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Remarks on e-locally fine spaces

Jan Pelant and Michael D. Rice[1]

A uniform space X is e-locally fine ($[Fr]_1$) (or locally sub-metric-fine ($[R]_1$)) if each cover, whose restriction to each member of some countable uniform cover is uniform, is itself a uniform cover. The e-locally fine spaces form a coreflective subcategory of uniform spaces - to each uniformity u one assigns the uniformity $m \circ u = u/\lambda eu$, where eu is the uniformity with the basis of countable u-covers, λ is the locally fine operator, and / denotes the operation defined in $[R]_1$: if u and v are families of covers, u/v denotes the family of covers refined by covers of the form $\{V_s \cap U_t^s\}$, where $\{V_s\} \in v$ and each $\mathcal{U}^s = \{U_t^s\} \in u$. This operation is a generalization of the Ginsburg-Isbell derivative defined in [GI]. In $[R]_1$ the second author asked whether each e-locally fine space is sub-metric fine (i.e., a subspace of a metric-fine space - see $[R]_3$). In this paper we will give two methods which negatively answer this question, as well as noting some new properties of e-locally fine speces. These methods also enable us to exhibit an RE space which is not an inverse limit of fine spaces, thus answering a question raised in [CI]. We remark that the second method is based on the procedure used by the first author in $[P]_2$ to establish that each locally fine space is subfine.

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We recall the following results from $[R]_1$ and $[R]_2$. Each e-locally fine space is an RE space and (clearly) each subspace of an e-locally fine space is e-locally fine. The metric-fine spaces are precisely the e-locally fine spaces which have the inversion property. Each e-locally fine space with a point-finite basis has a σ-disjoint basis.(a result which fails for general spaces see $[P]_3$). Finally, the e-locally fine operator m_o and the countable operator e commute: $m_o eu = em_o u$ for each uniformity u.

<u>Proposition 1:</u> X is e-locally fine if and only if each metric-valued mapping that is uniformly continuous on each member of some countable uniform cover is uniformly continuous.

Proposition 1 follows from the following result, which may be established by the proof technique found in [PPV]: if $\mathcal{U} \in u/eu$, there exists a countable uniform cover $\{A_n\}$ and a metric-valued mapping $f:X \to \langle M,d \rangle$ such that $f_{|A_n}$ is uniformly continuous, $n = 1, 2, \ldots$ and $f^{-1} \mathcal{S}_d(1) < \mathcal{U}$.

<u>Proposition 2:</u> If X is e-locally fine and has a point-finite basis, then each uniform cover may be sub-ordinated by an ℓ_1 - uniformly continuous partition of unity.

If $\mathcal{V}$ is a point-finite uniform cover, without loss of generality ([I],7.3) we may assume that there exists a uniform cover $\mathcal{W}$ such that each $W \in \mathcal{W}$ intersects only finitely many members from $\mathcal{V}$. By ([I], IV.10) there exists an equiuniformly continuous partition of unity $\{f_V : V \in \mathcal{V}\}$ sub-ordinate to $\mathcal{V}$ which generates the mapping $f: X \to \ell_1(|V|)$ defined by $f(x) = (f_V(x))$. To show that f is uniformly continuous, define $A_n = \{x: x$ belongs to at most n members of $V\}$, $n = 1, 2, \ldots$ Then $\mathcal{W} < \{A_n\}$, so $\{A_n\}$ is a countable uniform cover. Also, each $f_{|A_n}$ is uniformly continuous (given $\epsilon > 0$, let $\mathcal{U}_n$ denote the uniform cover $\wedge \{f_V^{-1} S_{||}(\epsilon/2n): V \in \mathcal{V}\}$; then $\mathcal{U}_{n|A_n} < (f_{|A_n})^{-1} \mathcal{S}_{\|\cdot\|}(\epsilon)$). Since X is e-locally fine, it follows from Proposition 1 that f is uniformly continuous.

A uniform space X has the <u>module property</u> ($[Fr]_2$) if $U(X,B)$ is a $U(X)$ - module for each normed space B (here $U(X,Y)$ denotes the family of uniform mappings and $U(X) = U(X,\mathbb{R})$). We can now state the following result.

<u>Proposition 3:</u> Each e-locally fine space has the module property. Each space with a finite dimensional basis which hereditarily possesses the module property is e-locally fine.

