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CONTINUOUS EXTENDERS FOR PSEUDOMETRICS

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This fourth lecture concerns results appearing in a joint paper by Teodor Przymusinski and myself [PL].

Recall that a space X is collectionwise normal if, given any discrete collection $\mathcal{F}$ of closed subsets of X, there is a corresponding disjoint collection $\mathcal{V} = \{V(F) : F \in \mathcal{F}\}$ of open sets in X having $F \subset V(F)$ for each $F \in \mathcal{F}$. This notion was introduced by Bing [Bi] in 1951 in his study of metrizability and normality in Moore spaces, questions which today are not completely settled and which make logic feel like a branch of applied mathematics.

But there are other uses of collectionwise normality. For a moment, consider the class of completely regular spaces. Completely regular spaces are:

(1) those spaces determined by families of continuous real--valued functions;

(2) those spaces determined by families of continuous pseudometrics.

If the space X is completely regular, then so is each (closed) subspace A and it is important to know what relationship, if any, exists between the continuous functions (respectively, continuous pseudometrics) which determine the topology of A and the continuous functions (respectively, continuous pseudometrics) on all of X. And that is where normality and collectionwise normality become important, as the next two theorems show. But first I must settle on some notation. For any space Y, C(Y) and P(Y) are the sets of continuous real-valued functions and continuous pseudometrics defined on Y, and the sets of bounded members of C(Y) and P(Y) are denoted by $C^*(Y)$ and $P^*(Y)$ respectively. I can now state the basic theorems about extending functions and pseudometrics as follows:

A. Theorem: A space X is normal if and only if for each closed $A \subset X$, each member of C(A) is the restrictions of some member of C(X). □

That result dates from the 1920's and is due to Urysohn. Extending members of P(A) to members of P(X) is more difficult:

B. Theorem: A space X is collectionwise normal if and only if for each closed $A \subset X$, each member of $P(A)$ is the restriction of some member of $P(X)$. $\square$

It is hard to say whose result Theorem B is. Half of it is usually ascribed to Arens [Ar], but a closely related result is implicit in a simultaneous paper by Dowker [Do]; Dowker's result was noted by E. Michael in his review of Dowker's paper, [M]. The other half of the theorem is usually ascribed to Shapiro [S] and Gantner [G].

There is a result midway between extending real-valued functions and extending pseudometrics. It is due to Alo and Sennott [AS], but the proof I'll give here is due to Pol (see [P, §3]) and neatly introduces many of the tricks of this trade. Also, I need it later.

C. Theorem: A T_1 space X is collectionwise normal if and only if every continuous mapping $f : A \to B$, where A is a closed subset of X and B is any Banach space, can be extended to a continuous mapping $F : X \to B$.

Proof: Suppose X is collectionwise normal and that $f : A \to B$ is given. Define a pseudometric ρ on A by $\rho(x,y) = \|f(x) - f(y)\|$, where $\|\cdot\|$ is the norm in B. According to Theorem B, there must be a continuous pseudometric $\tilde{\rho}$ on X which extendes ρ. Let $\mathcal{R}$ be the topology induced by $\tilde{\rho}$ on X, and let $C = cl_{\mathcal{R}}(A)$. Suppose $x \in C-A$; then there must be a sequence $a(n) \in A$ having $\lim \tilde{\rho}(x,a(n)) = 0$. But then, in the Banach space B, the sequence $\langle f(a(n)) \rangle$ is Cauchy so that $\bar{f}(x) = \lim_n f(a(n))$ exists and is independent of the choice of the approximating sequence $\langle a(n) \rangle$.

Defining $\bar{f}(a) = f(a)$ for each $a \in A$, we obtain a continuous function $\bar{f} : (C, \mathcal{R}_C) \to B$. According to the pseudometric generalization of the Dugundji Extension Theorem (for Banach-space-valued functions) [Du], there is a function $F : (X, \mathcal{R}) \to B$ which extends $\bar{f}$.

Conversely, suppose each Banach-space-valued continuous function on a closed subspace of X can be extended continuously over X. To prove that X is collectionwise normal, I again use Theorem B, so assume ρ is a continuous pseudometric defined on a closed subspace A of X. Then, identifying points of A at ρ-distance 0, I obtain a metric space (M, ρ) and a natural projection $\phi : A \to M$. Since M is metrizable, M can be isometrically embedded in a Banach space B so that there must be a continuous $F : X \to B$ which extends f. Let $||\cdot||$ denote the norm of B and define $\tilde{\rho}$ on $X \times X$ by $\tilde{\rho}(x,y) = ||F(x) - F(y)||$. Then $\tilde{\rho}$ is the required extension of ρ. □

In a earlier lecture [L] you heard me talk about sharpening Urysohn's result (Theorem A, above) to produce a function $e : C^*(A) \longrightarrow C^*(X)$, called an <u>extender</u>, satisfying

(1) if $f \in C^*(A)$ then $e(f)$ extends f; and

(2) e is continuous when both function spaces carry the sup-norm topology.

