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In [F] Z. Frolík introduced the notion of a refinement of a category and of fine and coarse objects with respect to this refinement. These notions have served him as a useful tool in his study of reflective and coreflective subcategories of the category of uniform spaces. In Vilímovský's paragraph [V] ibid a general theory of refinements of a category is developed and some special results valid in the category of uniform spaces are derived. We suppose that the reader is familiar with the notions from Vilímovský's paragraph [V]. We recall here only that for any modification κ from any concrete category $\mathcal{K}$ to its subcategory we can define a refinement, say $\mathcal{R}$, of $\mathcal{K}$ by putting

$\mathcal{R}(X,Y) = \{f \in \mathit{Set}(|X|,|Y|),\ \kappa_Y \circ f \in \mathcal{K}(X,Y)\}$.

By $|X|$ we denote the underlying set of an object X and by $\kappa_Y : Y \longrightarrow \kappa Y$ the modification of Y. Set is the category of sets and mappings, Set_1 the category of sets and bijective mappings. Similarly, Unif is the category of uniform spaces and uniformly continuous mappings, Unif_1 the category of uniform spaces and bijective uniformly continuous mappings. We use an abbreviated notation $\mathcal{U}(X,Y)$ instead of $\mathit{Unif}(X,Y)$.

In this note we turn our attention to the refinements of Unif and of Unif_1 given, in the way described above, by an arbitrary given modification $\varkappa$ of *Unif* and its restriction $\varkappa_1$ to Unif_1. We study the interrelation of classes $\mathcal{R}^f$ of $\mathcal{R}$-fine spaces and $\mathcal{R}_1^f$ of bijectively $\mathcal{R}$-fine spaces.

An easy reformulation of definitions yields this description of classes $\mathcal{R}^f$, $\mathcal{R}_1^f$:

$$\mathcal{R}^f = \{X \in \mathit{Unif}^0, (\forall Y \in \mathit{Unif}^0)(f \in \mathcal{U}(X, \varkappa Y) \Longrightarrow f \in \mathcal{U}(X, Y)\} .$$

$$\mathcal{R}_1^f = \{X \in \mathit{Unif}^0, (\forall Y \in \mathit{Unif}^0)(|X| = |Y| \mathbin{\&} \varkappa X = \varkappa Y) \longrightarrow X \leq Y)\} .$$

Obviously $\mathcal{R}^f \subseteq \mathcal{R}_1^f$ for every modification $\varkappa$. The converse inclusion fails to be generally true, as two examples given in this note illustrate. Thus the problem arises for which modifications $\varkappa$ of *Unif* $\mathcal{R}_1^f = \mathcal{R}^f$. We state one sufficient condition for a modification $\varkappa$ so that $\mathcal{R}_1^f = \mathcal{R}^f$. The author conjectures that this condition is far from being necessary, but no counterexample is known. This leads to, maybe, a difficult problem, whether there exists a cardinal reflection $\varkappa$ such that $\mathcal{R}^f \neq \mathcal{R}_1^f$. This problem is partially discussed at the end of our note and one interesting infinite combinatorial problem is given there.

I.

Let $\varkappa$ be the zerodimensional modification, i.e. for every $X \in \mathit{Unif}^0$, $\varkappa X$ has for its base all the partitions $\mathcal{V} \in X$.

I.1. Lemma. If $\varkappa$ is the zerodimensional modification then every member of $\mathcal{R}^f$ is a discrete uniform space.

Proof: For any $X \in \mathcal{R}^f$ denote by J_X the subspace of $\mathcal{L}_X(|X|)$ (see [F]) with the underlying set $|J_X| = \{\varepsilon \vec{x}, 0 \leq \varepsilon \leq 1, x \in |X|\}$. Define a mapping $f_X : |X| \longrightarrow |J_X|$ by $f_X(x) = 1\vec{x}$ for every $x \in |X|$. Since, clearly $\varkappa J_X$ is an indiscrete uniform space, it follows that $f_X \in \mathcal{U}(X, \varkappa J_X)$. Then $f_X \in \mathcal{U}(X, J_X)$ for $X \in \mathcal{R}^f$. Hence X is a discrete uniform space.

