Věra Kůrková-Pohlová
Concerning products of proximally fine uniform spaces

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1975. pp. 159–171.

Persistent URL: http://dml.cz/dmlcz/703126

CONCERNING PRODUCTS OF PROXIMALLY FINE UNIFORM SPACES

Věra KŮRKOVÁ-POHLOVÁ

A uniform space is called proximally fine if every proximally continuous mapping from this space to an arbitrary uniform space is uniformly continuous.

In [H] M. Hušek has proved that a product of an arbitrary system of proximally fine spaces is proximally fine if and only if a product of each of its finite subsystems is proximally fine. Thus the study of products of proximally fine spaces reduces to finite products. In [I_1],[I_2] and [P] there has been shown that in some special cases the product of two proximally fine spaces is proximally fine. First M. Hušek in [H] found an example of two proximally fine spaces, the product of which is not proximally fine. Thus the problem arises to characterize the pairs of proximally fine spaces, the product of which is proximally fine. This note is a contribution to this problem. The main result is the following: A product of an arbitrary proximally fine uniform space with a compact uniform space is proximally fine.

I.

By $\mathcal{U}nif$ we denote the category of uniform spaces and uniformly continuous mappings. For a uniform space X we denote by $|X|$ its underlying set, by tX the corresponding topological space, and by δ_X the corresponding proximity relation. By χ_X we denote the character of X, i.e. the smallest infinite cardinal number such that X has a base of its cardinality. If $\mathcal{U}$ is a covering of a set X and $Y \subseteq X$ then put $\mathcal{U}/Y = \{U \cap Y, U \in \mathcal{U}\}$. For arbitrary uniform spaces X, Y we denote by π_X, π_Y the projections $\pi_X : |X \times Y| \to |X|$, $\pi_Y : |X \times Y| \to |Y|$.

Recall from [R - Š] that a uniform space X is proximally fine if and only if X is the finest uniformity on the set X inducing the proximity δ_X.

If ϱ is a pseudometric on a set X, $x \in X$, $\varepsilon > 0$ then we denote by $B_\varrho(x, \varepsilon)$ the open ball with center x and diameter ε. By $\mathcal{B}_\varrho(\varepsilon)$ we denote the covering $\mathcal{B}_\varrho(\varepsilon) = \{B_\varrho(x, \varepsilon), x \in X\}$. For a subset Y of X put $B_\varrho(Y, \varepsilon) = \bigcup_{x \in Y} B_\varrho(x, \varepsilon)$.

For an arbitrary uniform space Y we denote by $\mathcal{P}_Y^*$ the class of all uniform spaces X such that for an arbitrary proximally continuous pseudometric ϱ on $X \times Y$ and arbitrary $\varepsilon > 0$ there exists $\mathcal{U} \in X$ with $\{U \times \{y\}, U \in \mathcal{U}, y \in |Y|\} < \mathcal{B}_\varrho(\varepsilon)$. Put

$\mathcal{P}^* = \bigcap \{\mathcal{P}_Y^*, Y \in \mathrm{Unif}^0\}$.

The reason for the introduction of these notions is clarified by the following easily proved Proposition.

I.1. Proposition. Let X, Y be arbitrary uniform spaces. Then $X \times Y$ is proximally fine if and only if $X \in \mathcal{P}_Y^*$ and $Y \in \mathcal{P}_X^*$.

Proof: If $\{U \times \{y\}, U \in \mathcal{U}, y \in |Y|\} < \mathcal{B}_\rho(\varepsilon)$ and $\{\{x\} \times V, V \in \mathcal{V}; x \in |X|\} < \mathcal{B}_\rho(\varepsilon)$ for some $\mathcal{U} \in X$ and $\mathcal{V} \in Y$ then $\{U \times V, U \in \mathcal{U}, V \in \mathcal{V}\} < \mathcal{B}_\rho(2\varepsilon)$.

We point out an evident consequence of the Proposition.

