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Distinguishable sets

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For each uniform space X denote by distg (X) the set of all subsets of X which are distinguished from the complement by a uniformly continuous mapping of X onto a metric space. Denote by distg (X,X) the set of all mappings of X into Y such that the preimages of distinguishable sets are distinguishable. Clearly distg is a refinement of the category of uniform spaces, and

$$U \hookrightarrow coz \hookrightarrow Ba \hookrightarrow distg.$$

We shall prove

$$distg_- = (distg^2)_F = metric - set_f = her\ (distg_-) = sub\ (distg_-)$$

$$distg_f = distg_{-+} = D_c$$

Moreover, usual results about $distg_f$ and $distg_c$ are proved. One possible aim of this investigation is to determine the properties of the finest metrically determined coreflection (namely $distg_-$). The results in [1] are assumed.

§ 1. Distg spaces. The refinement distg is generated by the functor distg from uniform spaces into paved spaces. Clearly, this functor is metrically determined (i.e. the value at X is projectively generated by all f: distg X $\hookrightarrow$ distg S with f: X $\longrightarrow$ S $\in$ U and S metric), and in addition it is the finest functor with this property. Recall that the other functors already studied were coz and Ba. The refinement was introduced by the present author in [2] in connection with the study of σ-uniformly refinable families.

Proposition 1. The distg-coarse spaces are just the set-coarse spaces (i.e. singletons in separated spaces). A space X is distg-fine if and only if every completely distg (X)-additive partition is uniform.

Proof. Obvious.

Proposition 2. A paved space $\langle X, \mathcal{X} \rangle$ is a distg-space iff it is a coz-space, and if f: $\langle X, \mathcal{X} \rangle \longrightarrow \langle R, coz\ R \rangle$

is measurable, then $f: \langle X, \mathfrak{X} \rangle \longrightarrow \langle R, \exp R \rangle$ is measurable.

Proof. Obvious.

§ 2. $(\mathrm{distg})^2$-fine spaces. The main result:

Theorem 1. For any X let mX be projectively generated by all $f: mX \longrightarrow Y$ such that

$$f \times f: X \times X \longrightarrow Y \times Y$$

is a distg-mapping. Then:

a) The set of all distinguishable sets in $X \times Y$ which contains the diagonal is a basis for uniform vicinities of the diagonal of MX, and mX is projectively generated by all $f: mX \longrightarrow \mathrm{set}_f\, S$ such that $f: X \longrightarrow S$ is uniformly continuous, and S is metric.

b) $\mathrm{distg}\,(mX \times mX) = \mathrm{distg}\,(X \times X)$, particularly,

$$\mathrm{distg}\ mX = \mathrm{distg}\ X.$$

c) $f: mX \longrightarrow Y$ is uniformly continuous iff $f \times f: X \times X \longrightarrow Y \times Y$ is a distg-mapping.

d) $f: mX \longrightarrow Y$ is uniformly continuous iff $f \times f: mX \times mX \longrightarrow Y \times Y$ is a distg-mapping.

e) m is a coreflection on distg^2-fine spaces, i.e.

$$m = (\mathrm{distg}^2)_f.$$

f). If S is metric, and if $f: mX \longrightarrow S$ is uniformly continuous, then so is $f: mX \longrightarrow \mathrm{set}_f\, S$, and m is the coreflection on the spaces with this property.

Proof. Follows the lines of the proof of a similar theorem for coz.

Corollary. distg^2-fine spaces coincide with metric-set_f spaces, and $(\mathrm{distg}^2)_f$ preserves distinguishable sets.

§ 3. distg^2-fine coreflection is distg_-.

Theorem 2. $(\mathrm{distg}^2)_f = \mathrm{distg}_-$.

Proof. Let m be the functor in Theorem 1. By corollary to Theorem 1

$$m \in \text{Inv}(\text{distg})$$

and

$$m = (\text{distg}^2)_f.$$

These two relations imply $m = \text{distg}_-$ by the following simple general result:

if $\mathcal{R}$ is any refinement of U, and if $(\mathcal{R}^2)_f \in \text{Inv}(\mathcal{R})$ then $\mathcal{R}_- = (\mathcal{R}^2)_f$.

§ 4. Plus functors.

Theorem 3. $\text{distg}_+ = \text{distg}_{-+} = D_c$.

Lemma 1. $\text{distg}\, X = \text{distg}\, D_c X$.

Proof. This follows from the fact that if S is metric then there exists a bijective uniformly continuous mapping of S onto a distally coarse metric space.

Corollary: $D \in \text{Inv}(\text{dist})$, $D \in \text{Inv}(\text{distg}^2)$.

The proof of the fact that D is the coarsest functor with the properties in Corollary seems to be long and uninteresting (essentially set-theoretical).

§ 5. Remarks. The usual questions are:

a) Is $\mathcal{R}_- \neq \mathcal{R}_f$?

b) When $\mathcal{R}_- X = \mathcal{R}_f X$?

c) When $\mathcal{R}_f X \in \langle X \rangle_{\mathcal{R}}$?

The answer to the first question is yes. Take any space X such that coz X = exp X. Then $\text{distg}_f X = \text{set}_f X$. On the other hand, if X is precompact, then the uniform partitions of $\text{distg}_- X$ are of cardinal at most $\exp \aleph_0$, and hence, $\text{distg}_- X \neq \text{distg}_f X$ if the cardinal of X is greater than $\exp \aleph_0$.

As concerns the second questions, two propositions hold; the first is trivial, the second requires Tashijan Lemma.

Theorem 4. $\text{distg}\, X = \text{distg}_f X$ iff every completely distg (X)-additive partition of X is uniformly discrete in

distg_- X.

Theorem 5. Assume that $\text{distg}_- X_a = \text{set}_f X_a$ for each a. Then

$$\text{distg}_- \prod\{X_a\} = \text{distg}_f \prod\{X_a\} .$$

Proof. First observe that every distinguishable set in the product of uniform spaces depends on a countable number of coordinates. Then apply the Tashijan Lemma.

For the third question we have just a formal statement.

Theorem 6. distg_f X is distinguishably equivalent to X iff for any two completely distg (X)-additive partitions $\{X_a\}$ and $\{Y_b\}$ the partition $\{X_a \cap Y_b\}$ is.

The further questions are:
what are sub and her functors of the functors involved.

Theorem 7. sub distg_- = her distg_- = distg_-

The proof follows from

Lemma 2. If $\text{distg}_- Z = Z$ then $\text{distg}_- X = X$ for each subspace X of Z.

Proof. It is enough to show that if S is metric then for any $Y \hookrightarrow S$, $\text{distg}_- Y \hookrightarrow \text{distg}_- S$, and this is obvious because $\text{distg}_- S = \text{set}_f S$.

Theorem 8. distg_- = sub distg_f.

Proof. Apply Theorems 5 and 7 to the following situation:

Embed given X into the product $\prod\{S_a\}$ of metric spaces.

Concluding problem:

What is her distg_f ?

References:

[1] Frolík Z.: Four functors into paved spaces, Seminar Uniform Spaces 1973-74, directed by Z. Frolík.

[2] Frolík Z.: Basic refinements of uniform spaces. Proc. 2nd Pitt. Conf., Lecture Notes