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Factorization of mappings (products of proximally fine spaces)

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By a space I always mean a uniform space (no separation axiom).

A space is called proximally fine if it is the finest member of a set of all spaces having the same proximity (or equivalently, if any proximally continuous mapping on it is uniformly continuous). It was shown in [E] and [VV] that any proximally continuous mapping defined on a metrizable space is uniformly continuous; consequently, metrizable spaces are proximally fine (this was explicitly stated in [RŠ] and [S]). More generally, [RŠ], any space with a linearly ordered base is proximally fine. The full subcategory of proximally fine spaces is coreflective in *Unif* ([P_1],[P_2] by transfinite induction, [H] by a categorial method) and the coreflection does not preserve proximity, [K] (moreover, there is no nontrivial coreflection in *Unif* preserving proximity - see Remark 2 at the end of this paper).

The problem when a product of proximally fine spaces is proximally fine was treated in [P_1],[P_2],[P_3] and [I]. In a talk in Spring 1973 during his stay in Prague A.W. Hager showed me a nice proof of the result that any product of separable metrizable spaces is proximally fine. The proof went as follows: If f is a proximally continuous mapping defined on a product of separable metrizable spaces and ranging in a metrizable space, then by Mazur's factorization theorem [Ma], f factorizes via a projection onto a countable subproduct; but the countable subproduct is metrizable, hence proximally fine, hence the factorized mapping is uniformly continuous and f, as the composition with the projection, is also uniformly continuous.

The factorization theorem used in the preceding proof was topological in its nature and it was necessary to suppose the factors to be separable (if metrizable). The purpose of this paper is to show that there is another factorization theorem with proximally-uniform character (purely uniform result is known, [V]) by means of which I can prove that any product of metrizable spaces is proximally fine. Then I shall prove two more factorization theorems of the same character and make use of them to prove that any product of spaces with linearly ordered bases is proximally fine and that any product of a proximally fine space with a product of proximally coarse (i.e. precompact) and proximally fine spaces is proximally fine. At the end I shall show that there is a countable topological space X with a unique accumulation point such that the product of the fine uniformity of X with the countable uniformly discrete space is not proximally fine.

By a uniform character (or pseudocharacter) of a space $\langle Y, \mathcal{V} \rangle$ we mean the least cardinal α such that there is a $\mathcal{V}' \subset \mathcal{V}$ with $\operatorname{card} \mathcal{V}' = \alpha$ and such that $\mathcal{V}'$ is a base for $\mathcal{V}$ (or $\bigcap \mathcal{V}' = 1_Y$, resp.). It is clear that the concept of the uniform pseudocharacter has a sense only for Hausdorff spaces; the definition can be stated for all spaces ($\bigcap \mathcal{V}' = \bigcap \mathcal{V}$) but we need the Hausdorff property to be included in it.

If $X \subset \prod_I X_i$, $f : X \longrightarrow Y$, we shall say that f depends on $J \subset I$ (or, if $\operatorname{card} J < \alpha$, on less than α coordinates) or that f factorizes via pr_J / X , if $fx = fy$ whenever $pr_J x = pr_J y$. That means, there exists a $g : pr_J[X] \longrightarrow Y$ such that $f = g \circ pr_{J/X}$, no continuity is required for g . If $pr_{J/X}$ is a quotient (uniform quotient, proximal quotient), which is the case e.g. when $X = \prod_I X_i$, then g is continuous (uniformly continuous, proximally continuous) provided f is. G. Vidossich proved in [V] that if f is uniformly continuous then always there exists a factorization $f = g \circ pr_{J/X}$ with g uniformly continuous and J with cardinality of at most the uniform character of Y (but not any such factorization is uniformly continuous), i.e. the factorisation does not depend on properties of the product, which is a big contradistinction with topological spaces. We want to show here that the proximal case lies in between: if $X = \prod_I X_i$, then the factorization depends only on properties of Y but if $X \subsetneqq \prod_I X_i$ it may depend also on properties of X . The main difference is the fact that in the first case there is the least set $J \subset I$ on which f depends and this set can be easily described - this was probably first stated by A. Miščenko [Mi]; this fact does not hold generally if X is a proper subset of the product (but it holds e.g. if X has a dense set open in the product).