I.2. Corollary. If $X, Y \in \mathcal{P}^*$ then $X \times Y$ is proximally fine.

II.

At the beginning of this section we shall summarize some results about the class $\mathcal{P}^*$.

First, we generalize a result from [R - Š].

II.1. Lemma. Let ρ be a pseudometric on a set X, $\varepsilon > 0$. Let $\{x_\alpha, \alpha \in \beta\}$ and $\{y_\alpha, \alpha \in \beta\}$ be families of points of X indexed by an ordinal number β such that $\rho(x_\alpha, y_\alpha) \geq \varepsilon$ for every $\alpha \in \beta$. Then there exists a cofinal subset I of β such that $\rho(x_\alpha, y_{\alpha'}) > \frac{\varepsilon}{4}$ for every $\alpha, \alpha' \in I$.

Proof: If for some $\alpha \in \beta$ either $I_\alpha = \{\gamma \in \beta, \rho(x_\alpha, y_\gamma) \leq \frac{\varepsilon}{4}\}$ or $J_\alpha = \{\gamma \in \beta, \rho(x_\gamma, y_\alpha) \leq \frac{\varepsilon}{4}\}$ is a cofinal subset of β we are ready since $\rho(x_\gamma, y_{\gamma'}) > \frac{\varepsilon}{2}$ for $\gamma, \gamma' \in I_\alpha$ or $\gamma, \gamma' \in J_\alpha$. Suppose that for every $\alpha \in \beta$ neither I_α nor J_α is a cofinal subset of β. Then we can construct by induction a cofinal set $I \subseteq \beta$ such that $\gamma \notin I_\alpha \cup J_\alpha$ whenever $\alpha < \gamma, \alpha, \gamma \in I$. Evidently $\rho(x_\alpha, y_\gamma) > \frac{\varepsilon}{4}$ for every $\alpha, \gamma \in I$.

II.2. Proposition. Let X be a uniform space with a linearly ordered base. Then $X \in \mathcal{P}^*$.

Proof: Let $(\mathcal{B}, <<)$ be a linearly ordered base for X. For an arbitrary uniform space Y consider a proximally continuous pseudometric ρ on $X \times Y$. Suppose that there is some $\varepsilon > 0$ with $\{U \times \{y\}, U \in \mathcal{U}, y \in |Y|\} \nprec \mathcal{B}_\rho(\varepsilon)$ for every $\mathcal{U} \in \mathcal{B}$. Then we have $a_{\mathcal{U}}, b_{\mathcal{U}} \in |X|$, $y_{\mathcal{U}} \in |Y|$ for every $\mathcal{U} \in \mathcal{B}$ with $\rho(\langle a_{\mathcal{U}}, y_{\mathcal{U}}\rangle, \langle b_{\mathcal{U}}, y_{\mathcal{U}}\rangle) > \varepsilon$ and $a_{\mathcal{U}} \in \mathrm{St}(b_{\mathcal{U}}, \mathcal{U})$. By II.1 there exists a cofinal subset $\mathcal{B}'$ of $\mathcal{B}$ with $\rho(\langle a_{\mathcal{U}}, y_{\mathcal{U}}\rangle, \langle b_{\mathcal{V}}, y_{\mathcal{V}}\rangle) > \frac{\varepsilon}{4}$ for every $\mathcal{U}, \mathcal{V} \in \mathcal{B}'$. Since $\mathcal{B}'$ is a cofinal subset of $\mathcal{B}$ it follows that $\{\langle a_{\mathcal{V}}, y_{\mathcal{V}}\rangle, \mathcal{V} \in \mathcal{B}'\} \delta_{X \times Y} \{\langle b_{\mathcal{V}}, y_{\mathcal{V}}\rangle, \mathcal{V} \in \mathcal{B}'\}$. This contradicts the assumption that ρ is a proximally continuous pseudometric on $X \times Y$.