A uniform space X is called an atom if it is a minimal element in the set of all non-discrete uniform spaces with the underlying set $|X|$. ($X \leq Y$ if X is finer than Y.)

I.2. Lemma. Let $\varkappa$ be the zerodimensional modification and X be an atom with $\varkappa X = X$. Then $X \in \mathcal{R}_1^f$.

Proof: Let $\varkappa X = \varkappa Y$ for some uniform space Y with $|Y| = |X|$. Since $\varkappa X = X$, it follows that $Y \leq X$. As X is not discrete and $\varkappa Y = \varkappa X$, Y is not discrete, too. Thus $Y = X$.

I.3. Corollary. Let $\varkappa$ be the zerodimensional modification then $\mathcal{R}^f \neq \mathcal{R}_1^f$.

Proof: By I.1 and I.2 it suffices to find a zerodimensional atom. As an example of such a space, the following atom described in [P - R] can serve: Let $\mathcal{F}$ be an arbitrarily chosen ultrafilter on the set ω_0, $F \in \mathcal{F}$.

Put

$\mathcal{U}_F = \{\{m\} \times 2, m \in F\} \cup \{\{\langle m, 0\rangle\}, m \in \omega_0 - F\} \cup \{\{\langle m, 1\rangle\}, m \in \omega_0 - F\}$.

Then $\{\mathcal{U}_F, F \in \mathcal{F}\}$ forms a base for a zerodimensional uniform space X with the underlying set $|X| = \omega_0 \times 2$. For the verification that X is an atom see [P - R].

II.

Recall that a covering of a set is called star-finite if each of its members meets only a finite number of the others. All star-finite uniform coverings of any uniform space form a base for a uniformity (see [I]). It is evident that this change of a uniformity is functorial - it is called the star-finite modification.

If ϱ is a pseudometric on a set X then we denote by $B_\varrho(x, \varepsilon)$ the open ball with center x and diameter ε. $\mathcal{B}_\varrho(\varepsilon)$ denotes the covering $\{B_\varrho(x, \varepsilon), x \in X\}$. Sometimes we write shortly $B(x, \varepsilon)$ and $\mathcal{B}(\varepsilon)$.

By p is denoted the precompact modification of $\mathcal{U}nif$, i.e. pX has for its base all the finite uniform coverings of X.

$\mathcal{P}$ denotes the corresponding refinement of $\mathcal{U}nif$; spaces from $\mathcal{P}^f$ are called proximally fine.

II.1. Lemma. Let Y be an arbitrary uniform space. Then every uniformly continuous real-valued function on

Y is bounded if and only if every star-finite uniform covering of Y is finite.

Proof: To verify the sufficiency consider a uniformly continuous real-valued function f on Y. Evidently $f^{-1}(\{B(n, 1), n \in Z\})$ (where Z denotes the set of all integers) is a star-finite uniform covering of Y. Clearly, if $f^{-1}(\{B(n, 1), n \in Z\})$ is finite then f is bounded.

To prove the necessity suppose that there exists an infinite star-finite uniform covering $\mathcal{U}$ of Y. Choose arbitrarily $U_0 \in \mathcal{U}$. Put $U_1 = \bigcup\{U \in \mathcal{U}, U \cap U_0 \neq 0\}$. Since $\mathcal{U}$ is star-finite it follows that $\mathcal{U}_1 = \{U \in \mathcal{U},$ $U \cap U_0 = 0\}$ is infinite. Put $U_2 = \bigcup\{U \in \mathcal{U}_1, U \cap U_1 \neq 0\}$. Again, $\mathcal{U}_2 = \{U \in \mathcal{U}_1, U \cap U_1 = 0\}$ is infinite. So, by induction, we obtain an infinite covering $\{U_n, n \in \omega_0\} >$ $> \mathcal{U}$ with $U_n \cap U_m = 0$ whenever $|n-m| > 1$. A countable covering indexed by integers with this property is called linear. Every linear uniform covering of any uniform space is realized by a real-valued function (see [F - H]).
So we have a uniformly continuous function f on Y with $f^{-1}(\{B(n, 1), n \in Z\}) < \{U_n, n \in \omega_0\}$. Since f is bounded and $\{U_n, n \in \omega_0\}$ linear, it follows that $\{U_n, n \in \omega_0\}$ is finite. This is a contradiction.

