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POINTWISE COMPACT SETS OF FUNCTIONS

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§ 1. Joint continuity. If not stated otherwise, the following notation will be used:

X is a set;

$\mathcal{H}$ is a non-void compact subset of the product space R^X where R is the space of the reals;

$X_{s\mathcal{H}}$ is the uniform space which is projectively generated by all $h : X \longrightarrow R$, $h \in \mathcal{H}$

$X_{\mathcal{H}}$ is the pseudometric space $\langle X, d_{\mathcal{H}} \rangle$ where

$d_{\mathcal{H}} \langle x,y \rangle = \sup \{ | hx-hy | \mid h \in \mathcal{H} \}$.

In what follows we say simply uniform to mean uniformly continuous. This applies to various derivatives, e.g. equi-uniform family of functions. The assertions in Proposition 1 are self-evident (from the definitions).

Proposition 1, $X_{\mathcal{H}}$ is the coarsest uniform space on X such that $\mathcal{H}$ is equi-uniform, the identity $X_{\mathcal{H}} \longrightarrow X_{s\mathcal{H}}$ is uniform, and

$$\mathcal{H} \subset U(X_{s\mathcal{H}}) \subset U(X_{\mathcal{H}}) .$$

Further notations:

$\tilde{\mathcal{H}}$ is the mapping of X into $C(\mathcal{H})$ defined as follows:

$$\tilde{\mathcal{H}} x = \{ h \longrightarrow hx \} : X \longrightarrow R .$$

Of course, $C(\mathcal{H})$ has the sup-norm. The set $C(\mathcal{H})$ with uniformity projeetively generated by the identity embedding into $R^{\mathcal{H}}$ is denoted by $C_s(\mathcal{H})$. It is well known, and easy to see that $C(\mathcal{H})$ is dense in $R^{\mathcal{H}}$.

Proposition 2. The mappings $\tilde{\mathcal{H}} : X_{s\mathcal{H}} \longrightarrow C_s(\mathcal{H})$ and $\mathcal{H} : X_{\mathcal{H}} \longrightarrow C(\mathcal{H})$ are uniform embeddings.

Proof. The first statement is self-evident, and for the second note:

$$d_{\mathcal{H}} < x,y > = \| \tilde{\mathcal{H}} x - \tilde{\mathcal{H}} y \| .$$

Here $\| \;\; \|$ stands for the sup-norm on $C(\mathcal{H})$.

Theorem 1. The topologies of $X_{s\mathcal{H}}$ and $X_{\mathcal{H}}$ coincide (i.e. $X_{s\mathcal{H}}$ and $X_{\mathcal{H}}$ are topologically equivalent) if and only if $\mathcal{H}$ is pointwise jointly continuous, i.e. the evaluation mapping

$$\mathcal{H} \times X_{s\mathcal{H}} \longrightarrow R$$

is continuous (the evaluation mapping assings to each $< h,x >$ the value of h at x).

Proof. It is well known, and easy to see that the evaluation map

$$\mathcal{H} \times X_{\mathcal{H}} \longrightarrow R$$

is continuous. Hence, if $X_{s\mathcal{H}}$ and $X_{\mathcal{H}}$ are topologically equivalent, then $\mathcal{H}$ is pointwise jointly continuous.

Conversely, assume that $\mathcal{H}$ is pointwise jointly continuous, and $X_{s\mathcal{H}}$ and $X_{\mathcal{H}}$ are not topologically equivalent.

There exists a positive real r and a net $\{x_a\}$, and an $x \in X$ such that $\{x_a\}$ converges to x in $X_{s\mathcal{H}}$, however, $d_{\mathcal{H}}<x_a,x> > r$ for each a. There exists a family $\{h_a\}$ in $\mathcal{H}$ such that $|h_a x_a - h_a x| > r$ for each a. Since $\mathcal{H}$ is compact, a subnet of $\{h_a\}$ converges to an $h \in \mathcal{H}$. We may and shall assume that $\{h_a\}$ converges to h. Clearly

$|h_a x_a - h_a x| \leqq |h_a x - hx| + |h_a x_a - hx|$.