The next lemma is the main assertion in factoring proximally continuous mappings defined on the whole product.

LEMMA 1. Let X_i $(i \in I)$ and Y be spaces, X a subspace of $\prod_I X_i$ and $f: X \longrightarrow Y$ be proximally continuous. Then for any symmetric uniform neighborhood $\mathcal{U}$ of the diagonal 1_Y the set $J_{\mathcal{U}} = \{ i \in I \mid$ there are $x_i, y_i \in X$ such that $\langle fx_i, fy_i \rangle \notin \mathcal{U}$, $pr_{I-(i)} x_i = pr_{I-(i)} y_i \}$ is finite.

Proof. Suppose there is a $\mathcal{U}$ such that $J_{\mathcal{U}}$ is infinite. Find a symmetric uniform neighborhood W of 1_Y such that $W^4 \subset \mathcal{U}$. Then there is an infinite $J \subset J_{\mathcal{U}}$ such that $\langle fx_i, fy_j \rangle \notin W$ for all $i, j \in J$, [E],[VV]. If we put $A = \{x_i \mid i \in J\}$, $B = \{y_i \mid i \in J\}$, then $f[A]$, $f[B]$ are distant in Y but A, B are proximal in X, which is impossible. Indeed, it suffices to prove that for any finite K in I and any uniform neighborhoods V_k of 1_{X_k}, $k \in K$, there are $x \in A$, $y \in B$ such that $\langle pr_k x, pr_k y \rangle \in V_k$ for all $k \in K$ - this is clear since there is $i \in J - K$ and we may put $x = x_i$, $y = y_i$.

PROPOSITION 1. Let X_i $(i \in I)$, Y be spaces and $f: \prod_I X_i \longrightarrow Y$ be a proximally continuous mapping. If the uniform pseudocharacter of Y is less than an infinite cardinal α, then f depends on less than α coordinates.

Proof. Let $\mathcal{V}$ be a set of symmetric uniform neighborhoods of 1_Y with $\operatorname{card} \mathcal{V} < \alpha$, $\bigcap \mathcal{V} = 1_Y$ and put $J = \bigcup \{ J_V \mid V \in \mathcal{V} \}$, where J_V are the sets from Lemma 1. Then, as one can easily prove (see [Mi]), f depends on J and $\operatorname{card} J < \alpha$.

As a direct consequence we get the following result which will be improved in Theorem 2.

PROPOSITION 2. A product of proximally fine spaces is proximally fine iff any countable subproduct is proximally fine.

Proof of the sufficiency. Let X_i $(i \in I)$ be proximally fine spaces and f be a proximally continuous mapping defined on $\prod_I X_i$ and ranging in a metrizable space Y. Then $\prod_I X_i$ is proximally fine iff any such f is uniformly continuous. By Proposition 1 there is a countable $J \subset I$ and a mapping $g : \prod_I X_i \longrightarrow Y$ such that $f = g \circ pr_J$. Since pr_J is a uniform, hence also proximal, quotient, the mapping g is proximally continuous; but by the assumption $\prod_J X_i$ is proximally fine, therefore g is uniformly continuous. Consequently, f is uniformly continuous.

THEOREM 1. Any product of pseudometrizable spaces is proximally fine.

The following Corollaries are connected with J. Vilímovský; the first one answers one of his problems posed in the seminar and the second one was suggested by him after knowing Theorem 1.

COROLLARY 1. Any space can be embedded into a proximally fine space.

COROLLARY 2. Any injective space is proximally fine.

Proof. Embeddings of injective spaces are coretractions.

To proceed further I need another factorization lemma which will be useful for mappings defined on subspaces of products.