II.2. Proposition. Let $\varkappa$ be the star-finite modification, N be a discrete uniform space with $|N| = \omega_0$ and Y be an arbitrary proximally fine uniform space with $\varkappa Y = p Y \neq Y$. Then there exists $X \in \mathcal{R}_1^f - \mathcal{R}^f$

with $\pi X = p Y \times N$.

Proof: If Y is proximally fine and $\pi Y = pY$ then $Y \in \mathcal{R}_1^f$. Namely, for every Y' with $|Y'| = |Y|$ and $\pi Y' = \pi Y = pY$ we have $pY' = pY$, and so $Y = Y'$.

Let X be a uniform space with $|X| = |Y|$ containing just those coverings $\mathcal{U}$ of $|Y| \times \omega_0$ for which there exist a finite $K \subseteq \omega_0$ and $\mathcal{U}_0 \in pY = \pi Y$ such that, denoting $\mathcal{U} / |Y| \times \{i\} = \mathcal{U}_i \times \{i\}$, we have $\mathcal{U}_i \in Y$ for every $i \in \omega_0$ and $\mathcal{U}_i > \mathcal{U}_0$ for every $i \notin K$. Clearly, $\pi X = \pi Y \times N = \pi Y \times N$. We shall verify that $X \in \mathcal{R}_1^f$.

First, let us show that every Z with $|Z| = |Y| \times \omega_0$ and $\pi Z = pY \times N$, has a base of some coverings of the form $\bigcup_{i \in \omega_0} \mathcal{V}_i \times \{i\}$, with $\mathcal{V}_i \in Y$ for every $i \in \omega_0$. $Z \leq pY \times N$ guarantees that $\{|Y| \times \{i\}, i \in \omega_0\} \in Z$. Denote by Z_i a subspace of Z with $|Z_i| = |Y| \times \{i\}$. Then, as $\pi Z = \pi Y \times N$, we have $\pi Z_i = \pi Y \times \{i\}$ .Hence $Z_i \geq Y \times \{i\}$.

Now, suppose that there exists some Z with $\pi Z = \pi X = pY \times N$ and $Z \not\leq X$. Consider some $\mathcal{V} \in Z - X$. Then the set I of all $i \in \omega_0$ such that $\mathcal{V}_i$ cannot be refined by a uniform covering of Y is infinite. Let ρ be a uniformly continuous pseudometric on Z with $\mathcal{B}_\rho(1) < \mathcal{V}$. For $i \in I$ let X_i be a maximal subset of $|Y|$ such that $\rho(\langle x, i\rangle, \langle y, i\rangle) > 1$ for every $x, y \in X_i$. Let $\mathcal{W} \in Z$ with $\mathcal{W} = \bigcup_{i \in \omega_0} \mathcal{W}_i \times \{i\}$, $\mathcal{W}_i \in Y$ for every $i \in \omega_0$ such that $\mathcal{W} < \mathcal{B}_\rho\left(\frac{1}{2}\right)$. Since X_i is infinite, we can choose distinct points $x_1^i, \ldots, x_i \in X_i$

for every $i \in I$. Put $\overline{\mathcal{W}}_i = \{St(\mathcal{W}_i, x_1^i), \dots, St(\mathcal{W}_i, x_i^i),$ $\bigcup\{W \in \mathcal{W}, (\forall k = 1,\dots,i)(x_k^i \notin W)\}\}$ and

$\overline{\mathcal{W}} = (\bigcup_{i\in I} \overline{\mathcal{W}}_i \times \{i\}) \cup (\bigcup_{i\in \omega_0 - I} |Y| \times \{i\})$.