A topological space is called a generalized sequential space if to every point of a closure of its arbitrary subset there converges a net of points of this subset indexed by an ordinal number. Poljakov (see [P]) proved that every uniform space X with tX compact generalized sequential is in the class $\mathcal{P}^*$. The following is a generalization of his result.

II.3. Proposition. Let X be a topologically fine uniform spaces. If tX is a generalized sequential topological space then $X \in \mathcal{P}^*$.

Proof: For an arbitrary uniform space Y consider a proximally continuous pseudometric ϱ on $X \times Y$. For $\varepsilon > 0$ put $\mathcal{U}_\varepsilon = \bigwedge_{y \in Y} \pi_X (\mathcal{B}_\varrho(\varepsilon) / (X \times \{y\}))$. Clearly, $\mathcal{U}_\varepsilon$ is a normal covering of the set X. If $\mathcal{U}_\varepsilon$ is an interior covering of tX for every $\varepsilon > 0$, we have $\mathcal{U}_\varepsilon \in X$ and hence $X \in \mathcal{P}^*$. Suppose that $\mathcal{U}_\varepsilon$ is not an interior covering of tX for some ε. Then there exists $x \in |X|$ such that for every neighbourhood σ of x we have $x_\sigma \in \sigma$ and $y_\sigma \in |Y|$ with $\varrho(\langle x_\sigma, y_\sigma \rangle, \langle x, y_\sigma \rangle) \geq \varepsilon$. In virtue of the fact that tX is a generalized sequential space, we can choose from points x_σ a net $\{x_\alpha, \alpha \in \beta\}$, indexed by an ordinal β, converging to x. By II.1 there exists a cofinal subset $I \subseteq \beta$ with $\varrho(\langle x_\alpha, y_\alpha \rangle, \langle x, y_{\alpha'} \rangle) > \frac{\varepsilon}{4}$ for every α, $\alpha' \in I$. But this is a contradiction with the proximal continuity of ϱ, since
$\{\langle x_\alpha, y_\alpha \rangle, \alpha \in I\} \, \delta_{X \times Y} \{\langle x, y_\alpha \rangle, \alpha \in I\}$.

Notice that the proofs of both Proposition II.2 and II.3 are based on the same idea. In the following we develop quite a different method for a verification that a uniform space is in the class $\mathcal{P}^*$.

II.4. Lemma. Let X be a proximally fine space and Y an arbitrary uniform space. If $X \notin \mathcal{P}^*_Y$ then there exists a proximally continuous pseudometric ϱ on $X \times Y$, $\varepsilon > 0$ and a non-void $Z \subseteq X$ such that

$\bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, \varepsilon) \cap (X \times \{y\}))$ is not a δ_X-neighbourhood of Z.

Proof: Consider a proximally continuous pseudometric ρ on $X \times Y$ and $\eta > 0$ such that $\{U \times \{y\}, U \in \mathcal{U}, y \in |X|\} \nprec \mathcal{B}_\rho(\eta)$ for every $\mathcal{U} \in X$. Put $\mathcal{V}_n = \bigwedge_{y \in Y} \pi_X(\mathcal{B}_\rho(\frac{\eta}{2^n}) / (X \times \{y\}))$. Clearly, $\{\mathcal{V}_n, n \in \omega_0\}$ forms a normal sequence of coverings of the set $|X|$ with $\mathcal{V}_n \notin X$ for every $n \in \omega_0$. If $\mathcal{V}_n$ are interior coverings of tX for all $n \in \omega_0$ then in virtue of the fact that X is proximally fine, we have a non-void $Z \subseteq X$ such that $\mathrm{St}(Z, \mathcal{V}_n)$ is not a δ_X-neighbourhood of the set Z for every $n \in \omega_0$. If $\mathcal{V}_n$ is not an interior covering of tX for some $n \in \omega_0$ then there exists $x \in |X|$ such that $\mathrm{St}(x, \mathcal{V}_{n+1})$ is not a neighbourhood of x in tX. Hence $\mathrm{St}(Z, \mathcal{V}_{n+1})$ is not a δ_X-neighbourhood of the set $Z = \{x\}$. Thus $\bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, \frac{\eta}{2^{n+1}}) \cap (X \times \{y\}))$ is not a δ_X-neighbourhood of Z, too.