LEMMA 2. Let X be a subspace of a product $\prod_{\xi<\alpha} X_\xi$, α be an infinite cardinal and f be a proximally continuous mapping on X into a space Y. Then for any symmetric uniform neighborhood $\mathcal{U}$ of 1_Y there is an $\eta<\alpha$ such that $\langle fx, fy\rangle \in \mathcal{U}$ provided $x, y \in X$ and $pr_\xi x = pr_\xi y$ for all $\xi<\eta$.

Proof. Suppose not. For any $\eta<\alpha$ there are x_η, $y_\eta \in X$ with $pr_\xi x_\eta = pr_\xi y_\eta$ for all $\xi<\eta$ and $\langle fx_\eta, fy_\eta\rangle \notin \mathcal{U}$. If V is a symmetric uniform neighborhood of 1_Y and $V^4 \subset \mathcal{U}$, then there is a cofinal set C in α such that $\langle fx_\xi, fy_\eta\rangle \notin V$ whenever $\xi, \eta \in C$, [RŠ1]. Thus if we put $A = \{x_\xi\}_C$, $B = \{y_\xi\}_C$, then A, B are proximal in X: let $\xi_1, \dots, \xi_n < \alpha$, pick out $\xi \in C$, $\xi > \xi_i$, $i = 1, \dots, n$; then $x_\xi \in A$, $y_\xi \in B$, $pr_{\xi_1,\dots,\xi_n} x_\xi = pr_{\xi_1,\dots,\xi_n} y_\xi$. But $f[A]$, $f[B]$ are distant in Y - a contradiction.

PROPOSITION 3. Let X be a subspace of a product $\prod_{\xi<\alpha} X_\xi$, α be an infinite cardinal and f be a proximally continuous mapping on X into a space Y. If the uniform pseudocharacter of Y is less than $\mathrm{cof}\,\alpha$, then f depends on less than α coordinates.

The next example shows that Proposition 3 cannot be generalized to the form that the factorized mapping is proximally continuous or that f depends on ω_0 coordinates even if Y is uniformly discrete.

EXAMPLE 1. Let Y be a uniformly discrete space of cardinality greater than 2^{ω_0}, X its proximally coarse (precompact) modification and $f: X \longrightarrow Y$ the identity mapping $fx = x$. Embed X into a power of $[0,1]$. The mapping f is proximally continuous; if f depended on a countably many coordinates of the power of $[0,1]$, then the image Y should have at most $(2^{\omega_0})^{\omega_0}$ points. Suppose now that $\mathrm{card}\, Y = \omega_2$, $2^{\omega_2} = \omega_3$, $2^{\omega_0} = \omega_1$. By Proposition 3 the mapping f depends on ω_2 coordinates (we may suppose that X is embedded into $[0,1]^{\omega_3}$). If the factorization is proximally continuous, then again by Proposition 3 the mapping f depends on ω_1 coordinates, but now certainly not proximally continuously because otherwise f would depend on ω_0 coordinates.

Now, we are prepared to generalize Proposition 2:

THEOREM 2. A product of proximally fine spaces is proximally fine iff any finite subproduct is proximally fine.

Proof. By Proposition 2 we have to prove that a countable product $\prod_N X_n$ of ~~proximally~~ fine spaces is proximally fine if $\prod_{n<k} X_n$ is proximally fine for all $k \in N$. Suppose that there is a proximally continuous mapping f on $\prod_N X_n = X$ into a metric space $\langle Y, d \rangle$ which is not uniformly continuous. Hence there exists an $\varepsilon > 0$ such that for any symmetric uniform neighborhood U of 1_X there is $\langle x_U, y_U \rangle \in U$ with $d\langle fx_U, fy_U \rangle \geq \varepsilon$. By Lemma 2 there is a $k \in N$ such that $d\langle fx, fy \rangle < \varepsilon/3$ whenever $pr_n x = pr_n y$ for all $n \leq k$. Let $a_n \in X_n$ for $n > k$, $Z = \prod_{n \leq k} X_n \times (\{a_n \mid n > k\})$, $g = f/Z$. Since

