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A STRENGTHENING OF THE PROXIMALLY FINE CONDITION

Michael D. Rice

Department of Mathematics
George Mason University
Fairfax, Virginia 22030

In this note a strengthening of the p-fine condition, called equi-p-fine, is investigated. This strengthening is significant since (i) each product of equi-p-fine spaces is equi-p-fine and (ii) most familiar p-fine spaces are equi-p-fine, with the notable exception of the fine spaces.

A uniform space X is equi-p-fine ($\aleph_0$-equi-p-fine) if a metric-valued family $F: X \to M$ ($|F| \leq \aleph_0$) is equi-unif. cont. whenever $h \circ F$ is equi-unif. cont. for each unif. cont. map $h: M \to [0,1]$. It is well known that if F is assumed to be a one element family, then the above condition is equivalent to the statement that X is proximally fine (p-fine) or finest in its proximity class ([RS]). Letting pM denote the precompact reflection of M, it is easily seen that X is equi-p-fine ($\aleph_0$-equi-p-fine) if and only if $F: X \to M$ ($|F| \leq \aleph_0$) is equi-unif. cont. whenever $F: X \to pM$ is equi-unif. cont. Further equivalent formulations are given in the following result.

Theorem 1: The following statements are equivalent.

(i) X is equi-p-fine (resp. $\aleph_0$-equi-p-fine).

(ii) $X \times \mathbb{D}$ is p-fine for every uniformly discrete space $\mathbb{D}$ (resp. $X \times \mathbb{N}$ is p-fine).

(iii) $X \to U(\mathbb{D}, pM)$ unif. cont. implies $X \to U(\mathbb{D}, M)$ unif. cont. for every uniformly discrete space $\mathbb{D}$ and metric space M (resp. $X \to U(\mathbb{N}, pM)$ unif. cont. implies $X \to U(\mathbb{N}, M)$ unif. cont. for each metric space M).

Comments.

(i) By ([I], VII, ex. 6) the equicharacter of a uniform space does not exceed the density character. Hence X is equi-p-fine if and only if $X \times \mathbb{D}$ is p-fine, where $\mathbb{D}$ is a uniformly discrete space of power $|X|$, and each $\aleph_0$-equi-p-fine space with separable topology is equi-p-fine.

(ii) One easily shows that the full subcategories of equi-p-fine spaces and $\aleph_0$-equi-p-fine spaces are coreflective in the category of uniform spaces (since they are closed under the formation of uniform sums and uniform quotients). In addition, each property is preserved by completion.

(iii) In [Ku], V. Kůrková-Pohlová defined the class P^* as follows: for each Y, P^*_Y is the class of spaces X such that for each proximally continuous pseudometric ρ on $X \times Y$ and $\epsilon > 0$, there exists a uniform cover $\mathcal{U}$ of X such that $\mathcal{U} \times \{\{y\}: y \in Y\} < B_\rho(\epsilon)$ (spheres of radius ϵ). Define P^* to be the intersection of the classes P^*_Y, as Y ranges over all uniform spaces. One may establish that P^* is the class of equi-p-fine spaces and that $P^*_{\aleph_0} = \cap\{P^*_Y: |Y| \leq \aleph_0\}$ is the class of $\aleph_0$-equi-p-fine spaces. Thus the results found in [Ku] complement the ones presented here.

Examples.

(i) Each metric space is equi-p-fine; in fact, each space with a totally ordered basis is equi-p-fine (for if X has a totally ordered basis and $\mathbb{D}$ is uniformly discrete, then $X \times \mathbb{D}$ has a totally ordered basis and is therefore p-fine).

(ii) Each fine space with P-space topology is $\aleph_0$-equi-p-fine (for by ([I], VII. 30) such a space admits $\aleph_0$, i.e. the meet of countably many uniform covers is uniform).

(iii) Each precompact p-fine space is equi-p-fine (by [Hu], Theorem 4).

Theorem 2: Each fine space with k-space topology is equi-p-fine.