Since $h_a \longrightarrow h$ in $\mathcal{H}$, $|h_a x - hx| \longrightarrow 0$, and since $\mathcal{H}$ is jointly pointwise continuous, $|h_a x_a - hx| \longrightarrow 0$. This contradicts the inequality $|h_a x_a - h_a x| > r > 0$.

The following Theorem 2 gives an important example of pointwise jointly continuous families.

Theorem 2. If $\mathcal{H}$ is a lattice (in the pointwise order), then $\mathcal{H}$ is pointwise jointly continuous.

Two self-evident lemmas are needed for the proof.

Lemma 1. If $\mathcal{H}$ is a lattice in a pointwise order, then $\mathcal{H}$ is a complete lattice in the pointwise order.

Proof. Let $\mathcal{L} \subset \mathcal{H}$, and let $\mathcal{K}$ be the smallest among all sub-lattices containing $\mathcal{K}$. Let k be the pointwise supremum of $\mathcal{L}$, or equivalently, of $\mathcal{K}$.

Cleary $\mathcal{K}$ is a net in the up-ward pointwise order, and converges to k in $\mathcal{H}$. Hence $k \in \mathcal{K}$, and $k = \sup \mathcal{K}$ in the pointwise order on all functions, and hence in the order of $\mathcal{H}$. Of course, $\sup \mathcal{K} = \sup \mathcal{L}$ in both meanings.

Lemma 2. Let $\{h_a\}$ be a net in $\mathcal{H}$. Put

$h'_a = \sup \{h_b \mid b \geqq a\}$,

$h''_a = \inf \{h_b \mid b \geqq a\}$.

Then $\{h_a\}$ converges if and only if the net $\{h'_a - h''_a\}$ pointwise decreases to zero.

Proof. Obvious by Lemma 1. It should be remarked that $\{h'_a\}$ decreases to the upper limit of $\{h_a\}$, and $\{h''_a\}$ increases to the lower limit of $\{h_a\}$.

Proof of Theorem 2. Assume that $\mathcal{H}$ is a lattice in the pointwise order. By Lemma 1, $\mathcal{H}$ is a complete lattice in the pointwise order, and Lemma 2 holds. Assume that $\{h_a\}$ converges to h in $\mathcal{H}$, and $\{x_b\}$ converges to x in $X_{s\mathcal{H}}$. We must show that $\{h_a x_b\}$ converges to hx. If h'_a and h''_a are defined as in Lemma 2, then $\{h'_a - h''_a\}$ is a net of continuous functions on $X_{s\mathcal{H}}$ decreasing to zero pointwise, and it is enough to show that

$$\{h'_a x_b - h''_a x_b\} \longrightarrow 0$$

because

$$h''_a x_b \leqq H_a x_b \leqq h'_a x_b$$

for each a and b. And this is quite easy. For convenience of the reader we shall state what is needed exactly.

Lemma 3. Assume that a net $\{f_a\}$ of functions on X decreases to zero pointwise. If $x \in X$, and $\{x_b\}$ is a net in X such that

$$\lim_{b} f_a x_b = f_a x$$

for each a. Then

$$\lim_{a,b} f_a x_b = 0 .$$

Note the following interesting corollary to Theorems 1 and 2.

Theorem 3. A topological space T is metrizable if and only if T is projectively generated by a pointwise compact lattice of functions (bounded if you want).

Proof. If T is projectively generated by a pointwise compact lattice $\mathcal{H}$ of continuous functions, then by Theorems 1 and 2 the pseudometric $d_{\mathcal{H}}$ metrizes T.

The converse is quite obvious. Assume that a pseudometric d metrizes T. Put

$$\mathcal{H}(d) = \{f \in R^T \mid \|f\| \leqq 1, \quad f \geqq 0, \quad |fx - fy| \leqq d\langle x,y\rangle$$
for each x and y in $T\}$.

It is well known and obvious that

Lemma 4. $\mathcal{L}(d)$ is a pointwise compact lattice, and

$d_{\mathcal{L}(d)}\langle x,y\rangle = \min(1, d\langle x,y\rangle)$

for each x and y.

Hence $\mathcal{L}(d)$ projectively generates T, because $T_{s\mathcal{L}(d)}$ and $T_{\mathcal{L}(d)}$ are topologically equivalent by Theorems 1 and 2.