the space Z is proximally fine there is a symmetric uniform neighborhood V of 1_Z such that $d\langle gx, gy\rangle < \varepsilon/3$ provided $\langle x, y\rangle \in V$. Put $\mathcal{U} = \{\langle x, y\rangle \in X \times X \mid \langle pr_Z x, pr_Z y\rangle \in V\}$, $x'_{\mathcal{U}} = pr_Z x_{\mathcal{U}}$, $y'_{\mathcal{U}} = pr_Z y_{\mathcal{U}}$. Then
$d\langle fx_{\mathcal{U}}, fy_{\mathcal{U}}\rangle \leqq d\langle fx_{\mathcal{U}}, fx'_{\mathcal{U}}\rangle + d\langle fx'_{\mathcal{U}}, fy'_{\mathcal{U}}\rangle + d\langle fy'_{\mathcal{U}}, fy_{\mathcal{U}}\rangle <$
$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ - a contradiction.

In the next results on products of proximally fine spaces I shall use the following idea: to prove that the product $X \times Y$ of proximally fine spaces is proximally fine, we embed Y into a product of metrizable spaces and show that any given proximally continuous mapping $f: X \times Y \to Z$, Z metrizable, factorizes via $1_X \times pr$ by means of a mapping g. If we know that g is proximally continuous and $X \times pr[Y]$ is proximally fine, then f is uniformly continuous. In the case when $pr[Y]$ is uniformly discrete, the mapping g is proximally continuous since $1_X \times pr$ is a proximal quotient in this case (if A, B are proximal in $X \times pr[Y]$ then $(1_X \times pr)^{-1}[A]$, $(1_X \times pr)^{-1}[B]$ are proximal in $X \times Y$). Recall that a space having a linearly ordered base is either metrizable or uniformly 0-dimensional.

THEOREM 3. Any product of spaces having linearly ordered bases is proximally fine.

Proof. By Theorem 2 it suffices to prove our theorem for finite products. Let $X = \prod_{i=0}^{n} X_i$, X_i have linearly ordered bases, X_0 be metrizable (e.g. a one-point space) and X_n nonmetrizable if $n > 0$. The proof goes by induction on n : our result is trivial for $n = 0$; suppose it is true for all $k < n$, $n > 0$ and let f be a proximally continuous mapping on X into a metrizable space Y . We must prove that f is uniformly continuous. The uniformity of X_n has a base $\mathcal{U}$ which is well-ordered by inclusion, has a regular cardinality α and is composed of equivalences. Therefore X_n is projectively generated by the canonical mapping into $\prod_{U \in \mathcal{U}} X_n / U$, where X_n / U is the quotient of X_n along U . Clearly, f factorizes via a subspace of $\prod_0^{n-1} X_i \times \prod_{\mathcal{U}} X_n / U$ and by Proposition 3 it depends on less than α coordinates - we may suppose that it depends on $(X_0, \ldots, X_{n-1}) \cup \mathcal{U}'$ for a $\mathcal{U}' \subset \mathcal{U}$, $\operatorname{card} \mathcal{U}' < \alpha$. If $U \in \mathcal{U}, U \subset \bigcap \mathcal{U}'$, then f factorizes via the product $\prod_0^{n-1} X_i \times X_n / U$ and the factorization is proximally continuous (see the paragraph before Theorem 3), hence uniformly continuous by inductive assumption.

Till now we have used factorization lemmas which hold generally and do not take into account special features of our problem. The next two lemmas make use of two such features: the subspace of the product is of the form $X \times Y$ and the investigated proximally continuous mapping is separately uniformly continuous.

LEMMA 3. Let X_i $(i \in I)$ and Y be spaces, X be a subspace of $\prod_I X_i$ and $f : X \times Y \longrightarrow Z$ be a proximally continuous mapping into a Hausdorff space Z . Suppose that there is an infinite dense set T in Y such that for any $t \in T$ the map $f/X \times (t)$ is uniformly continuous and the uniform character of Z is α. Then f depends on $\alpha . \mathrm{card}\, T$ coordinates and the factorization is proximally continuous provided one of the spaces X, Y is proximally coarse.

