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Point character of uniformities and completeness

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Introduction 0: Results contained in this paper generalize results from [P_1],[P_3] and [Š].

Definition 1: Let K b an infinite cardinal. Let n be a positive integer. We define $\mathcal{K}(K,n)$ as a set of all elements V of $(\exp K)^n$ such that $pr_1 V \subset pr_2 V \subset \dots \subset pr_n V$ and $pr_1 V \neq \emptyset$.

Notation 2: Let $n > 1$ be a positive integer. For $V \in \mathcal{K}(K, n-1)$, put $\mathcal{U}(V) = \{ U \in \mathcal{K}(K,n) \mid pr_1 U \subset pr_1 V \subset pr_2 U \subset \dots \subset pr_{n-1} V \subset pr_n U \}$.

Construction 3: Let α be an infinite cardinal. Denote $H_k = \{ \frac{i}{2^k} \mid i = 0,1,\dots,2^k \}$ for k non-negative integer, $H = \bigcup \{ H_k \mid k = 0,1,2,\dots \}$. Put $M(\alpha) = \{ f: H \longrightarrow \exp \alpha \mid (f(h_1) \supset f(h_2)$ for any $h_1, h_2 \in H$ such that $h_1 > h_2)$ and $f(0) \neq \emptyset$. For $f \in M(\alpha)$, $f \restriction H_k$ is an element of $\mathcal{K}(\alpha, 2^k + 1)$ in the fact. For $V \in \mathcal{K}(\alpha, 2^k)$ we define $\forall$ now a base of a pseudometric uniformity $\mathcal{V}'$ on $M(\alpha)$: $\mathcal{B}_i = \{ \tilde{V} \mid V \in \mathcal{K}(\alpha, 2^i) \}$, $i = 0,1,2,\dots$.

Put $(U(\alpha), \mathcal{V})$ for the Hausdorff reflection of the just defined pseudometric uniformity. Clearly, each point of $U(\alpha)$ can be represented by some point of $M(\alpha)$ and we shall suppose it.

Notation 4: Given a cardinal m, $S^+(m)$ denotes the positive part of the unit sphere in $\ell_\infty(m)$ with the uniformity induced by ℓ_∞-norm, (i.e. $S^+(m) = \{ f \in \ell_\infty(m) \mid \sup f = 1$ and $f(i) \geq 0$ for all $i \in m \}$).

Proposition 5: $(U(m), \mathcal{V})$ is uniformly homeomorphic to $S^+(m)$. Proof will be clear from the following:
Notation: For $a \in m$, define $M(a) = \{ f \in S^+(m) \mid a \in \operatorname{coz} f \}$. Put $\mathcal{B}_0 = \{ M(a) \mid a \in m \}$. For $V_1 \subset V_2 \subset \dots \subset V_n \subset m$, define $M(\{V_i\}_{i=1}^n) = \{ f \in S^+(m) \mid f^{-1}([\frac{n+1-i}{n}, 1]) \subset V_i \subset f^{-1}((\frac{n-i}{n}, 1])$,

$\forall$ $\tilde{V} = \{ f \in M(\alpha) \mid f \restriction H_k \in \mathcal{U}(V) \}$. We define

$i = 1,\dots,n\}$

Put $\overline{\mathcal{B}}_n = \{M(\{V_i\}_{i=1}^{n}) \mid V_1 \subset V_2 \subset \dots \subset V_n \subset m\}$

Lemma 6: $\mathcal{B}_n$ forms a base of $S^+(m)$.

Proof of Lemma 6: For $f \in S^+(m)$, put $B_\varepsilon(f) = \{g \in S^+(m) \mid \sup_{x\in m} |f(x) - g(x)| < \varepsilon\}$.