Clearly, $\overline{\mathcal{W}}$ is star-finite and as $\mathcal{W} < \overline{\mathcal{W}}$, we have $\overline{\mathcal{W}} \in \kappa Z$. Since $\{St(\mathcal{W}_i, x_k^i), k = 1,\dots,i\}$ is a disjoint system of sets for every $i \in I$, it follows that $\overline{\mathcal{W}} \notin pY \times N$. This is a contradiction.

Denote by π the projection $\pi : |Y| \times \omega_0 \longrightarrow |Y|$. Then, clearly, $\pi \in \mathcal{U}(X, \kappa Y) - \mathcal{U}(X, Y)$; hence $X \notin \mathcal{R}^f$. So we established $X \in \mathcal{R}_1^f - \mathcal{R}^f$.

II.3. Corollary. Let κ be the star-finite modification of $Unif$, then $\mathcal{R}^f \neq \mathcal{R}_1^f$.

Proof: By II.2 and II.1 it suffices to find some proximally fine uniform space Y with $pY \neq Y$ such that every uniformly continuous real-valued function on Y is bounded. Since every metric uniform space is proximally fine (see [I]), any non precompact metric space Y such that every uniformly continuous real-valued function on Y is bounded, will do. E.g. the subspace of $\ell_1(\omega_0)$ with the underlying set $\{\varepsilon \vec{n}, 0 \leq \varepsilon \leq 1, n \in \omega_0\}$ (see [F]).

III.

III.1. Proposition. Let κ be a modification of $\mathcal{Unif}$ fulfilling the following two conditions:
(a) If $f: X \longrightarrow Y$ is a projectively generating mapping between arbitrary uniform spaces X, Y then $f: \kappa X \longrightarrow \kappa Y$ is projectively generating, too.

(b) $\kappa(X \wedge \kappa Y) = \kappa X \wedge \kappa Y$ for arbitrary uniform spaces X, Y with $|X| = |Y|$.

Then $\mathcal{R}^f = \mathcal{R}_1^f$.

Proof: Let $X \in \mathcal{R}_1^f$ and $f \in \mathcal{U}(X, \kappa Z)$ for some Z. Denote by X' the uniform space projectively generated by $f: |X| \longrightarrow Z$. As $f: \kappa X' \longrightarrow \kappa Z$ is a projectively generating mapping, we have $X \leq \kappa X'$. Hence $\kappa(X' \wedge \kappa X) = \kappa X' \wedge \kappa X = \kappa X$. Since $X \in \mathcal{R}_1^f$, it follows that $X \leq X' \wedge \kappa X$ and so $f \in \mathcal{U}(X, Z)$.

For every infinite cardinal α denote by p^α the modification of $\mathcal{Unif}$ such that $p^\alpha X$ contains all the coverings $\mathcal{U} \in X$ for which there exists a normal sequence of coverings $\mathcal{U}_n \in X$, with $\operatorname{card} \mathcal{U}_n < \alpha$ for every $n \in \omega_0$ and $\mathcal{U} > \mathcal{U}_0$. It is known that under the assumption of the Generalized Continuum Hypothesis $p^\alpha X = \{\mathcal{U} \in X \mid (\exists \mathcal{V} \in X)(\mathcal{V} < \mathcal{U} \,\&\, \operatorname{card} \mathcal{V} < \alpha)\}$ for every infinite cardinal α and every $X \in \mathcal{Unif}^0$. It is usual to write p instead of $p^{\aleph_0}$.

It is easy to check that every cardinal reflection p^α fulfils the condition (a). The characterization of the cardinal reflections fulfilling the condition (b) is given by the following generalization of the known fact on p .

III.2. Proposition. (GCH) Let α be an infinite cardinal then $p^{\alpha}(X \wedge p^{\alpha} Y) = p^{\alpha} X \wedge p^{\alpha} Y$ for arbitrary uniform spaces X, Y with $|X| = |Y|$ if and only if $2^{\beta} < \alpha$ for every cardinal $\beta < \alpha$.