Remark 1. Let $\langle X, \leq\rangle$ be well ordered, and let

$h_x y = 0$ if $y \leq x$

$= 1$ if $y > x$.

Then the set $\mathcal{H}$ of all h_x, $x \in X$, given the pointwise order, is order isomorphic to $\langle X, \leq\rangle$, the isomorphism being $\{x \to h_x\} : X \to \mathcal{H}$. Thus if $\langle X, \leq\rangle$ is the space of all ordinals $\leq \omega_1$, then $\mathcal{H}$ is compact in the order topology (note that the order is the pointwise order). On the other hand, $\mathcal{H}$ is almost never compact in the pointwise topology. Also note that $d_{\mathcal{H}}$ is uniformly discrete ($d\langle x,y\rangle$ attains at most two values, namely 0 and 1).

Remark 2. A pointwise compact $\mathcal{H}$ is pointwise jointly continuous if and only if there exists a pointwise compact lattice $\mathcal{L} \supset \mathcal{H}$ such that $X_{s\mathcal{L}} = X_{s\mathcal{H}}$.

§ 2. Metrizability of $\mathcal{H}$.

Without any additional assumptions on $\mathcal{H}$ we have just the following simple result.

Theorem 1. The following four conditions are equivalent:

(a) $\mathcal{H}$ is metrizable

(b) $X_{\mathcal{H}}$ is a separable topological space (i.e. $X_{\mathcal{H}}$ has a countable dense set).

(c) $X_{s\mathcal{H}}$ is a separable topological space.

(d) There is a countable set $Y \subset X$ such that the projection of $\mathcal{H}$ onto $\mathcal{H}/Y$ is injective.

Proof. (a) $\Rightarrow$(b) : If $\mathcal{H}$ is metrizable, then $C(\mathcal{H})$ is separable, and hence $X_{\mathcal{H}}$ is separable (because $X_{\mathcal{H}}$ embeds in $C(\mathcal{H})$ by Proposition 1,2).

(b)$\Longrightarrow$(c) : Because the identity $X_{\mathcal{H}} \longrightarrow X_{s\mathcal{H}}$ is continuous.

(c)$\Longrightarrow$(d) : Let Y be a countable dense set in $X_{s\mathcal{H}}$. Since all functions in $\mathcal{H}$ are continuous on $X_{\mathcal{H}}$, if two of them are equal on Y, they are equal at all.

(d)$\Longrightarrow$(a) : Let Y be a countable subset of X such that the projection $\mathcal{H} \longrightarrow \mathcal{H}/Y$ is one-to-one. Since $\mathcal{H}$ is compact (and $\mathcal{H}/Y$ is Hausdorff) the projection is a homoemorphism. Finally, $\mathcal{H}/Y$ is metrizable as a subspace of a countable product of metric spaces, namely as a subspace of R^Y.

Remark : J.P.Lavigne, Approximation des fonctions uniformément continues. J. de Math. pures et Appl. 51 (1972), proved that a uniform space X is separable if and only if each uniformly equicontinuous family $\mathcal{H}$ on X is metrizable in the pointwise topology.

Corollary. If $\mathcal{H}$ is metrizable then there exists a σ-additive probability on $X_{s\mathcal{H}}$ (in fact a Radon measure on X endowed with the discrete topology) such that no two distinct elements of $\mathcal{H}$ are equal almost everywhere.

Proof. By Theorem 1 there exists a countable set $Y \subset X$ such that the projection $\mathcal{H} \to \mathcal{H}/Y$ is one-to-one. Arrange Y in a sequence $\{y_n\}$ and put

$$\mu = \sum 1/2\, n\, \delta_{y_n}$$

where δ_{y_n} is the Dirac measure at y_n, i.e. the evaluation at y_n, that means

$$\mu(h) = \sum_n 1/2\, n . h y_n \quad .$$

Obviously two functions h and k on X are equal almost everywhere if and only if $h|Y = k|Y$.

It is an open problem if the metrizability of $\mathcal{H}$ is implied by the existence of a probability on $X_{s\mathcal{H}}$ distinguishing the elements of $\mathcal{H}$. It is easy to see that the answer is yes if $\mathcal{H}$ is sequentially compact, and it has been proved by A.J.Tulcea-Below [5] that the answer is yes if $\mathcal{H}$ is convex.