A topological space is called α-compact, with α being an arbitrary cardinal, if every system of its non-void subsets with finite intersection property of the power at most α has a cluster point.

II. 5. Lemma. Let X be a uniform space and Y a proximally fine space such that tX is a χ_Y-compact topological space. If $Y \notin \mathcal{P}_X^*$ then there exists a proximally continuous pseudometric ρ on $X \times Y$, $\varepsilon > 0$ and non-void set $Z \subseteq X$ such that $\bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, \varepsilon) \cap \cap (X \times \{y\}))$ is not a δ_X-neighbourhood of Z.

Proof: Consider a proximally continuous pseudometric ρ on $X \times Y$, $\eta > 0$ such that $\{\{x\} \times V, x \in |X|, V \in \mathcal{V}\} \not< \not< \mathcal{B}_\rho(\eta)$ for every $\mathcal{V} \in Y$. Let $\mathcal{L}$ be a base for Y of the power at most χ_Y. Put

$X_{\mathcal{V}} = \{x \in |X|, (\exists y, y' \in |Y|)(y' \in \mathrm{St}(y, \mathcal{V}) \& \rho(\langle x, y\rangle, \langle x, y'\rangle) \geq \eta)\}$

for every $\mathcal{V} \in \mathcal{L}$. Then $\{X_{\mathcal{V}}, \mathcal{V} \in \mathcal{L}\}$ is a system of non-void sets with the finite intersection property. χ_Y-compactness of tX guarantees the existence of some $z \in \bigcap_{\mathcal{V} \in \mathcal{L}} \overline{X}_{\mathcal{V}}^{tX}$. Suppose that $\mathcal{O} = \bigcap_{y \in Y} \pi_X(B_\rho(\langle z, y\rangle, \frac{\eta}{4}) \cap \cap (X \times \{y\}))$ is a neighbourhood of z in tX. Then for every $\mathcal{V} \in \mathcal{L}$ we have $x_{\mathcal{V}} \in \mathcal{O}$ and $y_{\mathcal{V}}, y'_{\mathcal{V}} \in Y$ such that $y'_{\mathcal{V}} \in \mathrm{St}(y_{\mathcal{V}}, \mathcal{V})$ and $\rho(\langle x_{\mathcal{V}}, y_{\mathcal{V}}\rangle, \langle x_{\mathcal{V}}, y'_{\mathcal{V}}\rangle) \geq \eta$. By the triangle inequality it follows $\rho(\langle z, y_{\mathcal{V}}\rangle, \langle z, y'_{\mathcal{V}}\rangle) \geq \geq \frac{\eta}{2}$ for every $\mathcal{V} \in \mathcal{L}$. As the restriction of ρ on $\{z\} \times Y$ there is a proximally continuous pseudometric on the proximally fine space $\{z\} \times Y$, we have $\mathcal{W} \in \mathcal{L}$ with $\{z\} \times \mathcal{W} < \mathcal{B}_\rho\left(\frac{\eta}{4}\right) / (\{z\} \times Y)$. Then $\rho(\langle z, y_{\mathcal{W}}\rangle, \langle z, y'_{\mathcal{W}}\rangle) < \frac{\eta}{2}$ which is a contradiction. Thus $\mathcal{O}$ is not a neighbourhood of z in tX, and hence $\bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, \frac{\eta}{4}) \cap (X \times \{y\}))$ is not a δ_X-neighbourhood of the set $Z = \{z\}$.

II.6. Theorem. If X is a proximally fine separated uniform space such that tX is a countably compact topological space then $X \in \mathcal{P}^*$.