To prove the first statement (which is noted in [V], p.35), assume $f \in U(X)$ and $g \in U(X,B)$, where $(B,\|\cdot\|)$ is a normed space. Then $(f\cdot g)_{|A_{m,n}}$ is uniformly continuous for each member of the countable uniform cover $\{A_{m,n} | m,n = 1,2,\ldots\}$, where $A_{m,n} = f^{-1} S_{|\cdot|}(0,n) \cap g^{-1} S_{\|\cdot\|}(0,m)$. The second statement follows from a characterization of the hereditary module p operty r cently discovered by J. Vilímovský (see [V], Theorem 6.2).

It is an unsolved problem whether Proposition 3 is valid without the assumption of a finite dimensional basis.

We now turn our attention to the counterexamples.

Example 1: Let X be the complete space from $[P]_1$ such that eX is not complete. Then $m_\circ X$ has no point-finite basis, so it is not sub-metric-fine.

If $m_\circ X$ has a point-finite basis, then by $[R]_2$, $em_\circ X = m_\circ eX$ is complete, which implies that eX is complete. Since each inverse limit of fine spaces has a point-finite basis, $m_\circ X$ is an RE space which cannot be represented as such a limit. This negatively answers the question raised in [CI].

Before constructing the second example, we need the following transfinite construction of the e-locally fine modification of a uniformity u. Inductively define (for $\alpha < \omega_1$) $v^{(0)} = u$, $v^{(1)} = u/eu$, $v^{(2)} = v^{(1)}/eu, \ldots, v^{(\alpha+1)} = v^{(\alpha)}/eu, \ldots$, with $v^{(\alpha)} = \bigcup_{\beta<\alpha} v^{(\beta)}$ for α a limit ordinal.

Proposition 4: $m_\circ u = \bigcup_{\alpha<\omega_1} v^{(\alpha)}$.

To prove Proposition 4, we will need the following auxilliary transfinite process: inductively define (for $\alpha < \omega_1$) $w^{(1)} = u/eu$,

$w^{(2)} = w^{(1)}/e(w^{(1)}), \ldots, w^{(\alpha+1)} = w^{(\alpha)}/e(w^{(\alpha)}), \ldots$ with $w^{(\alpha)} = \bigcup_{\beta<\alpha} w^{(\beta)}$ for α a limit ordinal. It is easy to establish that $m \circ u = \bigcup_{\alpha<\omega_1} w^{(\alpha)}$. We will now establish (*): for each $\alpha < \omega_1$, there exists $\hat{\alpha} < \omega_1$ such that $w^{(\alpha)} \subset v^{(\hat{\alpha})}$. To prove (*), we need the following lemmas.

<u>Lemma 1:</u> For each $\alpha < \omega_1$, $e(v^{(\alpha)}) \subset (eu)^{(\alpha+1)}$, where $(eu)^{(\alpha+1)} = (eu)^{(\alpha)}/(eu)^{(\alpha)}$, $(eu)^{(1)} = eu/eu$, and $(eu)^{(\alpha)} = \bigcup_{\beta<\alpha} (eu)^{(\beta)}$ for α a limit ordinal.

<u>Lemma 2:</u> For all $1 \le \beta \le \gamma < \omega_1$, there exists $\tau = \tau(\gamma,\beta) < {}_1$ such that $v^{(\gamma)}/(eu)^{(\beta)} \subset v^{(\tau)}$.

Lemmas 1 and 2 are proved by induction using the basic facts that first, the operation / is associative, and second, $e(u/v) \subset eu/v$ for all families u and v.

We can now prove (*) by induction. For $\alpha = 1$, let $\hat{\alpha} = 1$. Assume that $w^{(\beta)} \subset v^{(\hat{\beta})}$ for $\beta < \alpha$. If α is a limit ordinal, let $\hat{\alpha} = \sup\{\hat{\beta} : \beta < \alpha\}$. If $\alpha = \beta + 1$, set $\hat{\alpha} = \tau(\hat{\beta}+1,\hat{\beta}+1)$; then $w^{(\alpha)} = w^{(\beta)}/e(w^{(\beta)}) \subset v^{(\hat{\beta})}/e(v^{(\hat{\beta})})$ and $v^{(\hat{\beta})}/e(v^{(\hat{\beta})}) \subset v^{(\hat{\beta}+1)}/(eu)^{(\hat{\beta}+1)} \subset v^{(\hat{\alpha})}$ using Lemmas 1 and 2, which completes the proof.