§ 3. Ascoli Theorem.

The main result says:

Theorem 1. A set $Y \subset X$ is precompact in $X_{\mathcal{H}}$ if and only if the uniform convergence in $\mathcal{H}/Y$ coincides with the pointwise convergence in $\mathcal{H}/Y$, and each function in $\mathcal{H}$ is bounded.

Corollary. $X_{\mathcal{H}}$ is precompact if and only if every pointwise converging net in $\mathcal{H}$ converges uniformly, and every function in $\mathcal{H}$ is bounded.

One part is an immediate consequence of the following Ascoli Theorem.

Lemma 1. Let X be a uniform space, and let $\mathcal{H}$ be an equi-uniform set of functions. Then if a net $\{h_a\}$ converges to h pointwise, then $\{h_a\}$ converges to h uniformly on each precompact subset of X.

It should be remarked that one may replace R by any uniform space in Lemma 1.

It remains to show that Y is precompact in $X_{\mathcal{H}}$ if the pointwise convergence implies the uniform convergence in $\mathcal{H}/Y$ and each function in $\mathcal{H}/Y$ is bounded. Since $d_{\mathcal{H}/Y}$ is a restriction of $d_{\mathcal{H}}$, it is enough to show that $X_{\mathcal{H}}$ is precompact whenever the pointwise convergence implies the uniform convergence in $\mathcal{H}$, and each function in $\mathcal{H}$ is bounded.

Let T be a completion of the uniform spaces $X_{s\mathcal{H}}$. Since each of the functions in $\mathcal{H}$ is bounded, $X_{s\mathcal{H}}$ is precompact, and hence T is a compact space. Since T is a completion of $X_{s\mathcal{H}}$, each function in $\mathcal{H}$ extends uniquely to a uniform function on T; let $\mathcal{K}$ be the set of all these extensions to T. Since the pointwise convergence coincides with the uniform convergence in $\mathcal{H}$, the same is true for $\mathcal{K}$, and the projection $\mathcal{K} \to \mathcal{H}$ is a homoemorphism. Since the pointwise convergence coincides with the uniform convergence in $\mathcal{K}$, necessarily $\mathcal{K}$ is pointwise jointly continuous, and hence by Theorem 1.1 $T_{\mathcal{K}}$ is topologically equivalent to $T_{s\mathcal{K}}$. But $T_{s\mathcal{K}}$ is the topology of T, and hence, $T_{\mathcal{K}}$ is a compact metric space. Since $X_{\mathcal{H}}$ is a subspace of $T_{\mathcal{K}}$, $X_{\mathcal{H}}$ is necessarily precompact. This concludes the proof.

Remark. In the proof we used completions of uniform spaces which are not necessarily Hausdorff. If one does ot want to do that, he may assume that $\mathscr{X}$ distinguishes points of X.

It may be worth noting:

Corollary. If $\mathscr{X}$ is a pointwise compact set of ontinuous functions on a compact space, then $\mathscr{H}$ is equicontinuous if and only if the pointwise convergence coincides with the uniform convergence in $\mathscr{H}$.

§ 4. When compact metrizable sets of uniform functions are equi-uniform.

The main result says:

Theorem 1. The following three properties of a uniform space X are equivalent:

(a) If $\mathscr{H} \subset U_b(X)$ is pointwise compact and metrizable, then $\mathscr{H}$ is equi-uniform (i.e. $d_{\mathscr{H}}$ is uniformly continuous on X).

(b) Every countable partition of X by Baire sets is a uniform cover of X.

(c) If $f : X \rightarrow Y$ is Baire measurable (i.e. the preimages of the Baire sets in Y are Baire sets in X), and if Y is a separable metric space, then $f : X \rightarrow Y$ is uniformly continuous.

The easiest part is (b) implies (c): One just uses the fact that every uniform cover of a separable metric space is refined by a countable open cover, and hence by a countable Baire partition. We assume the definition, and the elementary properties of Baire sets in uniform spaces.

The implication (c)$\Longrightarrow$(a) follows from the follow-

Lemma 1. Assume that K is a compact metrizable space. Then the Baire sets in $C(K)$ and $C_s(K)$ coincide (recall that $C(K)$ has the sup-norm, and $C_s(K)$ is a subspace of R^K).