If X is a separated uniform space and Y a proximally fine uniform space such that tX is a χ_Y-compact topological space then $Y \in \mathcal{P}^*_X$.

Proof: Suppose that in the first case $X \notin \mathcal{P}^*_Y$ for some uniform space Y and in the latter one $X \notin \mathcal{P}^*_Y$ for the mentioned Y. Then by the preceding Lemmas we have in both cases a proximally continuous pseudometric ρ on $X \times Y$, $\varepsilon > 0$ and non-void $Z \subseteq X$ such that $\bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, 2\varepsilon) \cap (X \times \{y\}))$ is not a δ_X-neighbourhood of Z. We shall construct by induction a sequence $\{X_m, m \in \omega_0\}$ of subsets of X such that $Z \delta_X X_m$ (hence $X_m \neq 0$) for every $m \in \omega_0$ and an infinite sequence $\{x_m, m \in \omega_0\}$ of points $x_m \in |X|$ and a sequence $\{y_m, m \in \omega_0\}$ of points $y_m \in |Y|$ such that $\rho(\langle x_m, y_m \rangle, Z \times \{y_m\}) \geq 2\varepsilon$ for every $m \in \omega_0$. Put $X_0 = X - \bigcap_{y \in Y} \pi_X(B_\rho(Z \times \{y\}, 2\varepsilon) \cap (X \times \{y\}))$, we have $Z \delta_X X_0$. Choose arbitrarily $x_0 \in X_0$ and $y_0 \in Y_0$ such that $\rho(\langle x_0, y_0 \rangle, Z \times \{y_0\}) \geq 2\varepsilon$. If we have X_i, x_i, y_i with the required properties for every $i < m$, put

$X_m = X_0 - \bigcup_{i<m} \pi_X(B_\rho(\langle x_i, y_i \rangle, \varepsilon) \cap (X \times \{y_i\}))$. Since ρ is a proximally continuous pseudometric on $X \times Y$ and $\rho(\langle x_i, y_i \rangle, Z \times \{y_i\}) \geq 2\varepsilon$ it follows that $(Z \times \{y_i\}) \neg\delta_{X \times Y} (B_\rho(\langle x_i, y_i \rangle, \varepsilon) \cap (X \times \{y_i\}))$, and hence

$Z \neg \delta_X \pi_X(B_\rho(\langle x_i, y_i\rangle, \varepsilon) \cap (X \times \{y_i\}))$ for every $i < n$. As $Z \,\delta_X\, X_0$ we have $Z \,\delta_X\, X_n$. Then we can choose $x_n \in X_n$ and $y_n \in Y$ such that $\rho(\langle x_n, y_n\rangle, Z \times \{y_n\}) \geq \varepsilon$. So we obtain an infinite sequence $M = \{x_n, n \in \omega_0\} \subseteq X$ and a sequence $\{y_n, n \in \omega_0\} \subseteq Y$ such that $B_\rho(\langle x_n, y_n\rangle, \varepsilon) \cap (M \times \{y_n\}) \subseteq \{x_1, \dots, x_n\} \times \{y_n\}$ for every $n \in \omega_0$. We shall show that we can suppose that $\rho(\langle x_n, y_n\rangle, \langle x_m, y_m\rangle) \geq 2\varepsilon$ whenever $n \neq m, n, m \in \omega_0$. Put
$I_n = \{i \in \omega_0, \rho(\langle x_i, y_i\rangle, \langle x_n, y_n\rangle) \leq \frac{\varepsilon}{4}\}$, $M = \{y_i, i \in I_n\}$

for every $n \in \omega_0$. Let us prove that M_n has no cluster point in tX for any $n \in \omega_0$ by contradiction. Consider $x \in X$ such that $x \in \overline{M_n - \{x\}}^{tX}$ for some $n \in \omega_0$. Then

$$\{\langle x_i, y_i\rangle, x_i \in M_n - \{x\}\} \,\delta_{X \times Y}\, (\{x\} \times \{y_j, x_j \in M_n - \{x\}\}).$$