1) $\{B_\varepsilon(f) \mid f \in S^+(m)\} \prec \overline{\mathcal{B}}_n$ for any $\varepsilon < \frac{1}{2n}$. Really, $f \in S^+(m)$, define $V_i(f) = f^{-1}((\frac{2(n-i)+1}{2n}, 1])$

$B_\varepsilon(f) \subset M(\{V_i(f)\}_{i=1}^{n})$ as if $g \in B(f)$ then $|f(x) - g(x)| < \frac{1}{2n}$ for each $x \in m$ and so $g^{-1}([\frac{n+1-i}{n}, 1]) \subset f^{-1}((\frac{2(n-i)+1}{2n}, 1]) \subset g^{-1}((\frac{n-i}{n}, 1])$

2) $\overline{\mathcal{B}}_n \prec \{B_\varepsilon(f) \mid f \in S^+(m)\}$ for any $n > \frac{2}{\varepsilon}$:

for $M(\{V_i\}_{i-1}^{n}) \in \mathcal{B}_n$ take any $f_o \in S^+(m)$ such that $f_o^{-1}((\frac{n-i}{n}, 1]) = V_i$, $i = 1,2,\dots,n$

hence $f_o^{-1}(0) = m - V_n$. Then $M(\{V_i\}_{i=1}^{n}) \subset B_\varepsilon(f_o)$: take $f \in M(\{V_i\}_{i=1}^{n})$ and $x \in m$. Find i such that $x \in V_{i+1} - V_i$ (we put $V_{n+1} = m$). Then $f_o(x) \in (\frac{n-(i+1)}{n}, \frac{n-i}{n}]$.

So $\mathrm{dist}(f_o, f) \le \frac{2}{n} < \varepsilon$.

Definition 7: If $\mathcal{A}$ is a family of sets, an order of $\mathcal{A}$ is defined $\mathrm{ord}\,\mathcal{A} = \sup\{|\mathcal{D}|^+ \mid \mathcal{D} \subset \mathcal{A}$ and $\bigcap\mathcal{D} \neq \emptyset\}$. For a uniform space $(X,\mathcal{U})$ a point-character $\mathrm{pc}(X,\mathcal{U})$ is defined as the least cardinal α such that there is a base $\mathcal{B}$ of $\mathcal{U}$ such that an order of each cover from $\mathcal{B}$ is less or equal to α.

Theorem 8: $\mathrm{pc}\, U(m) > \sup\{\xi \in \alpha \mid \xi$ is a regular cardinal$\}$ for each infinite cardinal m.

Corollary 9: The uniformity on $l_\infty(\omega_1)$ induced by sup-norm has not any point-finite base.

Remark 10: Corollary 9 improves results from [P_1] and [S

Outline of the proof of Theorem 8:

Notation: Suppose $W \in \mathcal{K}(K, n-1)$, $\{Y_i\}_{i=o}^{J}$ is a sequence of subsets of K, $j \leq n-1$. $W - \{Y_i\}_{i=1}^{j}$ is an element of $\mathcal{K}(K,n-1)$ such that $pr_t(W - \{Y_i\}_{i=1}^{j}) = pr_t W - \bigcup_{i=t}^{j} Y_i$, $t=1,\dots,n-1$.

$W \nabla \{Y_i\}_{i=o}^{j} = \{X \in \mathcal{U}(W - \{Y_i\}_{i=1}^{j} \mid pr_t X \cap Y_{t-1} \neq \emptyset,\ t=1,\dots,j\}$.

Lemma 11: We are given: 1) a mapping $c: \mathcal{K}(m,n) \to \mathcal{P}(m)$ such that $c(V) \neq \emptyset$ and $V(1) \subset c(V) \subset V(n)$ for each V. ($\mathcal{P}(m)$ is the set of all subsets of m)
2) an infinite cardinal m
3) a regular infinite cardinal $\xi < m$.

Notation: For $\mathcal{D} \subset \mathcal{P}(m)$, $|\mathcal{D}| < \xi$, $j \in \{1,\dots,n-1\}$ $V \in \mathcal{K}(m,n-1)$, $F(\mathcal{D},j,V)$ denotes the following formula:
$\exists X_j \forall Y_j$, $Y_j \supset X_j \exists X_{j-1} \forall Y_{j-1}$, $Y_{j-1} \supset X_{j-1} \dots \exists X_1 \forall Y_1$, $Y_1 \supset X_1 \exists Y_o : (V \nabla \{Y_i\}_{i=o}^{j}) - \mathcal{D} = \emptyset$ (X_i and Y_i denote members of $[m]^{\leq \xi}$).
If there are $V \in \mathcal{K}(m,n-1)$ such that $|V(1)| = m$ and $j \in \{1,2,\dots,n-1\}$ such that $F(\mathcal{D},j,V)$ does not hold for any $\mathcal{D} \subset [m]^{<\xi}$ then there is $W \in \mathcal{K}(m,n-1)$ such that $|c(\mathcal{U}(W))| \geq \xi$.