Proof. Since the identity $C(K) \longrightarrow C_s(K)$ is uniform, and since $C(K)$ is separable (because K is compact metrizable), it is enough to show that every open ball in $C(K)$ is a Baire set in $C_s(K)$. Choose a countable dense set Y in K. For each y in Y let ρ_y be a continuous pseudonorm on $C_s(K)$ defined as follows:

$$\rho_y h = |hy| .$$

Clearly

$$\|h\| = \sup \{\rho_y h | y \in Y\} .$$

It follows that the sup-norm is a Baire measurable function on $C_s(K)$. And this implies that the balls in $C(K)$ are Baire sets in $C_s(K)$.

Another proof of Lemma 1. Choose a countable dense set Y in K, and let C be $C(K)$ endowed with the topology of pointwise convergence in the points of Y. Then the identity mappings

$$C(K) \longrightarrow C_s(K) \longrightarrow C$$

are uniform, hence Baire measurable. Since $C(K)$ is analytic (because $C(K)$ is a separable metric space) and C is metrizable, the identity mapping

$$C(K) \longrightarrow C$$

is a Baire isomorphism by a deep theorem of N.Luzin. Hence

$$C(K) \longrightarrow C_s(K)$$

is a Baire isomorphism. It should be noted that the theorem used is much deeper than Lemma 1.

It remains to show (a) implies (b). The main step is in

Lemma 2. Condition (a) in Theorem 1 implies that the indicator function (= characteristic function) of each Baire set is uniform.

Proof. One checks immediately that it is enough to show:

There exists a compact metrizable $\mathcal{H} \subset C(X)$, where X is the unit interval $[0,1]$, such that the indicator function of the singleton (0) is uniform.

And this is easy. Construct a sequence $\{f_n\}$ of continuous functions such that $\{\text{coz } f_n\}$ is point-finite, and $f = \sup \{f_n\}$ is 0 at 0, and $\geqq 1$ otherwise. Then $\{f_n\}$ converges to the zero function pointwise, and the indicator function of (0) is 1-min (1,f), and this is uniform on $X_{\mathcal{H}}$, because all f_n are Lipschitz functions on $X_{\mathcal{H}}$, and so f is.

The most important example of spaces with the properties in Theorem 1 (called $\aleph_0$-measurable by the author in [2]) is the following:

Take any σ-algebra $\mathcal{A}$ on a set X and consider the uniformity on X which has all countable partitions ranging in $\mathcal{A}$ for a basis of uniform covers. The resulting space is denoted by $u_{\omega_1}\mathcal{A}$ in [2] . These measurable functions on $\langle X, \mathcal{A} \rangle$ coincide with the uniform function on $u_{\omega_1}\mathcal{A}$. This space can be used in studying measurable functions. For example, Theorem 1 says that every metrizable compact set of measurable functions is equi-

- uniform on $u_{\omega_1}\mathcal{A}$. As an application we shall prove the P.Meyer generalization of the Jegorov theorem [6] .

Theorem 2. If $\mathcal{H}$ is a compact metrizable set of bounded measurable functions on $\langle X, \mathcal{A} \rangle$ and if μ is a σ-additive probability on $\mathcal{A}$, then there exists a disjoint sequence $\{A_n\}$ in $\mathcal{A}$, such that

(1) $\mu(X - \bigcup \{A_n\}) = 0$.

(2) Pointwise convergence implies the uniform convergence in each $\mathcal{H} | A_n$.

Proof. Consider $u_{\omega_1}\mathcal{A}$ defined above. By Theorem $\mathcal{H}$ is equi-uniform, and hence $\tilde{\mathcal{H}} : u_{\omega_1}\mathcal{A} \longrightarrow C(\mathcal{H})$ is uniformly continuous. Hence μ is a σ-additive probability on a separable metric space $X_{\mathcal{H}}$, and hence a Radon-measure $\hat{\mu}$ on the completion $\hat{X}_{\mathcal{H}}$ of $X_{\mathcal{H}}$. Hence there exists a disjoint sequence $\{B_n\}$ of compact sets, such that its union supports $\hat{\mu}$. Put $A_n = X \cap B_n$, and apply Theorem 3.1.