As ρ is a proximally continuous pseudometric on $X \times Y$ we have $i, j \in I_n$ with $x_i, x_j \neq x$ and
$\rho(\langle x_i, y_i\rangle, \langle x, y_j\rangle) < \frac{\varepsilon}{2}$. Then $\rho(\langle x_j, y_j\rangle, \langle x, y_j\rangle) \leq$

$$\leq \rho(\langle x_j, y_j\rangle, \langle x_n, y_n\rangle) + \rho(\langle x_n, y_n\rangle, \langle x_i, y_i\rangle) + \rho(\langle x_i, y_i\rangle, \langle x, y_j\rangle) < \varepsilon,$$

and hence $B_\rho(\langle x_j, y_j\rangle, \varepsilon)$ is an open neighbourhood of $\langle x, y_j\rangle$ in $t(X \times Y)$. As
$\langle x, y_j\rangle \in \overline{(M_n - \{x\}) \times \{y_j\}}^{t(X \times Y)}$ we have

$$\langle x, y_j\rangle \in \overline{B_\rho(\langle x_j, y_j\rangle, \varepsilon) \cap ((M_n - \{x\}) \times \{y_j\})}^{t(X \times Y)}.$$

Since tX is a T_1 topological space and $B_\rho(\langle x_j, y_j\rangle, \varepsilon) \cap ((M_n - \{x\}) \times \{y_j\})$ is finite we have a contradiction. Thus, since tX is countably compact and

M is infinite, we have $\bigcup_{n \in K} I_n \neq \omega_0$ for every finite $K \subseteq \omega_0$. We can construct an infinite subsequence $\{n_i, i \in \omega_0\}$ of ω_0 by induction: $n_0 = 0, \dots,$ $n_k = \min(\omega_0 - \bigcup_{i<k} I_{n_i}), \dots$. Then $\varrho(\langle x_{n_k}, y_{n_k}\rangle, \langle x_{n_i}, y_{n_i}\rangle) \geq \frac{\varepsilon}{4}$ whenever $i \neq k$, $k \in \omega_0$. If we replace $\frac{\varepsilon}{8}$ by ε, $\{x_{n_i}, i \in \omega_0\}$ by $\{x_n, n \in \omega_0\}$ and $\{y_{n_i}, i \in \omega_0\}$ by $\{y_n, n \in \omega_0\}$ we obtain the sequences required.

Further, for $I \subseteq \omega_0$ put $M_I = \{x_i, i \in I\}$. We shall verify that for every $I \subseteq \omega_0$ there exists a finite $K \subseteq I^2$ such that $\bigcup_{\langle n,m\rangle \in K} \pi_X(B_\varrho(\langle x_n, y_n\rangle, \varepsilon) \cap (M_I \times \{y_m\})) = M_I$. As tX is countably compact it suffices to check that $\bigcup_{\langle n,m\rangle \in I^2} \pi_X(B_\varrho(\langle x_n, y_n\rangle, \varepsilon) \cap (X \times \{y_m\})) \supseteq \overline{M}_I^{tX}$. Since $\{\langle x_n, y_n\rangle, n \in I\}\, \delta_{X\times Y}\, \{\langle x, y_m\rangle, m \in I\}$ for every $x \in \overline{M}_I^{tX}$ and ϱ is proximally continuous on $X \times Y$ it follows that there exist $n, m \in I$ with $\varrho(\langle x_n, y_n\rangle, \langle x, y_m\rangle) < \varepsilon$.