Recall that a partially ordered set $(T,<)$ is called a <u>tree</u> if for each $x \in T$, the set $\hat{x} = \{y \in T : y > x\}$ is well-ordered by $<$. We will add the additional restrictions that a tree has a largest element 0 and each maximal chain is finite. For such a tree T, inductively define $T^{(1)} = \{x \in T$: Chains in T

originating at x have unbounded length$\}, \ldots, T^{(\alpha+1)} = (T^{(\alpha)})^{(1)}, \ldots$, and $T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)}$ if α is a limit ordinal. We define the <u>complexity of T</u> = comp T = $\inf\{\alpha : T^{(\alpha)} = \emptyset\}$ Furthermore, let $e(T) = \{x \in T : \nexists p < x\}$ (endpoints of T) and if $p \in T$, define $T(p) = \{x \in T : x < p$ and $\nexists y : x < y < p\}$. We say that T is an α-tree if $|T(p)| \leq \alpha$ for each $p \in T$. Now inductively define $\ell^{(1)}(T) = T - e(T), \ldots, \ell^{(\alpha+1)}(T) = \ell^{(\alpha)}(T) - e(\ell^{(\alpha)}(T)), \ldots$, and $\ell^{(\alpha)}(T) = \bigcap_{\beta<\alpha} \ell^{(\beta)}(T)$ if α is a limit ordinal. Then the <u>length complexity of T</u> = lengthcomp T = inf $\{\alpha : \ell^{(\alpha)}(T) = \emptyset\}$. The following assertion illustrates the relationship between the complexity and length complexity of a tree T.

<u>Proposition 5:</u>

(i) Both lengthcomp T and comp T are non-limit ordinals.

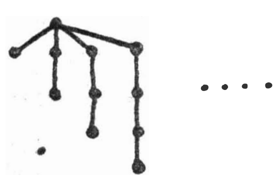
(ii) lengthcomp T $\geq$ comp T.

(iii) If α is an initial uncountable ordinal, then lengthcomp $T > \alpha$ implies comp $T > \alpha$.

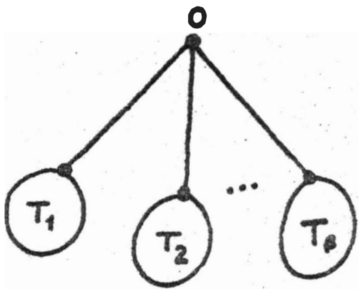
(iv) $\ell^{(\alpha\cdot w)}(T) \subset T^{(\alpha)}$ for all α.

(v) If T is an α-tree, then lengthcomp $T < \alpha^+$.

We comment that for any ordinal τ, there exists a tree $(T,<)$ of our special type with comp $T \geq \tau$. For $\tau = 1$, let T have the form pictured below - then comp T = 2.

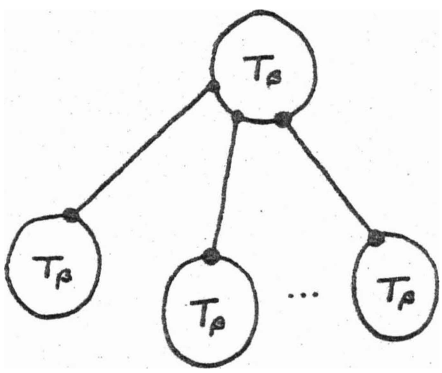


ssume that for each $\beta < \alpha$ there exists a tree T_β with omp $T_\beta \geq \beta$. If α is a limit ordinal, construct the tree T_α ndicated in the following diagram:



, $\beta < \alpha$.

here the largest element in each T_β precedes 0. Clearly comp $T_\alpha \geq \alpha$. If $\alpha = \beta + 1$, construct the tree T_α indicated in he diagram below, where one lets each endpoint of T_β act



as the largest element in a copy of T_β. Then comp $T_\alpha \geq 2\beta \geq \alpha$ (o see this, note that $T_\alpha^{(\beta)}$ is a copy of the tree T_β).

We will also use the following notation for a tree $(T,<)$ of our special type. Define $S_0 = \{0\}$, $S_1 = \{x \in T : x < 0$ and $\nexists y : x < y < 0\}, \ldots, S_{n+1} = \{x \in T : x < a, \exists a \in S_n$ and $\nexists y : x < y < a\}, \ldots, n = 1,2,\ldots$ By assumption $T = \bigcup_{n=0}^{\infty} S_n$. Define a metric uniformity on T using the family of covers $\mathcal{U}_n = \{\{x\} : x \in \bigcup_{k=0}^{n-1} S_k\} \cup \{(\leftarrow,p] : p \in S_n\}$, $n = 1,2,\ldots$, where $(\leftarrow,p] = \{x \in T : x \leq p\}$. The metric is complete since every maximal chain is finite. Whenever a tree T of our special type is considered, we assume that it is equipped with the complete metric uniformity described above.

<u>Example 2:</u> Let $(T,<)$ be a tree of special type with comp $T \geq \omega_1$. Then $m_0 T$ is a zero-dimensional e-locally fine space that is not sub-metric-fine.