Proof. By a theorem from [V], for each $t \in T$ there is a set $J_t \subset I$ such that $f/X \times (t)$ depends uniformly continuously on J_t and $\mathrm{card}\, J_t \leqq \alpha$ if α is infinite, J_t is finite if $\alpha = 1$. Put $J = \bigcup_T J_t$; then $\mathrm{card}\, J \leqq \alpha . \mathrm{card}\, T$ and f depends on J, i.e., $f = g \circ (pr_J / X \times 1_Y)$ for a $g : pr_J [X] \times Y \longrightarrow Z$ (indeed, if $x, x' \in X$, $pr_J x = pr_J x'$, $t \in T$, then $f\langle x, t\rangle = f\langle x', t\rangle$ since $J_t \subset J$; if $y \in Y$, then there is a net $\{t_a\}$ in T converging to y and clearly $\langle x, t_a\rangle \longrightarrow \langle x, y\rangle$, $\langle x', t_a\rangle \longrightarrow \langle x', y\rangle$, hence $f\langle x, y\rangle = f\langle x', y\rangle$). To prove that f has a proximally continuous factorization via $pr_J [X] \times Y$ we must show that the extension of f to Samuel compactifications factorizes via the Samuel compactification of $pr_J [X] \times Y$. We know that this is true for $f/pr_J [X] \times (t)$, $t \in T$. If one of the spaces X, Y is proximally coarse, the Samuel compactification of $X \times Y$ or $pr_J [X] \times Y$ is the product of Samuel compactifications and we can prove the required fact in the same way as we proved that f factorized via $pr_J [X] \times Y$.

<u>LEMMA 4.</u> Suppose that a mapping $f : X \times Y \longrightarrow Z$ of spaces is not uniformly continuous but that $f/X \times (y)$ is uniformly continuous for each $y \in Y$. Then there is $Y' \subset Y$ such that $f/X \times Y'$ is not uniformly continu-

ous and $\operatorname{card} Y' \leq \alpha \cdot \beta$, where α is the density of X and β is the uniform character of Y.

Proof. Let T be dense in X of cardinality α and $\mathcal{V}$ be a base of the uniformity of Y of cardinality β. Denote by $\mathcal{U}$ a base of the uniformity of X. There exists a uniform neighborhood W of 1_Z such that for any $U \in \mathcal{U}$, $V \in \mathcal{V}$ there are $\langle x_{UV}, x'_{UV} \rangle \in U$, $\langle y_{UV}, y'_{UV} \rangle \in V$ with $\langle f\langle x_{UV}, y_{UV} \rangle, f\langle x'_{UV}, y'_{UV} \rangle \rangle \notin W$. Since $f/X \times (y_{UV})$, $f/X \times (y'_{UV})$ are uniformly continuous we may suppose that x_{UV}, x'_{UV} belong to T. Pick out a pair $\langle y_{t,t',V}, y'_{t,t',V} \rangle \in Y \times Y$ for $\langle t, t' \rangle \in T \times T$, $V \in \mathcal{V}$ such that for a $U \in \mathcal{U}$ we have $y_{t,t',V} = y_{UV}$, $y'_{t,t',V} = y'_{UV}$ and $t = x_{UV}$, $t' = x'_{UV}$ if such a pair exists. Denote by Y' the set of all $y_{t,t',V}$, $y'_{t,t',V}$; then $\operatorname{card} Y' \leq \alpha \cdot \beta$ and $f/X \times Y'$ is not uniformly continuous; indeed, if $U \in \mathcal{U}$, $V \in \mathcal{V}$, $t = x_{UV}$, $t' = x'_{UV}$, then $\langle t, t' \rangle \in U$, $\langle y_{t,t',V}, y'_{t,t',V} \rangle \in V$ and $\langle f\langle t, y_{t,t',V} \rangle, f\langle t', y'_{t,t',V} \rangle \rangle \notin W$.

Now we are prepared to prove the last main theorem; it generalizes results proved in [P_1] (product of finitely many proximally fine and coarse spaces is proximally fine), [I_1] (any product of proximally fine and coarse spaces is proximally fine) and in [Ků] (product of a proximally fine space with a compact space is proximally fine).

