Jan Pelant; Jiří Reiterman

Atoms in uniformities

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1975. pp. 73–81.

Persistent URL: http://dml.cz/dmlcz/703119

ATOMS IN UNIFORMITIES

Pelant J., Reiterman J.

Introduction . Everybody knows that all topologies on a given set form a lattice with respect to the order $\preccurlyeq$ ("finer than"). One can describe easily atoms in this lattice, i.e. the finest topologies which are strictly coarser than the discrete topology on a given set: these atoms contain the only non-isolated point and reduced neighborhoods of this point form an ultrafilter. As all uniformities on a given set X also form a lattice, one can expect the question about atoms in this lattice. The aim of this note is to show that more complicated situation occurs in this "uniform" case.

2. Existence of atoms. If $\mathcal{U}$ is a uniformly non-discrete uniformity then there is an atom $\mathcal{A}$ with $\mathcal{A} \preccurlyeq \mathcal{U}$. Indeed, it is easily seen that the set of all uniformly non-discrete uniformities which are finer than $\mathcal{U}$ satisfies assumptions of Zorn's lemma with respect to the order $\preccurlyeq$. Minimal elements of this set are just the atoms which are finer than $\mathcal{U}$.

Let us introduce two examples of atoms. The former is trivival, the latter is due to P. Simon.

Example 1: Let $D \subset X$, $\operatorname{card} D = 2$. All covers of X which are coarser than the cover $\{D\} \cup \{\{x\} ; x \in X\}$ form a uniformity $\mathcal{A}_0$ which is an atom. (Proof is obvious.)

Example 2: Let $X = X_1 \cup X_2$ be a decomposition of an infinite set into two disjoint equipotent subsets. Let $f_i : X \longrightarrow X_i$, $i = 1, 2$ be bijections. Let $\mathfrak{U}$ be an ultrafilter on X. Then covers

$$\{\{f_1(u), f_2(u)\};\ u \in U\} \cup \{\{x\};\ x \in X\},\ (U \in \mathfrak{U}),$$

form a basis for a uniformity $\mathcal{A}_{\mathfrak{U}}$ which is an atom. (To be proved below.)

3. Ultraproducts of atoms. Let $\mathfrak{U}$ be an ultrafilter on a set I. Let a uniformity $\mathcal{M}_i$ on a set X_i be given for each $i \in I$. Let us assume for convenience that the sets X_i are pairwise disjoint. Then the covers

$$\{\{x\};\ x \in \bigcup X_i\} \cup \bigcup_{i \in U} P_i \quad (U \in \mathfrak{U},\ P_i \in \mathcal{M}_i)$$

form a base for a uniformity. We shall write $\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i$ and we shall say that $\mathcal{M}$ is an ultraproduct of $\mathcal{M}_i$. If $\mathcal{M}_i = \mathcal{N}$ for each $i \in I$, we shall simply write $\mathcal{M} = \mathcal{N}^{\mathfrak{U}}$.

Proposition: If $\mathcal{M}_i$ are atoms so is $\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i$.

Proof: Let $\mathcal{N} \prec \mathcal{M}$. Put $J = \{i \in I,\ \mathcal{N}/X_i = \mathcal{M}_i\}$. As clearly $\mathcal{N}/X_i \leq \mathcal{M}_i$ for each i, $\mathcal{N}/X_i$ is uniformly discrete for $i \in I - J$. If $J \in \mathfrak{U}$ then $\mathcal{N} = \mathcal{M}$. If $J \notin \mathfrak{U}$ then $I - J \in \mathfrak{U}$ hence $\mathcal{N}$ is discrete.

Corollary: $\mathcal{A}_{\mathfrak{U}}$ is an atom, for $\mathcal{A}_{\mathfrak{U}} = \mathcal{A}_0^{\mathfrak{U}}$ as easily seen (see Example 1).

4. Proximally non-discrete atoms. A uniformity is proximally discrete if it induces the discrete proximity, i.e. if it contains all finite covers; equivalently, all partitions into two sets.

The atom $\mathfrak{A}_{\mathfrak{U}}$ is proximally non-discrete, for the sets X_1, X_2 are proximal but disjoint (see Example 2). We shall show that it is the only example (note that $\mathfrak{A}_0 = \mathfrak{A}_{\mathfrak{U}}$ where $\mathfrak{U}$ is a trivial ultrafilter):

Proposition: $\mathfrak{M}$ is a proximally non-discrete atom iff $\mathfrak{M}$ is isomorphic to $\mathfrak{A}_{\mathfrak{U}}$ for some ultrafilter $\mathfrak{U}$.