Each fine space with k-space topology is the uniform quotient of the sum of its compact subspaces and by Example (iii each compact space is equi-p-fine, so the result is established. In particular, each fine space with first countable topology is equi-p-fine. More generally, ([Ku], II.3) establishes that each fine space with sequential topology is equi-p-fine.

Now let * denote the semi-uniform product defined in [I]. If U(-,-) denotes the usual function space operator, then U(X*Y,Z) is naturally isomorphic to U(X, U(Y,Z)) for all spaces X, Y, Z. Furthermore, * is an associative operation (for (X*Y)*Z and X*(Y*Z) have the same unif. cont. mappings to each uniform space) that is non-commutative (for example, [0,1] * $\mathbb{N}$ and $\mathbb{N}$ * [0,1] are not isomorphic).

Proposition 1: (i) If X is equi-p-fine and Y is p-fine, then X*Y is p-fine.

(ii) If X is $\aleph_0$-equi-p-fine and Y is a countable p-fine space, then X*Y is p-fine.

To establish (i), let $X * Y \xrightarrow{f} pM$ be unif. cont., where M is a metric space. By ([I], III.22), then $\{f_y\colon X \longrightarrow pM\}$ is equi-unif. cont. and each ${}_xf\colon Y \longrightarrow pM$ is unif. cont. Hence $\{f_y\colon X \longrightarrow M\}$ is equi-unif. cont. and each ${}_xf\colon Y \longrightarrow M$ is unif. cont., so once again applying ([I], III.22), it follows that $X * Y \xrightarrow{f} M$ is unif. cont., which completes the proof. The proof of (ii) is similar.

For our next result, we need the following result that may be established using ([I], III.21, 22). Let $X * Y \xrightarrow{i} X \times Y$ be the identity mapping and $Y * X \xrightarrow{h} X \times Y$ be defined by $h(y,x) = (x,y)$. A mapping $X \times Y \xrightarrow{f} Z$ is unif. cont. if and only if $f \circ i$ and $f \circ h$ are unif. cont.

Proposition 2: (i) If $X * Y$ and $Y * X$ are p-fine, then $X \times Y$ is p-fine.
(ii) $X * Y$ is equi-fine if and only if X and Y are equi-p-fine.

Part (i) follows at once from the preceding comments (or from ([Ku], I.1) since $X * Y$ p-fine implies $X \in P^*_Y$ and $X \times Y$ is p-fine if and only if $X \in P^*_Y$ and $Y \in P^*_Y$). To establish (ii), assume X and Y are equi-p-fine and $\mathbb{D}$ is uniformly discrete. Then $(X * Y) * \mathbb{D} = X * (Y * \mathbb{D}) = X * (Y \times \mathbb{D})$ is p-fine by Prop 1 (i) and Theorem 1, so $X * Y$ is equi-p-fine. Since the projection mappings $X * Y \longrightarrow X$ and $X * Y \longrightarrow Y$ have right inverses, they are quotient mappings; hence $X * Y$ equi-p-fine implies that X and Y are equi-p-fine.

<u>Proposition 3</u>: If X is $\aleph_0$-equi-p-fine and M is a separable metric space, then $X * M$ and $X \times M$ are p-fine.

Let $X * M \xrightarrow{f} pZ$ be unif. cont., where Z is a complete metric space. Let S be a countable dense subset of M. Since M is metric, $B = X * S$ is a dense uniform subspace of $X * M$ ([I], III.29). By Prop 1 (ii), $f_{|B}: B \longrightarrow Z$ is unif. cont., so it has a unique extension to a unif. cont. mapping $X * M \longrightarrow Z$, which must be f. Thus $X * M$ is p-fine. Since M is equi-p-fine, Prop 1 (i) implies $M * X$ is also p-fine, so Prop 2 (i) implies that $X \times M$ is p-fine.

It should be noted that Prop 3 may fail if M is not separable (i.e. there exist $\aleph_0$-equi-p-fine spaces that are not equi-p-fine); see example I.