§ 5. When compact metrizable lattices are equi- - uniform ?

The main result says:

Theorem 1. The following four properties of a uniform space are equivalent:

(a) If $\mathcal{H} \subset U_b(X)$ is a pointwise compact metrizable lattice then $\mathcal{H}$ is equi-uniform.

(b) If Y is a separable metric space, and if $f : X \longrightarrow Y$ is a coz-mapping (the preimages of cozero sets are cozero sets), then $f : X \longrightarrow Y$ is uniform.

(c) Every countable cozero cover of X is uniform.

(d) If Y is a separable metric space, and if $f : X \longrightarrow Y$ is uniform, then so is $f : X \longrightarrow t_f Y$ (where $t_f Y$ is the finest uniformity topolog'cally equivalent to Y).

The equivalence of the conditions (b), (c) and (d) is essentially due to A.Hager [4], for an obvious proof see e.g. [1] or [3]. Here we are going to prove that (a) is equivalent to (b). We start with more important part (b) implies (a). For the proof we need the follo ing

Lemma 1. If $\mathcal{H}$ is a compact lattice then the open balls in the matric space $X_{\mathcal{H}}$ are cozero sets in $X_{s\mathcal{H}}$. Consequently, if $X_{\mathcal{H}}$ is separable (i.e. $\mathcal{H}$ is metrizable by Theorem 2.1) then

$$\operatorname{coz} X_{\mathcal{H}} = \operatorname{coz} X_{s\mathcal{H}} .$$

Proof of (a)$\Longrightarrow$(b) (using Lemma 1): Assume (b), and let $\mathcal{H}$ be a compact metrizable lattice. By Lemma 1 the mapping $\mathcal{H} : X \longrightarrow X_{\mathcal{H}}$ is a cozero mapping, and $X_{\mathcal{H}}$ is separable, and hence, by Condition (b) the mapping $f : X \longrightarrow X_{\mathcal{H}}$ is uniform (because

$$\operatorname{coz} X_{s\mathcal{H}} \subset \operatorname{coz} X$$

in general), and this means by the definition that $\mathcal{H}$ is equi-uniform.

Example. If $X_{\mathcal{H}}$ is not separable, then

$$\operatorname{coz} X_{\mathcal{H}} \neq \operatorname{coz} X_{s\mathcal{H}}$$

in general.

Let X be a linearly ordered set, and let $\mathcal{H}$ be the let of all non-decreasing functions with the values between 0 and 1.

Since the indicator functions of intervals $[\![x, \rightarrow]\!]$ are in $\mathcal{H}$, d attains just two values, 0 and 1, and hence $X_{\mathcal{H}}$ is uniformly discrete. Thus $\operatorname{coz} X_{\mathcal{H}} = \exp X$.

On the other hand, it is easy to check that $\operatorname{coz} X_{s\mathcal{H}}$ consists of countable unions of intervals, and therefore

$$\operatorname{coz} X_{s\mathcal{H}} \neq \operatorname{coz} X_{\mathcal{H}}$$

whenever X is measurable.

It should be noted that in many cases the equality holds even in the space $X_{\mathcal{H}}$ being not separable. A general example is the set of Lipschitz functions (with respect to a pseudometric). I do not know of any condition which is close to being necessary and sufficient. It may be a wrong question, anyway.

The proof of Lemma 1 follows easily from the following basic

Theorem 2. Let $\mathcal{H}$ be a compact lattice, and let $x \in X$. For each $r > 0$ there exists a finite subset $\mathcal{K}$ of $\mathcal{H}$ such that

$$d_{\mathcal{H}} \langle x,y \rangle \leq d_{\mathcal{K}} \langle x,y \rangle + r$$

for each y in Y. Hence, all functions $\{y \rightarrow d_{\mathcal{H}} \langle z,y \rangle\}$, $z \in X$, are uniform on $X_{s\mathcal{H}}$.

Remark. Theorem 2 implies that if $\mathcal{H}$ is a compact lattice, then $X_{\mathcal{H}} = X_{s\mathcal{H}}$ topologically.