Since the smallest sub-metric-fine uniformity containing a complete metric uniformity is fine, it suffices to show that $m_0 T$ does not contain all covers of T (for T is topologically discrete). We will present two different methods for showing this fact. The first method is based on the following lemma.

<u>Lemma 3:</u> For each $\alpha < \omega_1$ and for all $p \in T^{(\alpha+1)}$, each $\mathcal{V} \in v^{(\alpha)}$ contains a member V and $x \in V$ such that (i) $x < p$, (ii) $(\leftarrow,x] \subset V$, and $|(\leftarrow,x]| \geq 2$.

We will prove the lemma by induction. Let $\alpha = 0$ and $p \in T^{(1)}$. Suppose $p \in S_k$; if $\mathcal{V} \in v^{(0)} = u$, then without loss of generality we may assume that $\mathcal{V} = \mathcal{U}_r$ for some $r > k$. Since $p \in T^{(1)}$,

$\exists p_2 < p_1 < p$, $p_2 \in S_{r+1}$, $p_1 \in S_r$. Then $V = (\leftarrow, p_1] \in \mathscr{V}$ and $x = p_1$ satisfy conditions (i)-(iii). Assume that the assertion is true for each $\beta < \alpha$. If α is a limit ordinal, then the assertion is easily established. If $\alpha = \beta + 1$, choose $p \in T^{(\alpha+1)}$ and suppose $p \in S_k$. If $\mathscr{V} \in v^{(\alpha)}$, then without loss of generality we may assume that $\mathscr{V} = \{A_n \cap V : V \in \mathscr{V}_{k(n)}\}$, where $\{A_n\} \in eu$ is based on $\mathscr{U}_r$, $r > k$, and each $\mathscr{V}_{k(n)} \in v^{(\beta)}$ (that is, $S_r = \bigcup_{n=1}^{\infty} S_{r,n}$, and for each n, there exists $j(n)$ such that $A_{j(n)} = \bigcup\{(\leftarrow, p] : p \in S_{r,n}\}$). Since $p \in T^{(\alpha+1)}$, there exists $p_1 < p$ belonging to $T^{(\alpha)} \cap S_r$. Suppose $p_1 \in S_{r,n}$. By the induction assumption, there exists $x < p_1$ and $V \in \mathscr{V}_{k(j(n))}$ such that $|(\leftarrow, x]| \geq 2$ and $(\leftarrow, x] \subset V$. Thus $(\leftarrow, x] \subset A_{j(n)} \cap V$, so the proof is complete.

To prove that $m_\circ T$ is not fine (where comp $T \geq \omega_1$), we now use Proposition 4. If the countable cover $\mathscr{U} = \{S_0, S_1, \ldots\} \in m_\circ T$, then $\mathscr{U} \in v^{(\alpha)}$ for some $\alpha < \omega_1$, there exists $p \in T^{(\alpha+1)}$; then by Lemma 3 some member of $\mathscr{U}$ must contain a set of the form $(\leftarrow, x]$, with $|(\leftarrow, x]| \geq 2$, which is impossible.

Our second proof of example 2 is based on the lemma given below, which uses the following notation. Given a tree S of our special type and a uniform space X, let $f : S \to \mathscr{P}(X)$ be a mapping such that for each $p \in S - e(S)$, the family $\{f(q) : q \in S(p)\}$ is a uniform cover of X. Define $[S,f] = \{\cap\{f(p) : p \in C\} : C \subset S$ is a maximal chain$\}$. By induction define R: Ordinals $\to$ Ordinals by $R(0) = R(1) = 0$, $R(2) = 3, \ldots, R(\alpha) = \sup\{R(\beta) : \beta < \alpha\} + 2, \ldots$

<u>Lemma 4:</u> (i) For each uniform space uX, $\{[S,f] : S$ is an ω_o-tree, f as above$\}$ is a basis for λeu.

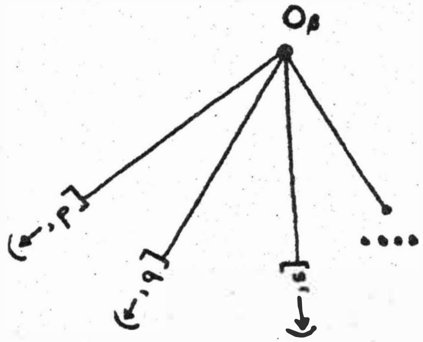
(ii) Let S and T be trees of our special type with lengthcomp $S \leq \alpha$ and comp $T \geq R(\alpha)$. If $f : S \to \mathcal{P}(T)$ is a mapping of the above type, then there exists $p \in T^{(1)}$ such that $(\leftarrow,p] \subset U$, for some $U \in [S,f]$.

