Jan Pelant; Pavel Pták

On σ-discreteness in uniform spaces

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1976. pp. 115–119.

Persistent URL: http://dml.cz/dmlcz/703150

Seminar Uniform Spaces 1975-76

On σ-discreteness in uniform spaces

Jan Pelant, Pavel Pták

This paper has two parts. The first one is an investigation of the plus and minus functors associated with the refinement $\mathcal{D}^{\sigma}$ and $\mathcal{D}^{\sigma d}$. Both refinements were introduced by Z. Frolík in [F_1]. The refinement $\mathcal{D}^{\sigma}$ respects σ-discrete collections and $\mathcal{D}^{\sigma d}$ the σ-discretely decomposable ones. It is shown that $\mathcal{D}^{\sigma}_{+}$ and $\mathcal{D}^{\sigma d}_{+}$ is the distally coarse functor $\mathcal{D}_c$, $\mathcal{D}^{\sigma}_{-}$ is the identity and $\mathcal{D}^{\sigma d}_{-}$ adds the σ-discrete partitions. Further $\mathcal{D}^{\sigma 2}_{+} = \mathcal{D}^{\sigma 2}_{-} = \mathrm{Id}$, $\mathcal{D}^{\sigma d 2}_{+} = \mathcal{D}_c$ and $\mathcal{D}^{\sigma d 2}_{-} = \mathcal{D}^{\sigma d}_{-}$.

The symbols $\mathcal{D}^{\sigma 2}_{+}$, $\mathcal{D}^{\sigma 2}_{-}$, $\mathcal{D}^{\sigma d 2}_{+}$ and $\mathcal{D}^{\sigma d 2}_{-}$ are to be read as $((\mathcal{D}^{\sigma})^2)_{+}$, $((\mathcal{D}^{\sigma})^2)_{-}$, $((\mathcal{D}^{\sigma d})^2)_{+}$ and $((\mathcal{D}^{\sigma d})^2)_{-}$. As we will use the above symbols only in this sense, we shall write in the simplified form.

Finally two examples are given, the second one of principle importance for σ-discreteness (compare with [F_1]).

The second paragraph brings an example of a metric fine space which is not $\mathcal{D}^{\sigma d} \wedge$ coz fine. This question was stated by Z. Frolík in Seminar Uniform Spaces 1973-74, p. 63 (and in [F_1]).

This paper overlaps sometimes with the one [P], ibid and the reader is invited to consult [P] before.

§ 1. The refinement $\mathcal{D}^{\sigma}$ has for the morphisms the mappings $f: X \longrightarrow Y$ such that $\{f^{-1}(Y_{\alpha}) \mid \alpha \in I\}$ is σ-discrete (abbr. σ-d.) in X whenever $\{Y_{\alpha} \mid \alpha \in I\}$ is σ-d. in Y. For the definition of the refinement $\mathcal{D}^{\sigma d}$ we replace σ-d. by σ-d.d. (σ-discretely decomposable). Recall that a collection $\{X_{\alpha} \mid \alpha \in I\}$ is called σ-discretely decomposable in a space X if we may write $X_{\alpha} = \bigcup X^{n}_{\alpha}$ such that any collection $\{X^{n}_{\alpha} \mid \alpha \in I\}$ is discrete.

For the intuition, in the hedgehog H(I) on I with uncountable cardinality the "thorns" form a σ-d.d. collection but not a σ-d. one.

For the definition of the plus and minus functors consult [F_4].

Theorem 1: It holds $\mathcal{D}_+^\sigma = \mathcal{D}_c$ and $\mathcal{D}_-^\sigma = \mathrm{Id}$.

Proof: The equality $\mathcal{D}_+^\sigma = \mathcal{D}_c$ can be obtained from the Lemma 2.3 in [P] (we prove that any $F \in \mathrm{Inv}_+ \mathcal{D}^\sigma$ is identical on all hedgehogs on a sequentially regular cardinality). We shall prove that $\mathcal{D}_-^\sigma = \mathrm{Id}$, in fact, that $\mathrm{Inv}_- \mathcal{D}^\sigma = \{\mathrm{Id}\}$. Let $F \in \mathrm{Inv}_- \mathcal{D}^\sigma$ and let FX be strictly finer than X for a space X. Take a covering $\mathcal{X} \in FX - X$ and further take the set $T_{\mathcal{X}} = \{(x,y) \mid y \notin \mathrm{St}(x, \mathcal{X})\}$. Put $Y = X \times T_{\mathcal{X}}$ ($T_{\mathcal{X}}$ understood as a uniformly discrete space). Finally put $Z = Y \times \omega_1 \times \omega_1$ (ω_1 the first uncountable ordinal as u.d. space). We show that FZ has more σ-d. collections than Z.