The new material to be presented today concerns similar sharpenings of Theorems B and C. I'll start with a special case of Theorem C, but first I'll introduce some notation. For any space Y and any Banach space B, $C(Y,B)$ denotes the set of all continuous $f : Y \longrightarrow B$, and $C^*(Y,B)$ is the set consisting of all <u>bounded</u> members of $C(Y,B)$, i.e., functions $f : Y \longrightarrow B$ such that the set $f[Y]$ has finite diameter with respect to the norm $||\cdot||$ of B. The set $C^*(Y,B)$ is itself a linear space (as is the larger set $C(Y,B)$) and becomes a Banach space if we define the <u>sup-norm</u> by

$$||f|| = \sup\{||f(y)|| : y \in Y\}$$

for each $f \in C^*(Y,B)$. (The obvious notational abuse could cause endless confusion later on, but to be notationally precise is often messy. Just to illustrate the real price of precision, later today I'll introduce norms $||\cdot||_i$, $1 \le i \le 5$, on five different spaces appearing in the same proof.) I can now present

D*. <u>Theorem:</u> Let A be a closed subset of a collectionwise normal space X, and let B be any Banach space. Then there is a continuous extender $e : C^*(A,B) \longrightarrow C^*(X,B)$ such that for each $f \in C^*(A,B)$, $||e(f)|| = ||f||$.

<u>Proof:</u> As in my third lecture [L], I will use the <u>Bartle--Graves Theorem</u> which asserts that if $R : E \longrightarrow F$ is a continuous linear surjection between Banach spaces, then there is a continuous $e : F \longrightarrow E$ such that $e(y) \in R^{-1}\{y\}$ for each $y \in F$ [BG]. According to Theorem C, the restriction operator $R : C^*(X,B) \longrightarrow C^*(A,B)$ is a surjection, and R is obviously continuous and linear. Hence there is a continuous $\hat{e} : C^*(A,B) \longrightarrow C^*(X,B)$ which is an extender. Of course, this $\hat{e}$ may fail to be norm-preserving and in that case we modify $\hat{e}$ by defining, for $f \in C^*(A,B)$ and $x \in X$,

$$e(f)(x) = \begin{cases} \hat{e}(f)(x) & \text{if } ||\hat{e}(f)(x)|| \le ||f|| \\ \dfrac{\hat{e}(f)(x)}{||\hat{e}(f)(x)||}\,||f|| & \text{if } ||\hat{e}(f)(x)|| > ||f|| . \end{cases}$$

Note that $||\hat{e}(f)(x)||$ is the norm of an element of B, while $||f$ is the sup-norm of an element of $C^*(A,B)$. □

Now Theorem C can be sharpened. For any space Y and any Bana space B let $C(Y,B)$ carry the topology of uniform convergence. Then $C^*(Y,B)$, with the sup-norm topology, is a closed linear subsp ce of $C(Y,B)$ and $C(Y,B)$ is metrizable (although $C(Y,B)$ is usu ly not a topological vector space).

D. Theorem: Let A be a closed subset of a collectionwise no mal space X and let B be any Banach space. Then there is a conti nuous extender $\tilde{e} : C(A,B) \longrightarrow C(X,B)$.

Proof: Let $F \subset C(A,B)$ be chosen so that the collection $\{f + C^*(A,B) : f \in F\}$ is exactly the family of cosets of the linear subspace $C^*(A,B)$ in the vector space $C(A,B)$. According to Theorem C, I may choose an extension $\hat{f} \in C(X,B)$ for each $f \in F$. According to Theorem D* there is a continuous extender $e : C^*(A,B) \longrightarrow C(X,B)$. Now define $\tilde{e} : C(A,B) \longrightarrow C(X,B)$ by the rule that, for $g \in C(A,B)$

$\tilde{e}(g) = \hat{f} + e(g-f)$ where f is the unique member of F having $g \in f + C^*(A,B)$. Because $C^*(A,B)$ is an open and closed subspace of $C(A,B)$, this $\tilde{e}$ is continuous. □

Now let me try to sharpen Theorem B. First I must describe the topology on the set $P^*(Y)$. Each $\rho \in P^*(Y)$ is, of course, a member of $C^*(Y \times Y)$, and we topologize $P^*(Y)$ as a subspace of $C^*(Y^2)$. It is easily seen that $P^*(Y)$ is a closed, convex subset of $C^*(Y^2)$, indeed even a cone in $C^*(Y^2)$ in the sense of [KN], but $P^*(Y)$ is not a linear subspace of $C^*(Y^2)$. That unfortunate fact makes a Bartle-Graves proof of the next theorem impossible, at least using known tools.