<u>THEOREM 4.</u> Let X_i $(i \in I)$ be proximally fine spaces and all except at most one of them be proximally coarse. Then

$\prod_I X_i$ is proximally fine.

Proof. It suffices to prove that $X \times Y$ is proximally fine provided X and Y are proximally fine and Y is proximally coarse since that, by induction, any finite product of proximally fine and coarse spaces is proximally fine, and, of course, proximally coarse; consequently by Theorem 2, any product of proximally fine and coarse spaces is proximally fine and coarse and the product of this product with a proximally fine space is proximally fine.

Let then X be proximally fine and Y be proximally fine and coarse. Suppose that $f : X \times Y \longrightarrow Z$ is proximally continuous mapping into a complete metrizable space Z ; then f has a proximally continuous extension to $X \times \tilde{Y}$ into Z , where $\tilde{Y}$ is a completion of Y . Thus we may and shall suppose that Y is compact although the next proof is almost the same also for proximally fine and coarse Y . By dP we denote the density and by uP the uniform character of the space P . We shall prove by induction on dX that f is uniformly continuous provided the following condition ($*$) holds:

X is a space, Y is a compact space, Z is a metrizable space and $f : X \times Y \longrightarrow Z$ is a proximally continuous mapping which is uniformly continuous on every $X \times (y)$ for y from a dense set Y' in Y with $\operatorname{card} Y' = dY$.

Our assertion clearly holds if $dX < \omega_0$; suppose it is true for all X with $dX < \alpha$, where α is an uncountable cardinal. Let now $dX = \alpha$ and Y, Z, f

satisfy (*). First let $uY < \alpha$; since $dY \leq uY$, by Lemma 3 f factorizes proximally continuously via $X' \times Y$, where $uX' \leq uY$ and the factorized mapping f' satisfies again (*). If f' is not uniformly continuous, then by Lemma 4 there exists $X'' \subset X'$ such that $f'/X'' \times Y$ is not uniformly continuous and $card\, X'' \leq u Y$ but this is a contradiction with the inductive assumption. If $uY = \alpha$ and $cof\, \alpha \neq \omega_0$, then by Lemma 2 f factorizes via $X \times Y'$ where $uY' < dX$, Y' is an image of Y (i.e. $X \times Y'$ is a proximal quotient [P] and, hence, uniform quotient of $X \times Y$) and again the factorized mapping satisfies (*) and we have the preceding case. If $cof\, \alpha = \omega_0$ and T is dense in X with $card\, T = \alpha$, $T = \bigcup_N T_n$ with $card\, T_n < \alpha$, then $f/T_n \times Y$ is uniformly continuous for each n by the inductive assumption; thus if $Y \subset \prod_I Y_i$, Y_i pseudometrizable compact, $f/T_n \times Y$ factorizes via a countable J_n . Consequently, f factorizes via $\bigcup_N J_n$ (proximally continuously as in the preceding case) and we have again the first case. If, at last, $uY > \alpha$, then by Lemma 3 f factorizes proximally continuously via $X \times Y'$ with $uY' \leq \alpha$ and we obtain the previous cases. The proof is finished.

The following example shows that not any product of proximally fine (even topologically fine) spaces is proximally fine. The spaces can be found countable, one of them uniformly discrete and the other with a unique accumulation point.

EXAMPLE 2. Let P be a space and D (or E) a uniformly discrete (or indiscrete, respectively) two-point space with points c, d . Put $P_U = \Sigma\{E_{\langle p,q\rangle} \mid \langle p,q\rangle \in U - 1_p\} + \Sigma\{D_{\langle p,q\rangle} \mid \langle p,q\rangle \in P \times P - U\}$, where $E_{\langle p,q\rangle} = E$, $D_{\langle p,q\rangle} = D$ and U belongs to a base $\mathcal{U}$ composed of symmetric uniform neighborhoods of 1_p . Finally put $\tilde{P} = \inf\{P_U\}$ and $h\langle\langle p,q\rangle,c\rangle = p$, $h\langle\langle p,q\rangle,d\rangle = q$. Then $h : \tilde{P} \longrightarrow P$ is a quotient mapping ([I_2],p. 52 and [Č],p. 699). Consequently, if P is not proximally fine then $\tilde{P}$ is not proximally fine too and, therefore, there exists a proximally continuous mapping g defined on $\tilde{P}$ which is not uniformly continuous.