For each $(x,y) \in T$ and for each $(\alpha, \beta) \in \omega_1 \times \omega_1$ we take two points $x(\alpha, \beta) = (x, (x,y), (\alpha, \beta))$, $y(\alpha, \beta) = (y, (x,y), (\alpha, \beta))$ in $X \times (x,y) \times (\alpha, \beta)$. We shall define a collection $\{S_\gamma \mid \gamma \in \omega_1\}$. For any $\gamma \in \omega_1$ and for any $(\alpha, \beta) \in \omega_1 \times \omega_1$ we put in the set S_γ the points $x(\alpha, \beta)$ as soon as $\gamma = \min\{\alpha, \beta\}$. If $\gamma = \max\{\alpha, \beta\}$ then we put in the set S_γ the points $y(\alpha, \beta)$. Then $\{S_\gamma \mid \gamma \in \omega_1\}$ is discrete in FZ (as F is a functor) but it is not σ-d. in Z as S_{γ_1}, S_{γ_2} are near for any different indices γ_1, γ_2 (according to the construction they are near on the set $Y \times (\gamma_1, \gamma_2)$).

Recall that the symbol $\mathcal{R}^2$ for a refinement $\mathcal{R}$ denotes the refinement having for the morphisms the mappings $f: X \rightarrow Y$ such that $f \times f: X \times X \rightarrow Y \times Y$ is in $\mathcal{R}$.

Theorem 2: It holds $\mathcal{D}_+^{\sigma^2} = \mathcal{D}_-^{\sigma^2} = \mathrm{Id}$.

Proof: The equality $\mathcal{D}_-^{\sigma^2} = \mathrm{Id}$ is evident. The proof of $\mathcal{D}_+^{\sigma^2}$ = Id is in fact an interplay of Lemma 2.6 in [P] and the idea of the proof of Theorem 1. Similarly as in Theorem 2.3 in [P] it suffices to show that for any $F \in$ $\in \mathrm{Inv}_+ \mathcal{D}^{\sigma^2}$ and for any space X with a discrete subset D fulfilling card D = card X we have FX = X. Suppose the contrary. Take a covering $\mathcal{X} \in X - FX$ and the set $T_{\mathcal{X}}$. We can assume that card D $> \omega_0$. Let $\{D_\alpha \mid \alpha \in \omega_1\}$ be a partition of D such that card D_α = card D (= card $T_{\mathcal{X}}$) for all $\alpha \in \omega_1$. For any D_α we construct a discrete set M_α of points in $X \times X$ which is not discrete in $FX \times FX$ and the projection of M_α is D_α (see Lemma 2.6 in [P]). Put M = $= \bigcup M_\alpha$. By a suitable joint of the points in M we obtain a discrete collection in $X \times X$ which is not σ-discrete in $FX \times FX$.

The analogous observations of $\mathcal{D}^{\sigma d}$ are more varied.

Statement 1: The $\mathcal{D}^{\sigma d}$ fine functor $\mathcal{D}_f^{\sigma d}$ is that which assigns to any space X the σ-d. partitions of X.

Proof: Denote by $\mathcal{P}$ the described functor. We show that, for any X, $\mathcal{P}X$ is $\mathcal{D}^{\sigma d}$ fine and that $\mathcal{P}X = X$ whenever X is $\mathcal{D}^{\sigma d}$ fine. Take a $\mathcal{D}^{\sigma d}$ continuous mapping f: : $\mathcal{P}X \to M$ into a metric space. Let $\mathcal{X} \in M$. According to the Stone theorem we can refine $\mathcal{X}$ by a σ-d. partition $\mathcal{R}$. So $f^{-1}(\mathcal{R})$ is σ-d.d. in $\mathcal{P}X$ and then it is σ-d.d. in X (as X and $\mathcal{P}X$ have the same σ-d.d. collections).

If X is $\mathcal{D}^{\sigma d}$ fine then $\mathcal{P}X = X$ - it is easy.

Theorem 3: It holds $\mathcal{D}_+^{\sigma d} = \mathcal{D}_c$ and $\mathcal{D}_-^{\sigma d} = \mathcal{D}_f^{\sigma d}$.

Proof: The first part follows similarly as the equality $\mathcal{D}_+^{\sigma} = \mathcal{D}_c$ because a collection of points is σ-d.d. iff it is σ-d. The second part is obvious.

Theorem 4: It holds $\mathcal{D}_+^{\sigma d^2} = \mathcal{D}$ and $\mathcal{D}_-^{\sigma d^2} = \mathcal{D}_f^{\sigma d}$.

