Vojtěch Rödl

Fineness in the category of all 0-dimensional uniform spaces

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1975. pp. 139–143.

Persistent URL: http://dml.cz/dmlcz/703123

Fineness in the category of all 0-dimensional uniform spaces

V. Rödl

Let α be an infinite cardinal. Let $(X,\mathcal{U})$ be a uniform space. Define $p^{\alpha}\mathcal{U} \subset \mathcal{U}$ by $p^{\alpha}U = \{V_0 \mid V_0 \in \mathcal{U}$ & & $(\exists \{V_i\}_{i=1}^{\infty} \subset \mathcal{U} \quad V_{i+1} \lesseqgtr V_i , \quad i = 0,1,2,\dots)$ & & $(\text{card } V_i < \alpha \quad , \quad i = 0,1,2,\dots)\}$. Clearly p^{α} defines a reflection from UNIF into UNIF .

Definition: Let $\mathcal{K}$ be a subcategory of UNIF , α be an infinite cardinal.

1) A uniform space $(X,\mathcal{U}) \in \mathcal{K}$ is said to be $\mathcal{K}-\alpha$-simple if for each uniform space $(X,\mathcal{V}) \in \mathcal{K}$, $p^{\alpha}\mathcal{U} = p^{\alpha}\mathcal{V}$ implies that $id_X: (X,\mathcal{U}) \longrightarrow (X,\mathcal{V})$ is uniformly continuous.

2) A uniform space $(X,\mathcal{U}) \in \mathcal{K}$ is said to be $\mathcal{K}-\alpha$-fine if for each uniform space $(Y,\mathcal{V}) \in \mathcal{K}$ a mapping $f: (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is uniformly continous if and only if $f: (X, p^{\alpha}\mathcal{U}) \longrightarrow (Y, p^{\alpha}\mathcal{V})$ is uniformly continuous.

Convention: We shall write α-simple (α-fine) instead of UNIF α-simple (UNIF α-fine resp.).

α-fineness (α-simplicity) implies $\mathcal{K}-\alpha$-fineness ($\mathcal{K}-\alpha$-simplicity resp.) for any $\mathcal{K} \subset$ UNIF . Obviously a uniform space $(X,\mathcal{U})$ is $\mathcal{K}-\alpha$-simple if it is $\mathcal{K}-\alpha$-fine.

Theorem [25 B 22] in [1] asserts that the converse is true for $\alpha = \omega_c$. This theorem is extended in [2] to all inaccessible cardinals α . The question what situation occurs for other cardinals α remains open. As I know there is a conjecture due to Frolík, Hušek, Hager, Kůrková and others that for

α not inaccessible such a theorem fails to be true. A possible example which could support this feeling for $\alpha = \omega_1$ was mentioned in [2]. Using the method of this construction we are going to construct spaces $(X_\alpha, \mathcal{U}_\alpha)$; $\alpha > \omega_0$ not inaccessible which are not $\mathcal{O}-\alpha$-fine but they are $\mathcal{O}-\alpha$-simple ($\mathcal{O}$ is a category of all zerodimensional uniform spaces).

Construction: Let α, β, γ be infinite cardinals such that $\alpha > \beta > \gamma$ and $\prod_{\iota \in \delta} \xi_\iota < \alpha$ for each $\delta < \gamma$ and $\{\xi_\iota\} < \alpha$. We define a uniform space $(X(\alpha, \beta, \gamma),$ $\mathcal{U}(\alpha, \beta, \gamma))$ by the following way: $X(\alpha, \beta, \gamma) = \alpha \times \beta$ and $\mathcal{U}(\alpha, \beta, \gamma)$ is a set of all partitions $\{\mathcal{U}_a\}_{a \in A}$ of $\alpha \times \beta$ such that there is a partition $\{R_b\}_{b \in B}$ of α, card $B < \alpha$ and for each $b \in B$ there is $M_b \subset \beta$, card $M_b < \gamma$ and a partition $\{V_c^b\}_{c \in C_b} = V^b$ of R_b, card $C_b < \alpha$ such that a partition

$$V = \bigcup_{b \in B} \{\{(x,y)\} \mid x \in R_b \,\&\, y \in M_b\} \cup \bigcup_{b \in B} \{\{V_c^b \times \{y\}\} \mid c \in C_b \,\&\, y \in \beta - M_b\}$$ refines $\{\mathcal{U}_a\}_{a \in A}$.