Put now X to be a uniformly discrete space with the underlying set $P \times P - 1_p$ and Y the set $(P \times P - 1_p) \cup (x)$, $x \notin P \times P$, endowed with the fine uniformity of the topology having the only accumulation point x with the base of neighborhood system $\{(U - 1_p) \cup (x) \mid U \in \mathcal{U}\}$. We shall prove that if P is not proximally fine then $X \times Y$ is not proximally fine.

We embed $\tilde{P}$ into $X \times Y$ in the following way:

$$i\langle\langle p,q\rangle,c\rangle = \langle\langle p,q\rangle,x\rangle, \quad i\langle\langle p,q\rangle,d\rangle = \langle\langle p,q\rangle,\langle p,q\rangle\rangle$$

and define f on $X \times Y$ into the range of g :

$$f\langle\langle p,q\rangle,\langle p',q'\rangle\rangle = g\langle\langle p,q\rangle,c\rangle \text{ if } \langle p,q\rangle \neq \langle p',q'\rangle$$

$$f\langle\langle p,q\rangle,\langle p,q\rangle\rangle = g\langle\langle p,q\rangle,d\rangle$$

$$f\langle\langle p,q\rangle,x\rangle = g\langle\langle p,q\rangle,c\rangle .$$

We have $f \circ i = g$ and, hence, f is not uniformly continuous and it remains to prove that f is proximally continuous. Suppose that $A, B \subset X \times Y$, A, B are proximal in $X \times Y$ and $f[A], f[B]$ are distant in the range of f. Then for each $U \in \mathcal{U}$ there are $\langle p_U, q_U \rangle \in P \times P - 1_p$ and $a_U, b_U \in Y$ with $\langle\langle p_U, q_U\rangle, a_U\rangle \in A$, $\langle\langle p_U, q_U\rangle, b_U\rangle \in B$, $(a_U, b_U) \subset U \cup (x)$. Clearly, $(f\langle\langle p_U, q_U\rangle, a_U\rangle, f\langle\langle p_U, q_U\rangle, b_U\rangle) = (g\langle\langle p_U, q_U\rangle, c\rangle, g\langle\langle p_U, q_U\rangle, d\rangle)$. Put $A_1 = \{\langle\langle p_U, q_U\rangle, a_U\rangle\}_{\mathcal{U}}$, $B_1 = \{\langle\langle p_U, q_U\rangle, b_U\rangle\}_{\mathcal{U}}$; then A_1, B_1 are proximal in $X \times Y$ and $f[A_1], f[B_1]$ are distant. One of the points $\langle\langle p_U, q_U\rangle, a_U\rangle, \langle\langle p_U, q_U\rangle, b_U\rangle$ must belong to the diagonal 1_X, say $\langle\langle p_U, q_U\rangle, a_U\rangle \in 1_X$ and we denote $\bar{a}_U = a_U$, $\bar{b}_U = x$ and similarly for the other case; then $f\langle\langle p_U, q_U\rangle, \bar{a}_U\rangle = f\langle\langle p_U, q_U\rangle, a_U\rangle$, $f\langle\langle p_U, q_U\rangle, \bar{b}_U\rangle = f\langle\langle p_U, q_U\rangle, b_U\rangle$. If we put $A_2 = \{\langle\langle p_U, q_U\rangle, \bar{a}_U\rangle\}_{\mathcal{U}}$, $B_2 = \{\langle\langle p_U, q_U\rangle, \bar{b}_U\rangle\}_{\mathcal{U}}$, then A_2, B_2 are proximal in $X \times Y$, in fact in $i[\tilde{P}]$, and $f[A_2], f[B_2]$ are distant in the range of f, which is a contradiction because $f \circ i$ is proximally continuous.