<u>Theorem 3</u>: (i) Each product of equi-p-fine spaces is equi-p-fine.

(ii) Each product of an $\aleph_0$-equi-p-fine space with a product of separable metric spaces is $\aleph_0$-equi-p-fine.

By Prop 2, the equi-p-fine spaces form a class that is finitely productive. Since ([Hu], Theorem 2) a product of p-fine spaces is p-fine if and only if each finite subproduct is p-fine, part (i) is established. Part (ii) follows from the same result and Prop 3.

More generally, one may establish the following result ($[R]_2$): let S be a (full) coreflective subcategory of uniform spaces. Then S is a finitely productive class if and only if $X \in |S|$ and $\mathbb{D}$ uniformly discrete implies $X \times \mathbb{D} \in |S|$.

Corollary: Each injective space is equi-p-fine. Each injective space is a uniform retract of a product of metric spaces, so the Corollary follows from Theorem 3.

Theorem 4: (i) Assume that $X \times Y$ is p-fine, where Y is uniformly zero-dimensional. Then either Y is precompact or X is $\aleph_0$-equi-p-fine.

(ii) Assume that $X \times \alpha Y$ is p-fine, where αY is the fine space associated with a zero-dimensional metic space Y. Then either Y is compact or X is $\aleph_0$-equi-p-fine.

For (i), if Y is not precompact, there exists a unif. cont. retraction of Y onto $\mathbb{N}$; thus $X \times \mathbb{N}$ is a unif. cont. retract of $X \times Y$, so X is $\aleph_0$-equi-p-fine. Part (ii) is established in a similar manner.

Comment: I have been informed by M. Hušek that in connection with Theorem 4 he has established the following conjecture of Z. Frolik: if $X \times Y$ is p-fine for all p-fine spaces Y, then X is precompact and p-fine.

Counterexamples.

The following useful new construction is found in [Hu]. Let P be any space and $\mathcal{U}$ the family of entourages of P. Let Y be the uniformly discrete space on the set $(P \times P)$ (where Δ is the diagonal) and let X be the set $[(P \times P) - \Delta] \cup \{z\}$, $z \notin P \times P$, where the local basis for the topology at z is $\{(U \quad \Delta) \cup \{z\} : U \in \mathcal{U}\}$, and the other points in X are isolated. Let αX be the fine space associated with X. [Hu] shows that $\alpha X \times Y$ is not p-fine whenever P is not p-fine.

Example I: Let $|P| = 2^c$ and let $\mathcal{U}$ be generated by the partitions of P of power $\leq c$. Then P is not p-fine, but $(P, \mathcal{U})$ admits $\aleph_0$. Hence X is a P-space, so by example (ii) αX is $\aleph_0$- equi-p-fine. However, using the above result from [Hu], $\alpha X \times Y$ is not p-fine, so αX is not equi-p-fine.

Example II: Modify example I by letting $|P| = \aleph_0$ and $\mathcal{U}$ be generated by all finite partitions of P. Then P is not p-fine, so $\alpha X \times \mathbb{N}$ is not p-fine; hence a fine space need not be $\aleph_0$-equi-p-fine. Furthermore, since αX is uniformly zero-dimensional, it follows from Theorem 4 (or Prop 2(iii)) that $\alpha X \times \alpha X$ is not p-fine.

<u>Example III:</u> Let $X = \dot{\bigcup}_{\alpha<\Omega} \{0_\alpha, 1_\alpha\}$ and consider the uniformity on X with the basis of covers

$\mathcal{U}_\alpha = \{\{p\} : p = 0_\beta$ or $p = 1_\beta, \beta \le \alpha\} \cup \{\{0_\gamma, 1_\gamma\} : \gamma > \alpha\}, \alpha < \Omega.$

Then X admits $\aleph_0$ and has a totally ordered basis, so X is a <u>measurable</u> ($[R]_1$) and <u>(hereditarily) equi-p-fine</u> space (by example (i)) that is <u>not locally</u> <u>fine</u>.

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