$X(\alpha, \beta, \gamma)$ is obviously a zerodimensional uniform space.

Proposition: Let α, β, γ be like in the construction. Denote by $D(k)$ a uniformly discrete space of cardinality k. Then: 1) $p^\alpha X(\alpha, \beta, \gamma) = p^\alpha D(\alpha) \times D(\beta)$

2) $X(\alpha, \beta, \gamma)$ is not α-fine.

Proof: 1) Consider that $\beta < \alpha$ and $\prod_{\iota \in \delta} \xi_\iota < \alpha$ for each $\delta < \gamma$ and $\{\xi_\iota\} \subset \alpha$.

2) Take pr: $\alpha \times \beta \longrightarrow \alpha$ and $D(\alpha)$.

Theorem: Let α be a cardinal such that either there exist $\xi < \alpha$ and $\tau < \mathrm{cf}\,\alpha$ with $\xi^\tau \geq \alpha$ or $\mathrm{cf}\,\alpha < \alpha$

(i.e. α is not inaccessible). Then there are cardinals β, γ such that $X(\alpha, \beta, \gamma)$ is defined by the construction and it is $\sigma - \alpha$ -simple.

Proof: I) Suppose there is $\xi < \alpha$ and $\tau < \mathrm{cf}\, \alpha$ with $\xi^{\tau} \geq \alpha$ (this case involves isolated cardinals). Put $\gamma = \min \{ \tau \mid \text{there is } \xi < \alpha \text{ such that } \xi^{\tau} \geq \alpha \}$. Clearly $\gamma < \mathrm{cf}\, \alpha$ hence $\prod_{\iota \in \sigma} \xi_{\iota} < \alpha$ for each $\sigma < \gamma$ and $\{\xi_2\} \subset \alpha$. Put $\beta = \min \{ \xi \mid \alpha > \xi \geq \omega_0 \ \& \ \xi^{\gamma} \geq \alpha \}$. We shall prove now that $X(\alpha, \beta, \gamma)$ is $\sigma - \alpha$ -simple. Take P a partition of $\alpha \times \beta$. Denote a uniformity generated by $\mathcal{U}(\alpha, \beta, \gamma) \cup \{P\}$ by $\mathcal{V}$. We are going to show that either $\mathcal{V} = \mathcal{U}(\alpha, \beta, \gamma)$ or $p^{\alpha} \mathcal{V} \neq p^{\alpha} \mathcal{U}(\alpha, \beta, \gamma) = p^{\alpha} D(\alpha) \times D(\beta)$. Let us suppose that $p^{\alpha} \mathcal{V} = p^{\alpha} \mathcal{U}(\alpha, \beta, \gamma)$. Put $P^k = P/\alpha \times \{k\}$ $k \in \beta$ we may suppose that $P = \bigcup_{k \in \beta} P^k$.

There are two possibilities:

1) $\exists M \subset \alpha$, $\mathrm{card}\, M < \alpha$ $\forall x \in \alpha$ $\exists J \subset \beta$, $\mathrm{card}\, J < \gamma$
$\forall k \in \beta - J$ $(x, k) \in \mathrm{st}(M \times \{k\}, P^k)$

2) $\forall M \subset \alpha$, $\mathrm{card}\, M < \alpha$ $\exists x \in \alpha$ $\exists K \subset \beta$, $\mathrm{card}\, K \geq \gamma$
$\forall k \in K$ $(x, k) \notin \mathrm{st}(M \times \{k\}, P^k)$.

Case 1) Take $M \subset \alpha$ from the formula.

For $J \subset \beta$, $\mathrm{card}\, J < \gamma$ define $R_J = \{x \,;\, x \in \alpha \ \& \ (x,k) \notin \mathrm{st}(M \times \{k\}, P^k) \text{ for each } k \in J) \ \& \ ((x,k) \in \mathrm{st}(M \times \{k\}, P^k) \text{ for each } k \in \beta - J)$

$\{R_J\}$ is a partition of α which has at most $\sum_{\tau < \gamma} \beta^{\tau}$ equivalence classes and $\sum_{\tau < \gamma} \beta^{\tau} < \alpha$.