E*. Theorem: Let A be a closed subspace of the collectionwise normal space X. Then there is a continuous extender $E : P^*(A) \longrightarrow P^*(X)$ such that for each $\rho \in P^*(A)$, $||E(\rho)|| = ||\rho||$.

Proof: The proof requires a sequence of lemmas which are not difficult if only one can keep straight which norm refers to whic Banach space. To that end, norms will be subscripted, $||\cdot||_i$ $(1 \le i \le 5)$. For example, in the statement of the theorem, $||E(\rho)||$ and $||\rho||$ are really norms in different spaces.

(E 1) Lemma: There is a continuous mapping (indeed, an isometry) $L : P^*(A) \longrightarrow C^*(A, C^*(A))$ defined by the rule that if $\rho \in P^*(A)$ and if $x,y \in A$ then $(L(\rho)(x))(y) = \rho(x,y)$.

Proof: Since ρ is bounded and continuous, the real-valued function $L(\rho)(x)$ defined on the space A is continuous and bounded, as is the function $L(\rho) : A \to C^*(A)$. Thus L is well-defined. Let $\|\cdot\|_1$ denote the sup-norm in $C^*(A)$ and let $\|\cdot\|_2$ denote the sup-norm in $C^*(A,C^*(A))$ defined from $\|\cdot\|_1$. Let $\|\cdot\|_3$ be the sup-norm in $P^*(A)$. To prove that L is an isometric embedding, let $\rho, \delta \in P(A)$. Then

$$\|L(\rho) - L(\delta)\|_2 = \sup \{\|L(\rho)(a) - L(\delta)(a)\|_1 : a \in A\} =$$

$$= \sup \{\sup\{|L(\rho)(a)(b) - L(\delta)(a)(b)| \quad b \in A\} : a \in A\} =$$

$$= \sup \{|\rho(a,b) - \delta(a,b)| : a,b \in A\} = \|\rho - \delta\|_3 . \square$$

Now by Theorem D^*, because $C^*(A)$ is a Banach space, there is a continuous extender $e : C^*(A,C^*(A)) \to C^*(X,C^*(A))$. Continuing the painful notation of Lemma (E 1), we let $\|\cdot\|_4$ denote the sup-norm in $C^*(X,C^*(A))$ defined by the sup-norm $\|\cdot\|_1$ on $C^*(A)$.

(E 2) <u>Lemma:</u> For each $f \in C^*(X,C^*(A))$ and each $x,y \in X$ define $S(f)(x,y) = \|f(x) - f(y)\|_1$. Then $S(f)$ is a bounded, continuous pseudometric on X and the function $S : C^*(X,C^*(A)) \to P^*(X)$ is continuous.

<u>Proof:</u> For a fixed f, $S(f)$ is clearly a continuous bounded pseudometric on X. To prove that S is continuous we show that if $f,g \in C^*(X,C^*(A))$ then $\|S(f) - S(g)\|_5 \le 2\|f-g\|_4$, where $\|\cdot\|_5$ is the sup-norm on $P^*(X)$. We have

$$\|S(f) - S(g)\|_5 = \sup \{|S(f)(x,y) - S(g)(x,y)| : (x,y) \in X^2\} .$$

But $|S(f)(x,y) - S(g)(x,y)| =$

$$= | \|f(x) - f(y)\|_1 - \|g(x) - g(y)\|_1 | \le$$

$$\le \|f(x) - f(y) - g(x) + g(y)\|_1 \le$$

$$< \|f(x) - g(x)\|_1 + \|f(y) - g(y)\|_1 \le 2\|f - g\|_4$$

where the last inequality follows from the definition of $\|\cdot\|_4$, while the preceding two inequalities are consequences of the triangle inequality for the norm $\|\cdot\|_1$. $\square$

(E 3) <u>Lemma:</u> With L, e and S as above, the composite function $E = S \circ e \circ L \quad P^*(A) \to P^*(X)$ is a continuous extender.