Remark 12: Lemma 11 is Lemma in $[P_1]$, p. 150.

For proving Theorem 8, it is enough to show that the $U(m)$-uniform cover $\mathcal{B}_o$ (see Construction 3) has no $U(m)$-uniform refinement of order less than ξ^+. We can employ Lemma 11 and the following definition:

Construction 13: We are going to construct a mapping c for Lemma 11. Suppose $\mathcal{W}$ is $U(m)$-uniform cover such that $\mathcal{W} < \mathcal{B}$ and an order of $\mathcal{W}$ is less than m. Choose a mapping $d : \mathcal{W} \to \mathcal{B}_o$ such that $d(P) \supset P$ for each $P \in \mathcal{W}$. $\mathcal{W}$ is refined by some $\mathcal{B}_q$. Choose $f : \mathcal{B}_q \to \mathcal{W}$ such that $f(P) \supset P$ for each $P \in \mathcal{B}_q$. Define $\bar{c} : \mathcal{B}_q \to \mathcal{B}_o$ by $\bar{c} = d \circ f$. Now define $c : \mathcal{K}(m,2^q+2) \to \mathcal{P}(m)$ by

$$c(\overbrace{(V_1,\dots,V_{2^q+2})}) = \bar{c}(\overbrace{(V_2,\dots,V_{2^q+1})}).$$

Comment 14: Uniform spaces of point-character less than an infinite cardinal α form an epireflective clas in UNIF containing all praecompact spaces, b^{α} denotes the corresponding modification. In $[P_3]$, I promised to prove that b^{α} does not preserve Cauchy filters. Now I going to do it.

Notation 15: Inv^{+}(Cauchy) denotes the class of all functors $F : \mathrm{UNIF} \longrightarrow \mathrm{UNIF}$ such that $id : X \longrightarrow F(X)$ is uniformly continuous for each uniform space X and X and $F(X)$ have the same set of Cauchy filters for each

Problem 16: The question was raised by Z.Frolík wheher there is a member of Inv^{+}(Cauchy) distinct from the identical functor. This problem remains to be open and the following theorem shows that a non-identical member of Inv^{+}(Cauchy) should be pretty wild.

Theorem 17: If $F \in \mathrm{Inv}^{+}$(Cauchy) then $\mathrm{pc}\, F(U(m)) > \xi$ for each infinite regular cardinal $\xi < m$.

Corollary 18: $b^{\alpha} \notin \mathrm{Inv}^{+}$(Cauchy) for any cardinal α.

Remark 19: As the identical functor is contained in Inv^{+}(Cauchy), Theorem 17 may generalize Theorem 8 (see Problem 16). So proving Theorem 17 we shall reprove Theorem 8.

Proof of Theorem 17: The basic fact is the validity of the following formula T:
T: There is a point $f \in U(m)$, $|f(h)| = m$ for each $h \in H$ such that for each cover $\mathcal{P} \in (U(m), \mathcal{V})$, $\mathrm{ord}\, \mathcal{P} < \xi$, there is a member P of $\mathcal{P}$ such that there is an integer n_0 such that for all integers n greater than n_0, the following holds: $\forall D_{2^n}\ \exists H_{2^n-1}$, $H_{2^n-1} \supset D_{2^n}\ \forall D_{2^n-1}$, $D_{2^n-1} \supset H_{2^n-1} \ldots (f^{(n)} \nabla \{D_i\}_{i=1}^{2^n}) \subset P$. $(D_i, H_i \in [m]^{\leq \xi})$, $f^{(n)} = f \restriction (H_n - \{0\})$. Really, if T holds then one can construct a filter $\mathcal{F}$ that is Cauchy w.r.t. $b^{\xi}(U(m), \mathcal{V})$ and does not converge to any point in $U(m)$ but $U(m)$ i

a complete uniform space.

Proof of T will be done by the way of contradiction.

Put $Z(m) = \{f \in U(m) \mid |f(h)| = m$ for each $h \in H\}$.