Case 2) In this case, we shall find a subset S of α

such that card $S = \alpha$ and there is a partition $\{T_\iota\}_{\iota \in I} = T$ of $\alpha \times \beta$, card $I < \alpha$ refined by P such that for any two distinct points x, y of S there is $k \in \beta$ such that $(x,k) \notin st((y,k), T^k)$ (T^k has the same sense as P^k above) and it will contradict the assumption that $p^\alpha \mathcal{V} \geq p^\alpha \mathcal{U}(\alpha, \beta, \gamma)$

T will be such that $T = \bigcup_{k \in \beta} T^k$ and $T^k = \{\bigcup f_k^{-1}(q)\}_{q \in \beta}$ where f_k is a mapping from P^k into β.

We construct simultaneously S and f_k's by transfinite induction.

The ξ-th induction step: Suppose we have defined $s_\delta \in \alpha$ and partial mappings $\varphi_\delta^k : P^k \longrightarrow \beta$ for $\delta < \xi$ such that:

1) $\delta_1 \leq \delta_2 < \xi$ & $\mathcal{U} \in P^k$ & $\varphi_{\delta_1}^k(\mathcal{U})$ is defined then $\varphi_{\delta_1}^k(\mathcal{U}) = \varphi_{\delta_2}^k(\mathcal{U})$

2) φ_δ^k is defined for $\mathcal{U} \in P^k$ iff there is s_ν, $\nu \leq \delta$ such that $s_\nu \in \mathcal{U}$

3) $\delta_1 < \delta_2 < \xi$ then there is $k \in \beta$ such that $\varphi_k^{\delta_i}(\mathcal{U}_1) \neq \varphi_k^{\delta_i}(\mathcal{U}_2)$ where $s_{\delta_i} \in \mathcal{U}_i \in P^k$, $i = 1,2$.

A set $M = \{s_\delta, \delta < \xi\}$ has cardinality less than α. There exist $x \in \alpha$ and $k \in \beta$, card $k \geq \gamma$ such that $(x,k) \notin st(M \times \{k\}, P^k)$ for each $k \in K$. Put $x = s_\xi$. As card $K > \gamma$, $\beta^\gamma \geq \alpha$ and card $M < \alpha$ there is a mapping $\psi : K \longrightarrow \beta$ such that for any $\nu < \xi$ there is $k \in K$ such that $\psi(k) \neq \varphi_\nu^k(\mathcal{U})$ $s_\nu \in \mathcal{U} \in P^k$. For $k \in K$ define $\varphi_\xi^k(\mathcal{U}) = \psi(k)$. For other $\mathcal{U}'$s and k's define φ_ξ^k's in any way which does not contradict 1), 2), 3) (it can be done). Proof of I is finished.

II) If $\xi^{\tau} < \alpha$ for each $\xi < \alpha$ and $\tau < \mathrm{cf}\,\alpha$ ($\mathrm{cf}\,\alpha < \alpha$) then we can proceed quite analogously like in I) (for choice $X(\alpha, \beta, \gamma)$ put $\beta = \mathrm{cf}\,\alpha$, $\gamma = \mathrm{cf}\,\alpha$) except the discussed possibilities which are following.

1) $\exists\, M \subset \alpha$, card $M < \alpha$ $\exists\, \tau_M < \mathrm{cf}$ $\forall\, x \in \alpha$

$\exists\, J \subset \mathrm{cf}\,\alpha$ card $J \leq \tau_M$ $\forall\, k \in J - \mathrm{cf}\,\alpha : x \in$
$\in \mathrm{st}\ (M, P^k)$

2) $\forall\, M$ card $M < \alpha$ $\forall\, \tau < \mathrm{cf}\,\alpha$ $\exists\, x \in \alpha$ $\exists\, K$

card $K > \tau$

$\forall\, k \in K$: $x \notin \mathrm{st}\ (M, P^k)$

(f_k is to be defined as a mapping from P^k into δ_k where $\{\delta_k\}_{k \in \mathrm{cf}\,\alpha}$ is an increasing transfinite sequence such that $\sup \delta_k = \alpha$).

Acknowledgment. I am grateful to J. Pelant for his kind help during the final preparation of this paper.

References:

[1] Čech E.: Topological spaces (revised by Z. Frolík and M. Katětov), Academia, Prague, 1966.

[2] Kůrková V.: Fine and bijectively fine uniform spaces, this volume, p. 127-137.