Now, put $A_{n,m} = \pi_X(B_\varrho(\langle x_n, y_n\rangle, \varepsilon) \cap (X \times \{y_m\}))$. Let us summarize the properties of $\{A_{n,m}, \langle n,m\rangle \in \omega_0^2\}$:

(i) $A_{n,n}$ is finite for every $n \in \omega_0$

(ii) $A_{n,k} \cap A_{m,k} = 0$ whenever $n \neq m$, $n, m \in \omega_0$

(iii) for every $I \subseteq \omega_0$ there exists a finite $K \subseteq I^2$ such that $\bigcup_{\langle n,m\rangle \in K} A_{n,m} \supseteq M_I$.

It is proved in [P] that no such system of sets exists. (We recall the idea of the proof: By (iii) we have an infinite A_{n_0,m_0} for some $n_0, m_0 \in \omega_0$ by (i) $n_0 \neq m_0$. Similarly we obtain $n_1 \neq m_1$ such that $n_1, m_1 \in$ $\in A_{n_0,m_0} - \langle 0, n_0 + m_0 \rangle$ and $A_{n_1,m_1} \cap (A_{n_0,m_0} - \langle 0, n_0+m_0 \rangle)$ is infinite. So by induction we can construct sequences $\{n_i, i \in \omega_0\}$ and $\{m_i, i \in \omega_0\}$ such that $n_0 \neq m_0 <$ $< \ldots\ldots < n_i \neq m_i < \ldots$ and $n_i, m_i \in A_{n_k,m_k}$ whenever $i \geq$ $\geq k$. For $I = \{m_i, i \in \omega_0\}$ we have by (iii) a finite $K \subseteq I^2$ with $\bigcup_{\langle m_i, m_j \rangle \in K} A_{m_i,m_j} \supseteq I$. But by (ii) $A_{m_i,m_j} \cap A_{n_j,m_j} = 0$ and since $m_k \in A_{n_j,m_j}$ for every $k \geq j$ it follows that $A_{m_i,m_j} \cap I$ is finite for every $i, j \in \omega_0$. This is a contradiction.)

II.7. Corollary. If X and Y are proximally fine uniform spaces such that tX is χ_Y-compact T_2 topological space then $X \times Y$ is proximally fine.

Proof: By II.6 $X \in \mathcal{P}^*$ and $Y \in \mathcal{P}^*_X$; hence by I.1 $X \times Y$ is proximally fine.

II.8. Corollary. If X is a proximally fine uniform space and Y a separated uniform space with tY being a compact topological space then $X \times Y$ is proximally fine.

II.9. Remark. At the end of the proof of II.5 we used a trick which was used by Poljakov (see [P]) in his proof that $N \times \beta N$ is the finest zerodimensional uniform space with the underlying set $|N \times \beta N|$ inducing the proximity $\delta_{N \times \beta N}$ (N denotes the discrete uniform space with $|N| = \omega_0$; a uniform space is called zerodimensional if it has a base formed by partitions). From our results there follows that $N \times \beta N$ is proximally fine. Thus every uniformity on the set $|N \times \beta N|$ inducing the proximity $\delta_{N \times \beta N}$ is zerodimensional. This solves the Poljakov's problem from [P]: a big proximal dimension of the proximity $\delta_{N \times \beta N}$ is equal to zero.

References:

[E] Efremovič V.A.: Geometry of proximity (Russian), Mat. Sb.31(1952),189-200

[I_1] Isbell J.R.: Uniform spaces, Amer.Math.Soc.(1964)

[I_2] Isbell J.R.: Spaces without large projective subspaces, Math.Scand.17(1965),89-105

[H] Hušek M.: Factorization of mappings (products of proximally fine spaces) (ibid)

[P] Poljakov V.Z.: On regularity of a proximal product of regular spaces (Russian), Mat.Sb.68(1965),242-250

[R - Š] Ramm I., Švarc A.S.: Geometry of proximity, uniform geometry and topology (Russian), Mat.Sb.33(1953),157-180

[V - V] Vilhelm V., Vitner Č.: Continuity in metric spaces (Czech),Čas.pěst.mat.77(1952),147-173