Proof: Both equalities follow immediately from the following observation: If $\mathcal{D}_f^{\sigma d} X = \mathcal{D}_f^{\sigma d} Y$ then

$\mathcal{D}_f^{\sigma d}(X \times X) = \mathcal{D}_f^{\sigma d}(Y \times Y)$. To prove this, take a discrete collection $\{Z_\alpha \mid \alpha \in I\}$ in $X \times X$. Let $\{Z_\alpha \mid \alpha \in I\}$ be discrete of the order $\mathcal{X} \times \mathcal{X} \in X \times X$. The covering $\mathcal{X}$ can be refined by a σ-d. partition in Y. Then $\{Z_\alpha \mid \alpha \in I\}$ is σ-d.d. in $Y \times Y$ and the proof is concluded.

We finish the first paragraph with two examples.

Example 1: Let X be of a nonmeasurable cardinality and let F be a free ultrafilter on X. Then any disjoint collection in X is σ-d. in the space X_F.

Proof: Let $\{A_\alpha \mid \alpha \in I\}$ be a disjoint collection in X. If a set A_α belongs to F then it is clear. Suppose the contrary. Take a mapping $f: X \to I$ such that $f(A_\alpha) = \alpha$. As I is nonmeasurable and $f(F)$ is a free ultrafilter on I then there is a countable family $\{G_n \mid n \in N\}$ of sets of $f(F)$ such that $\bigcap G_n = \emptyset$. The covering of X_F given by $f^{-1}(G_n)$ realize the σ-discreteness of $\{A_\alpha \mid \alpha \in I\}$.

Going over all free ultrafilters we obtain a family of uniformities such that any of these induce the same (trivial) σ-d. structure but the greatest lower bound of these induce some nontrivial σ-d. structure (as the space given by the Fréchet filter).

Example 2: This example shows that there is a σ-d. fine space which is not σ-d.d. fine.

Put $Y = \bigvee_{n=1}^{\infty} w_n$ where $|w_n| = \omega_1$ for all $n \in N$. Endow the set Y with a uniformity $\mathcal{U}$ such that a covering $\mathcal{X}$ belongs to the base of it iff the trace of $\mathcal{X}$ on at most finitely many ω_n is discrete and the trace of $\mathcal{X}$ on the remainder is a countable partition. Then Y is not σ-d.d. fine as Y is σ-d.d. into itself. We shall show that Y is σ-d. fine. Take a pseudometric ϱ on Y such that each σ-discrete family wrt ϱ is σ-discrete wrt $\mathcal{U}$. We are to prove that ϱ belongs to $\mathcal{U}$. Suppose the contrary. Then for an infinite number $n \in N_0 \subset N$, there is an uncountable ε-discrete (wrt ϱ) family S_n contained in w_n. Put $S_n = \{ s^n \mid \alpha \in \omega_1 \}$ for each $n \in N_0$. Define a transfinite sequence $\{ a_\iota \}_{\iota \in \omega_1}$ by induction:

$a_0 = 0$

$$a_\iota = \min \{ \omega_1 - \{ \alpha \in \omega_1 \mid \text{there is } j < \iota \text{ such that } \rho(s^n_\alpha, s^m_{a_j}) < \frac{\varepsilon}{3} \text{ for some } n, m \in N_0 \}$$

Put $P_\iota = \{ s^n_{a_\iota} \mid n \in N_0 \}$ for each $\iota \in \omega_1$. The collection $\{ P_\iota \}_{\iota \in \omega_1}$ is $\frac{\varepsilon}{3}$ -discrete wrt ρ but it is not σ - discrete wrt $\mathcal{U}$.

§ 2. Theorem: There is a metric fine space which is not $\mathcal{D}^{\sigma d} \wedge$ coz fine.

Proof: Let card $X > \omega_1$ and let X have for a base the partitions on at most ω_1 classes. It is easy to check that X is metric fine and not proximally fine. Take the "prequotient" $\tilde{X}$ to $\tilde{X}$ (see e.g. [P], Th.2.2). From the construction of $\tilde{X}$ we have that $\tilde{X}$ is metric fine too and that any mapping from $\tilde{X}$ is $\mathcal{D}^{\sigma d}$ continuous (any disjoint collection in $\tilde{X}$ is σ-d.d.). So, it suffices to find a cozero continuous mapping from $\tilde{X}$ which is not uniformly discrete. But $\tilde{X}$ is not proximally fine (as X is not) and, being metric fine, it is not cozero fine (see [F_3]). The proof is concluded.

References

[F_1] Z. Frolík: Basic refinements of uniform spaces, Proc. 2nd Top. Conf. in Pittsburgh, Lecture Notes in Math. 378, 140-158

[F_2] Z. Frolík: Four functors into paved spaces, SUS 1973-74 directed by Z. Frolík, Publ. Math. Inst. Prague, Prague 1975

[F_3] Z. Frolík: A note on metric fine spaces, Proc. A.M.S. 46(1974), 111-119

[F_4] Z. Frolík: Three technical tools in uniform spaces,