Proof of Lemma 1 (using Theorem 2): It is obvious that if $\mathcal{L}$ is a finite set then

$$C(\mathcal{L}) = C_s(\mathcal{L})$$

uniformly, and hence

$$\operatorname{coz} C(\mathcal{L}) = \operatorname{coz} C_s(\mathcal{L}),$$

and hence

$$\operatorname{coz} X_{\mathcal{L}} = \operatorname{coz} X_{s\mathcal{L}}.$$

Hence, if $\mathcal{K}$ is finite subset of $\mathcal{H}$, then

$$\operatorname{coz} X_{\mathcal{K}} \subset \operatorname{coz} X_{s\mathcal{H}} .$$

It follows immediately from Theorem 2 that every ball in $X_{\mathcal{H}}$ is a countable union of balls in various $X_{\mathcal{K}}$, $\mathcal{K} \subset \mathcal{H}$ finite, and hence, each ball in $X_{\mathcal{H}}$ is a cozero set in $X_{s\mathcal{H}}$.

Remark. Lemma 1 follows immediately from the last statemant in Theorem 2.

Proof of Theorem 2. The set $[\mathcal{H}]x$ of all hx, $h \in \mathcal{H}$, is compact because $\mathcal{H}$ is pointwise compact. Hence $[\mathcal{H}]x$ is contained in a compact interval J. Sub-divide J into a finite number of intervals, say $\{J_n | n \leq k\}$ such that the length of each J_n is less than r. Put

$$\mathcal{H}_n = \{h | h \in \mathcal{H} , hx \in J_n\} .$$

If $\mathcal{H}_n \neq \emptyset$, put

$$h_n' = \sup \mathcal{H}_n, \quad h_n'' - \inf \mathcal{H}_n .$$

Let $\mathcal{K}$ be the set of all h_n' and h_n'' with $\mathcal{H}_n \neq \emptyset$. It is obvious that if $h \in \mathcal{H}_n$ then

$$|hx-hy| \leq r + \max(|h_n'x-h_n'y| , |h_n''x-h_n''y|),$$

for each y in X, and hence

$$d_{\mathcal{H}} \langle x,y \rangle \leq d_{\mathcal{K}} \langle x,y \rangle + r$$

for each y.

This completes the proof of (b) implies (a) in Theorem 1. For (a) implies (b) we need the following

Lemma 2. Condition (c) in Theorem 1 implies that X is inversion closed (every cozero function is uniform).

Proof of (a) implies (b) (using Lemma 2): Let $\mathcal{L}$ be the set of all Lipschitz functions on Y between 0 and 1. Since $Y_{\mathcal{L}} = Y$ uniformly, and since Y is separable, the compact lattice $\mathcal{L}$ is metrizable. Let $\mathcal{H}$ be the set of all $\ell \circ f$, $\ell \in \mathcal{L}$. Then $\mathcal{H}$ is a compact metrizable lattice of functions on X, and each of these functions is a cozero function. By Lemma 2, all these functions are uniform. Hence $\mathcal{H}$ is equi-uniform by Condition a, this means that $X \longrightarrow X_{\mathcal{H}}$ is uniform, and since $f : X_{\mathcal{H}} \longrightarrow Y$ $(=Y_{\mathcal{L}})$ is uniform, necessarily $f : X \longrightarrow Y$ is uniform.

Proof of Lemma 2. If $\{f_n\}$ is a sequence in $U_b(X)$ pointwise decreasing to zero, then $\{f_n\}$ is a compact metrizable lattice, and hence, $\{f_n\}$ is equi-uniform by Condition (a) in Theorem 1. This statement implies that X is inversion-closed (in fact it is equivalent), see, e.g., [7]. This completes the proof.

In conclusion we state a problem (which may be wrong):

Under what conditions on $\mathcal{H}$ the uniform space $X_{\mathcal{H}}$ is distal ?

A sufficient condition is given in [3]:

$\sum \{|hx| \mid h \in \mathcal{H}\}$

is a bounded function.

This is a consequence of the result [3] that the unit ball ℓ_1 is a distal subspace of ℓ_∞.

Another example: If $\mathcal{H} = \{h_n \downarrow 0\}$, $h_1 \leq 1$, then $X_{\mathcal{H}}$ is distal, implicitely in [7].

This text will be followed, and essentially completed, by a paper on uniformly weakly compact sets of uniformly continuous functions.

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