, moreover, a $\Phi\sigma$-algebra in the following sense:

[7]) Notice that, of course, $x = x_n$ for a suitable positive integer n.

[8]) It is remarkable that the result of each of the defined infinite operations in A is given by a finite inference process of the calculus (in each individual case).

Consider all the multiple sequences of the forms $\{|F^*(x_n)|\}_{n=1}^{\infty}$, $\{|G^*(x_{n_1}, x_{n_2})|\}_{n_1, n_2=1}^{\infty}$, $\{|H^*(x_{n_1}, x_{n_2}, x_{n_3})|\}_{n_1, n_2, n_3=1}^{\infty}$, ... where $F(.)$, $G(.\,,.)$, $H(.\,,.\,,.)$, ... are predicate variables. Let us form the smallest family Φ containing these sequences and fulfilling the conditions (1) to (5) of the introduction. Then (6) is true too (by the already stated interpretation of the quantifiers by countably infinite joins and meets resp. in A) and Φ becomes the family of all the sequences of the form $\{|\mathfrak{A}^*(x_{n_1}, x_{n_2}, \ldots, x_{n_k})|\}_{n_1,\ldots,n_k=1}^{\infty}$ where $\mathfrak{A}(x, y, \ldots, w)$ is a formula containing the k different free individual variables $x, y, \ldots, w$. Therefore we can conclude (with the help of the above thm. 1) by

Theorem 2. *The Lindenbaum algebra A* (of the lower predicate calculus of HILBERT-BERNAYS) *is a countable $\Phi\sigma$-algebra with the (already defined) countable family Φ of marked sequences and A is σ-isomorphic* (in the generalised sense) *with a $\Psi\sigma$-field $\mathfrak{a}$ of sets.*[9])

Now, notice that any prime $\Phi\sigma$-ideal P of A generates a proper $\Phi\sigma$-homomorphic mapping of the algebra A (of classes of logically equivalent formulae) onto the two element algebra (0, 1) of the "truth" = 1 and the "falsehood" = 0, i. e. P gives an interpretation of formulae as sentences. To any predicate variable, we namely define a k-tuple relation $K(.\,,.\,,\ldots,.)$ between positive integers $n_1, n_2, \ldots, n_k$ as fulfilled when $|K(x_{n_1}, \ldots, x_{n_k})| \in P$.

By induction (under the usual interpretation of logical junctives and quantifiers) we easily see that any formula $\mathfrak{A}$ becomes then a sentence about positive integers which should be taken for true if and only if $|\mathfrak{A}| \in P$. The rigourous details of these considerations would require the introduction of certain semantical notions being of less interest here. Nevertheless, it may be sufficiently clear that the (somewhat strengthened) mathematical kernel of the GÖDEL'S completeness theorem[10]) (for the lower predicate calculus) can be now stated as follows:

Theorem 3. *To each non-zero element $|\mathfrak{A}| \neq 0$ of the Lindenbaum $\Phi\sigma$-algebra A there is a prime $\Phi\sigma$-ideal P which contains $|\mathfrak{A}|$.* The (somewhat weaker) formulation in terms of logic is as follows:

Each formula $\mathfrak{B}$ of the lower predicate calculus either is a derived formula (then it is "identically true" in each interpretation) *or $\mathfrak{B}$ can be interpreted as a false sentence about positive integers* (by means of a suitable interpretation.) (We put, of couse, $\mathfrak{A} = \overline{\mathfrak{B}}$.)

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[9]) It can be shown, that $\mathfrak{a}$ is a subfield of the set-field of Borel subsets of Cantor discontinuum, see RIEGER [10].

The finitary isomorphism is essentially well known, comp. e. g. MOSTOWSKI [9].

[10]) In its whole sense (of theoretical logic), this theorem is a syntactically-semantical assertion the complete formulation of which is complicated and, moreover, till now not wholly unified, for certain rather philosophical aspects not to be discussed here.

(The (unessential) strengthening in comparison with the original result of GÖDEL [2] (essentially restated in HILBERT-ACKERMANN [3] and in HILBERT-BERNAYS [4]) consists in the fact that our algebraical formulation gives, indeed, a simultaneous consistent interpretation of all formulae (this interpretation including the being true of $\mathfrak{A}$) whereas originally, the true intrpretation of the isolated formula $\mathfrak{A}$ was constructed only.)[11])

P r o o f of the theorem 3.[12])

Apply theorem 2 in choosing a point ξ in the nonvoid set $S \epsilon \mathfrak{A}$ corresponding to the $|\mathfrak{A}| \epsilon A$ in the set representation of the LINDENBAUM $\Phi\sigma$-algebra A of the lower predicate calculus. Then the prime $\Phi\sigma$-ideal P_ξ in A, corresponding to the prime $\Psi\sigma$-ideal P^*_ξ in $\mathfrak{a}$, where P^*_ξ consists of all the sets of $\mathfrak{a}$ containing the point ξ, furnishes an example of a prime ideal of the desired kind.

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[11]) In the semantical terminology of CARNAP, one could take the prime $\Phi\sigma$-ideal P (of A) in question for an expression of the so called real state including the truth of the sentence $\mathfrak{A}$. An analogous situation (in other terms) is to be found in L. HENKIN [5].

[12]) The first essential modification of the original GÖDEL'S method (avoiding the use of the s.c. SKOLEM normal form) has been given by A. MOSTOWSKI [8]. An essentially new proof was recently given by L. HENKIN [5]. Another proof (using topological means) of R. SIKORSKI and H. RASIOWA appeared in Fund. Math. 1950.