REMARKS. By modifying the space $\tilde{P}$ we can obtain examples where X is not uniformly discrete (any uniformity coarser than $\tilde{P}$ and finer than the uniformity projectively generated by h is all right for h to be quotient).

The space $\tilde{P}$ need not be proximally fine but at any case it is proximally minimal, i.e., any strictly finer uniformity induces a strictly finer proximity. Indeed, let W be a uniformizable neighborhood of $1_{\tilde{P}}$

which does not belong to $\mathcal{U}$ and is smaller than the basic neighborhood of $1_{\tilde{P}}$ in $\Sigma\{E_{\langle p,q\rangle} \mid \langle p,q\rangle \in P\times P - 1_P\}$. Then for each $U \in \mathcal{U}$ there exists $\langle p_U, q_U\rangle \in U$ such that $\langle\langle\langle p_U,q_U\rangle, c\rangle, \langle\langle p_U, q_U\rangle, d\rangle\rangle \notin W$. Put $A = \{\langle\langle p_U, q_U\rangle, c\rangle\}_{\mathcal{U}}$, $B = \{\langle\langle p_U, q_U\rangle, d\rangle\}_{\mathcal{U}}$. Then A, B are proximal in $\tilde{P}$ but they are V-distant. This property of $\tilde{P}$ was found when I tried to solve the following problem posed by J. Vilímovský in the seminar: Does there exist a nontrivial coreflection in Unif preserving proximity? Since we have just proved that any space is a quotient of a proximally minimal space, there exists no such coreflection.

By the method used to prove Theorems 3 and 4 one can prove other results on products of proximally fine spaces. E.g., if X has a linearly ordered base and the density of X is α and if Y is proximally fine and the intersection of α uniform neighborhoods of 1_Y is again a uniform neighborhood of 1_Y, then $X \times Y$ is proximally fine.

REFERENCES

[Č] Čech E.: Topological Spaces, Academia Prague, 1966.

[E] Efremovič V.A.: Geometry of proximity (Russian), Mat.Sb.31(1952),189-200.

[F] Friedler L.: Regular maps and product of p-quotient maps, Fund.Math.80(1973),295-303.

[H] Hušek M.: Categorial methods in topology, Proc.2nd Prague Top.Symp.1966(Academia Prague, 1967),190-194.

[I_1] Isbell J.R.: Spaces without large projective subspaces, Math.Scand.17(1965),89-105.

[I_2] - - - - - : Uniform spaces (Amer.Math.Soc.,Providence,1964).

[Ka] Katětov M. Über die Berührungsräume, Wiss.Z.Humboldt Univ.Math.Nat.9(1959),685-691.

[Ků] Kůrková V.: Concerning products of proximally fine uniform spaces, this volume,pp.159-171

[Ma] Mazur S.: On continuous mappings in product spaces, Fund.Math.39(1952),229-238.

[Mi] Miščenko A.: Several theorems on products of topological spaces (Russian), Fund.Math. 58(1966),259-284.

[P_1] Poljakov V.Z.: Regularity, products and spectra of proximity spaces (Russian), Doklady Akad.Nauk SSSR 154(1964),51-54.

[P_2] - - - - - - : Regularity and product of proximity spaces (Russian), Mat.Sb.67(1965), 428-439.

[P_3] - - - - - - : On regularity of a proximal product of regular spaces (Russian), Mat. Sb.68(1965),242-250.

[RŠ] Ramm I., Švarc A.S.: Geometry of proximity, uniform geometry and topology (Russian), Mat.Sb.33(1953),157-180.

[S] Smirnov J.M.: On proximity spaces (Russian), Doklady Akad.Nauk SSSR 84(1952),895-898.

[V] Vidossich G.: Two remarks on a Gleason´s factorization theorem, Bull.Amer.Math.Soc. 76(1970),370-371.

[VV] Vilhelm V., Vitner Č.: Continuity in metric spaces (Czech), Čas.pěst.mat.77(1952),147-173.