Proof: Obviously E is continuous; we sho it as an extender. To that end, let $a,b \in A$ and let $\rho \in P^*(A)$. Then, from definitions, and with notation as above,

$E(\rho)(a,b) = S(e(L(\rho)))(a,b) =$

$= \|e(L(\rho)(a) - e(L(\quad (b\|_1$. But

$L(\rho)\in C^*(A,C^*(A))$ and e is an extender so that $e(L(\rho))(a) = L(\rho)(a)$ and $e(L(\rho))(b) = L(\rho)(b)$. Hence, from definitions,

$$\tilde{E}(\rho)(a,b) = ||L(\rho)(a) - L(\rho)(b)||_1 =$$
$$= \sup\{||L(\rho)(a)(z) - L(\rho)(b)(z)|| : z\in A\}$$
$$= \sup\{|\rho(a,z) - \rho(b,z)| : z\in A\} = \rho(a,b)$$

the last equality following from the triangle law for the pseudometric ρ plus the fact that $b\in A$. □

It is not guaranteed that $||\tilde{E}(\rho)||_5 = ||\rho||_3$, as required by the theorem, where $||\cdot||_5$ denotes the norm of $P^*(X)$. However $\tilde{E}$ can be modified as follows: for $\rho\in P^*(A)$ and for $x,y\in X$, let $E(\rho)(x,y) = \min\{\tilde{E}(\rho)(x,y), ||\rho||_3\}$. Then $E : P^*(A)\longrightarrow P^*(X)$ is a continuous extender and $||E(\rho)||_5 = ||\rho||_3$ for each $\rho\in P^*(A)$, as asserted by Theorem E^*. □

Having proved Theorem E^*, I should present a Theorem E asserting that there is a continuous extender from $P(A)$ to $P(X)$ where the latter two sets are equipped with the topology of uniform convergence inherited from $C(A^2)$ and $C(X^2)$ respectively. Unfortunately I do not have such a theorem. A moment's reflection will convice you that the coset approach used to obtain Theorem D from Theorem D^* has no hope of working. Currently I tend to favor a selection-theoretic approach to a possible Theorem E. For each
$\delta\in P(A)$, let $E(\delta) = \{\Delta\in P(X) \mid \Delta$ extends $\delta\}$.
It is easy to see that $E(\delta)$ is a closed convex subset of $P(X)$ and $C(X^2)$. However there are two problems:

(1) Most selection theorems would require that $E(\delta)$ be a subset of a Banach space, while $C(X^2)$ is not even a topological vector space (except in the unlikely event that X^2 is pseudocompact).

(2) Most currently existing selection theorems treat only those situations where the correspondence $\delta\longrightarrow E(\delta)$ is lower semi-continuous. This also presents difficulties.

Indeed, more can be said about selection-theoretic proofs, and about that second problem even if only <u>bounded</u> pseudometrics are considered. For each $\delta\in P^*(A)$, define a subset $F(\delta)$ of $P^*(X)$ by $F(\delta) = \{\Delta\in P^*(X) \mid \Delta$ extends $\delta\}$. Pryzmusinski has discovered a topological proof that F is lower-semi-continuous, but it is at least as hard as the above proof of Theorem E^*. There ought to be an easier functional-analytic proof for lower-semi-continuity of F.

To see the possible shape of such a proof, consider the selection theoretic proof of the Bartle-Graves theorem: take a continuous linear surjection $r : X \longrightarrow Y$ where X and Y are Banach spaces and consider the set-valued function (=carrier) R given by $R(y) = r^{-1}\{y\}$ for each $y \in Y$. Lower-semi-continuity of R is exactly equivalent to the fact that r is an open mapping and that is exactly the classical Open Mapping Theorem. (The continuous selection $e : Y \longrightarrow X$ for R is exactly the function required in the Bartle-Graves theorem.) Such reflections raise a natural question: is there an open mapping theorem for affine surjections between closed cones in Banach spaces ? The desired application, of course, is to the restriction operator $r : P^*(X) \longrightarrow P^*(A)$: one wants to say that if U is a relatively open subset of $P^*(X)$ then $r[U]$ is a relatively open subset of $P^*(A)$ and that would be enough to guarantee that the carrier F, defined above, is lower-semi-continuous.

Let me make two comments in closing. First, the paper [Pl] contains more technical results than the ones mentioned today - for example, pseudometrics of weight λ and Banach spaces of weight λ are considered instead of arbitrary pseudometrics and Banach spaces. Second, let me mention an untouched area for future research. Neither Przymusinski nor I have studied it seriously, beyond determining that a lot of people who should know about any results in the area don't know them. We conclude that the area is a virging one. Here is the general question: Find conditions on X so that for each closed $A \subset X$, there is a continuous <u>affine</u> extender $E : P^*(A) \longrightarrow P^*(X)$ (i.e., if $\delta, \rho \in P^*(A)$ and if $s,t \in [0,1]$ have $s + t = 1$, then $E(s\delta + t\rho) = sE(\rho) + tE(\delta))$ such that for each $\rho \in P^*(A)$, $||E(\ )|| = ||\rho||$. A paper obtaining positive results in that area would be aptly titled "Dugundji Extension Theory for Pseudometrics".

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