Proof: Evidently $p^{\alpha}(X \wedge p^{\alpha} Y) \leq p^{\alpha} X \wedge p^{\alpha} Y$. To prove the converse inequality consider some covering $\mathcal{U} = \{U_i, i \in \beta\}$ with $\beta < \alpha$ such that $\mathcal{U} > \mathcal{V} \wedge \mathcal{W}$, where $\mathcal{V} \in X$ and $\mathcal{W} \in \{W_j, j \in \gamma\} \in Y$ with $\gamma < \alpha$. Then for every $V \in \mathcal{V}$ choose some mapping $\varphi_V : \gamma \longrightarrow \beta$ such that $V \cap W_j \subseteq U_{\varphi_V(j)}$ for every $j \in \gamma$. For a mapping $\varphi : \gamma \longrightarrow \beta$ put $V_{\varphi} = \bigcup\{V \in \mathcal{V}, \varphi_V = \varphi\}$, $\tilde{\mathcal{V}} = \{V_{\varphi}, \varphi \in \mathrm{Set}(\gamma, \beta)\}$. Since $card\, \tilde{\mathcal{V}} = \beta^{\gamma} < \alpha$, we have (under GCH) $\tilde{\mathcal{V}} \in p^{\alpha} X$. Evidently $\mathcal{U} > \tilde{\mathcal{V}} \wedge \mathcal{W}$.

Now, assume that there is some $\beta < \alpha$ with $2^{\beta} \geq \alpha$, i.e. there exists a one-to-one mapping $\psi : \alpha \longrightarrow \mathrm{Set}(\beta, 2)$. Denote by X the coarsest uniform space with $|X| = \alpha \times \beta$ containing the covering $\{\{i\} \times \beta, i \in \alpha\}$, and by Y the coarsest uniform space with the same underlying set containing the covering $\{\alpha \times \{j\}, j \in \beta\}$. Put

$U_0 = \{\langle i, j\rangle \in \alpha \times \beta, \psi(i)(j) = 0\}$, $U_1 = \{\langle i, j\rangle \in \alpha \times \beta, \psi(i)(j) = 1\}$.

Then $\mathcal{U} = \{U_0, U_1\} \in p^{\alpha}(X \wedge Y) = p^{\alpha}(X \wedge p^{\alpha} Y)$. Since ψ

is one-to-one, it follows that $\mathcal{U} \notin p^{\alpha} X \wedge p^{\alpha} Y$. Thus $p^{\alpha}(X \wedge p^{\alpha} Y) \neq p^{\alpha} X \wedge p^{\alpha} Y$.

Thus, many cardinal reflections do not fulfil the condition (b). But nothing more is known. E.g. for $\kappa = p^{\aleph_1}$. A. Hager suggested this interesting problem: Denote by N, M discrete uniform spaces with underlying sets ω_0, ω_1. Does there exist $X \in \mathcal{R}_1^f$ with $|X| = \omega_0 \times \omega_1$ and $p^{\aleph_1} X = N \times p^{\aleph_1} M$? If there would exist such a space, then $\mathcal{R}_1^f \neq \mathcal{R}^f$ since evidently the projection $\pi : X \longrightarrow M$ is not uniformly continuous but $\pi \in \mathcal{R}$.

This leads to the following infinite combinatorial problem: Let a system of equivalences $\{\varrho_n, n \in \omega_0\}$ on the set ω_1 be given such that for every distinct x, $y \in \omega_1$ the set $\{n \in \omega_0, \langle x,y\rangle \notin \varrho_n\}$ is infinite. Do there always exist equivalences $\{\tilde{\varrho}_n, n \in \omega_0\}$ on ω_1 fulfilling the following three conditions:

(1) $\varrho_n \subseteq \tilde{\varrho}_n$ for every $n \in \omega_0$

(2) $\tilde{\varrho}_n$ has only countably many classes for every $n \in \omega_0$

(3) there exists an uncountable $X \subseteq \omega_1$ such that for every distinct x, $y \in X$ there is some $n \in \omega_0$ with $\langle x,y\rangle \notin \tilde{\varrho}_n$?

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