Choose a (m)-uniform cover $\mathscr{P}$ such that $\mathscr{B}_{n_0} < \mathscr{P}$ (see Construction and each member of $\mathscr{B}_{n_0}$ intersects less than m members of $\mathscr{P}$, so ord $\mathscr{P} < \xi$.

Fix an integer $n > n_0$. Define $i \subset \mathscr{P}(U(m)) \times \mathscr{P}$ by $i(A) = \{P \in \mathscr{P} \mid P \supset A\}$. Because of (2), the following formula $\tau(1)$ holds: $\forall f \in Z(m)\ \exists X_1 \subset \mathscr{P}$, $|X_1| < \xi\ \forall D_1\ \exists H_1$, $H_1 \supset D_1 : i(f^{(n)}\ \{H_1\})\nabla\{D_1\}) - X_1$.

$\tau(1)$'s a corolary of the following formula forced by (2):

$\forall f \in Z(m)\ \exists Y_1 \subset \mathscr{P}$, $|Y_1| < \xi\ \forall D_1\ \exists H_1$, $H_1 \supset D_1 :$

$: i(f^{(n)} - \{H_1\})\nabla\{D_1\}) \subset Y_1$

and the fact that for each $f \in Z(m)$, Y_1 can be divided into two disjoint sets Y_1^1, Y_1^2 so that:

(1): $\forall P \in Y_1^1\ \forall D_1\ \exists H_1$, $H_1 \supset D_1 : i(f^{(n)} - \{H_1\})\nabla\{D_1\}) \ni$

$\ni P$

$\forall P \in Y_1^2\ \exists D_1\ \forall H_1$, $H_1 \supset D_1 : i(f^{(n)} - \{H_1\})\nabla\{D_1\}) \not\ni P$,

and the regularity of m is very useful, as well.

For each $p \in \{1,\dots,2^n\}$ denote by $\tau(p)$ the following formula: $\forall f \in Z(m)\ \exists X_p \subset \mathscr{P}$, $|X_p| < \xi\ \forall D_p\ \exists H_p$, $H_p \supset D_p$ ..

... $\forall D_1\ \exists H_1$, $H_1 \supset D_1 : i((f^{(n)} - (\{H_i\}_{i=1}^{p})\nabla\{D_i\}_{i=1}^{p})) =$

$= X_p$.

We will show that $\tau(p)$ implies $\tau(p+1)$ for $p=1,\dots$

$\dots,2^n-1$.

We an use again the formula

(3) $\forall f \in Z(m)\ \exists Y_{p+1} \subset \mathscr{P}$, $|Y_{p+1}| < \xi\ \forall D_{p+1}\ \exists H_{p+1}$,

$H_p \supset D_p$ ... $\exists H_1$, $H_1 \supset D_1 : i(f^{(n)} - \{H_i\}_{i=1}^{p+1})\nabla\{D_i\}_{i=}$)

$\subset Y_{p+1}$.

The formula (3) i true as $\tau(p$ and (2) hold.

Now divide Y_{p+1} into two classes Y^1_{p+1} and so that a condition similar to (1) is satisfied. Clearly, $\tau(2^n)$ implies T.

Remark 20: 1) We conclude again the paper by promises: there is a reasonable hope to remove "cornets" from the above proofs. We can do it even so that we are able to prove more general statements concerning point--character of uniformities: (All spaces are assumed not to be 0-dimensional.)
If ξ is a regular infinite cardinal less than and $\alpha \leq |I|$ then the point-character of $\Pi^u_{\alpha^+}(X)_I$ is greater than ξ. Moreover, if $\alpha \geq \omega_1$, $\xi < \alpha$, ξ regular and $\alpha \leq |I|$ then the point-character of $\Pi_{\alpha^+}\{X_i\}_{i\in I}$ is greater than ξ. $\Pi^u_\alpha(X)_I$ ($\Pi_\alpha\{X_i\}_{i\in I}$, resp.) is a uniform space on an underlying set X^I ($\prod_{i\in I} X_i$, resp.) whose base is formed by all covers of the form $\bigcap_{i\in A} \pi_i^{-1}(\mathcal{U})$ ($\bigcap_{i\in A} \pi_i^{-1}(\mathcal{U}_i)$, resp.) where $\mathcal{U}$ (resp. $\mathcal{U}_i$) is a uniform cover of X (X_i, resp.) and A is a subset of I such that $|A| < \alpha$.

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