The proof of part (i) is similar to the proof of an analogous result found in $[P]_2$. We will prove part (ii) by induction. The result is clear for $\alpha = 0$ or 1. If $\alpha = 2$, then $[S,f]$ is a uniform cover of T. Suppose $\mathcal{U}_n < [S,f]$. Since comp $T \geq 3$, there exists $p \in T^{(1)}$ such that $(\leftarrow,p] \subset U$, for some $U \in \mathcal{U}_n$. Now assume that the assertion has been proved for each $\beta < \alpha$. Given a mapping $f : S \to \mathcal{P}(T)$ of the above form, $\{f(s) : s \in S(0)\}$ is a uniform cover of T. Since comp $T \geq R(\alpha)$, there exists $q \in T$ ($\hat{\alpha} = \sup \{R(\beta) : \beta < \alpha\}$) and $s \in S(0)$ such that $(\leftarrow,q] \subset f(s)$. Now lengthcomp $(\leftarrow,s] < \alpha$ and comp $(\leftarrow,q] \geq \hat{\alpha}$, so by the induction assumption there exists $p \in T^{(1)}$ such that $(\leftarrow,p] \subset U$, for some $U \in [S,f]$.

Now to prove in example 2 that m_oT is not fine, assume $\mathcal{U} = \{S_0, S_1, \ldots\} \in m_oT$. Then by Lemma 4 (ii), $\mathcal{U} \in em_oT = \lambda eT$ implies that $[S,f] < \mathcal{U}$ for some ω_o-tree S and f of the above form. Then lengthcomp $S = \alpha < \omega_1$ (Proposition 5(v)) and comp $T \geq \omega_1 > R(\alpha)$, so by Lemma 4 (ii) some member of $\mathcal{U}$ contains a set of the form $(\leftarrow,p]$, $p \in T^{(1)}$, which is impossible.

Proposition 6: Assume that T is a tree of special type with comp T $< \omega_1$. Then $m_\circ T$ is the fine uniformity on T.

Assume comp T = 1; then for some n, $\mathcal{U}_n$ consists of singleton sets, so the metric uniformity is fine. Now assume that the statement is true for all trees with complexity $< \alpha$ and assume comp T = α. By Proposition 5(i), $\alpha = \delta + 1$, for some δ, so there exists n such that the length of each chain in $T^{(\delta)}$ does not exceed n. Then each tree $(\leftarrow,p]$, $p \in S_n$, has complexity $< \alpha$. Now for each $\beta < \alpha$, consider the tree S_β constructed in the following manner:



where we use every predecessor set $(\leftarrow,p]$ with complexity = β. Then comp $S_\beta = \beta$, so the induction hypothesis implies that the cover $\mathcal{P}_\beta$ of S_β consisting of singleton sets is a member of $m_\circ(S_\beta)$. Also, $\{P_\beta = \bigcup\{(\leftarrow,p] : p \in S_n, \text{comp }(\leftarrow,p] = \beta\} : \beta < \alpha\}$ is a countable uniform cover of $\bigcup_{i \geq n} S_i$, so the cover $\mathcal{V}$ formed by the restriction of $\mathcal{P}_\beta - (0_\beta)$ to each P_β consists of singleton sets and is a member of $m_\circ(\bigcup_{i \geq n} S_i)$ (for $m_\circ u/eu = m_\circ u$); hence $\mathcal{V} \wedge \mathcal{U}_n$, the cover of T consisting of singleton sets, is a member of $m_\circ T$.

dded in proof (24th Jan. 1978):

) The problem of whether each space which hereditarily possesses he module property is e-locally fine is connected with other questions:

) Does the distal modification d (distal space = space with a finite dimensional base) preserve Cauchy filters in the class of spaces with a point-finite base? (It was proved by the first author that the answer is negative if we consider all uniform spaces instead of those with a point-finite base.)

1´) Do $d(c_0(\omega))$ and $c_0(\omega)$ have the same collection of Cauchy filters? (As shown by G. Reynolds and the second author, this question is equivalent to the question 1 provided that there is no measurable cardinal.)

2) Is the mapping id: $\lambda d(c_0(\omega)) \longrightarrow c_0(\omega)$ uniformly continuous? (The affirmative answer implies the affirmative answer to 1´)

3) Is the mapping id: $\lambda d(c_0(\omega)) \longrightarrow pt_f(c_0(\omega))$ uniformly continuous? The first author conjectures that the answers to questions 0, 1, 1´ and 2 are negative.

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