Proof: The part "if" has been proved above. Let $\mathfrak{M}$ be proximally non-discrete. Then there exist two disjoint proximal sets A_1, A_2. We may assume that $\{A_1, A_2\}$ is a cover. As $\{A_1, A_2\} \notin \mathfrak{M}$, the uniformity

$$\mathfrak{N} = \{\{\mathfrak{U}_i \cap A_1\}_{i\in I} \cup \{\mathfrak{U}_i \cap A_2\}_{i\in I} ; \{\mathfrak{U}_i\}_{i\in I} \in \mathfrak{M}\}$$

is strictly finer than $\mathfrak{M}$. If $\mathfrak{M}$ is an atom then $\mathfrak{N}$ is discrete. Thus there is a cover $\mathfrak{U} = \{\mathfrak{U}_i\}_{i\in I} \in \mathfrak{M}$ such that $\{\mathfrak{U}_i \cap A_1\}_{i\in I} \cup \{A_2$ is a cover by one point sets. Then each $\mathfrak{U}_i$ must be at most two points set: if it is the case one of these points must be im A_1 and the other in A_2. The same applies to any $\mathcal{V} \prec \mathfrak{U}$. Let $\mathcal{V} \in \mathfrak{M}$ star-refine $\mathfrak{U}$; then $\mathcal{V}$ is a partition (if a cover is not a partition then $\mathrm{St}\,\mathcal{V}$ contains a set with at least three elements). We may assume that $A_1, A_2, \mathcal{V}, X$ are pairwise equipotent. Thus there are bijections $f_i : X \longrightarrow A_i$, $i = 1, 2$ such that

$$\mathcal{V} = \{\{x\} ; x \in C\} \cup \{\{f_1(x), f_2(x)\} ; x \in D\}$$

for some $C, D \subset X$. For each $W \prec \mathcal{V}$ put $F_W \{x \in X ; \{f_1(x), f_2(x)\} \in W\}$. Obviously $F_W \cap F_Z = F_{W\cap Z}$ for any $W, Z \prec \mathcal{V}$. Hence the sets F form a base of a filter $\mathfrak{R}$ on X. If $\mathfrak{F}$ is an ultrafilter, $\mathfrak{F} \supset \mathfrak{R}$, then clearly

$\mathfrak{A}_{\mathfrak{F}} \preccurlyeq \mathfrak{M}$. As $\mathfrak{M}$ is an atom, $\mathfrak{A}_{\mathfrak{F}} = \mathfrak{M}$.

5. <u>Proximally discrete atoms.</u> Let $\mathfrak{F}$ be a filter on X . Define a uniformity $\mathfrak{U}_{\mathfrak{F}}$ on X as follows: A cover $\{U_i\}_{i \in I}$ belongs to $\mathfrak{U}_{\mathfrak{F}}$ iff there is $i \in I$ with $U_i \in \mathfrak{F}$. If $\mathfrak{F}$ is a principal ultrafilter then $\mathfrak{U}_{\mathfrak{F}}$ is uniformly discrete. Otherwise $\mathfrak{U}_{\mathfrak{F}}$ is a proximally discrete zero-dimensional uniformly non-discrete uniformity.

$\mathfrak{U}_{\mathfrak{F}}$ need not be an atom, even if $\mathfrak{F}$ is an ultrafilter. However, each proximally discrete atom refines some $\mathfrak{U}_{\mathfrak{F}}$:

<u>Proposition:</u> For any proximally discrete atom $\mathfrak{M}$ there is an ultrafilter such that $\mathfrak{M} \preccurlyeq \mathfrak{U}_{\mathfrak{F}}$.

Proof: Denote $N_{\mathfrak{M}} = \{A \subset X ; \mathfrak{M}/A$ is not uniformly discrete $\}$.

Then

$$(*) \qquad A \subset B , \ A \in N_{\mathfrak{M}} \Longrightarrow B \in N_{\mathfrak{M}} .$$

Let $\mathfrak{F}$ be a maximal $N_{\mathfrak{M}}$ -filter, i.e. a maximal family $\mathfrak{F} \subset N_{\mathfrak{M}}$ such that

(i) for any $A, B \in \mathfrak{F}$ there is $C \in \mathfrak{F}$ with $C \subset A \cap B$,

(ii) if $A \in \mathfrak{F}$ and $B \in N_{\mathfrak{M}}$, $B \supset A$, then $B \in \mathfrak{F}$.

According to $(*)$, $\mathfrak{F}$ forms a filter in the ordinary sense. Moreover, $\mathfrak{F}$ is an ultrafilter. Indeed, let $A \subset X$. Consider the following two cases: 1) $A \cap F \, N_{\mathfrak{M}}$ for any $F \in \mathfrak{F}$. Put

$\mathcal{R} = \{A \cap F ; F \in \mathcal{F}\}$. Then $\mathcal{R}$ is a $N_{\mathfrak{m}}$-filter and $\mathcal{R} \supset \mathcal{F}$, so that $\mathcal{R} = \mathcal{F}$ because of the maximality of $\mathcal{F}$. Thus $A \in \mathcal{F}$.

2) $A \cap F \notin N_{\mathfrak{m}}$ for some $F \in \mathcal{F}$. Then $A \cap G \notin N_{\mathfrak{m}}$ for each $G \in \mathcal{F}$, $G \subset \mathcal{F}$. Then necessarily $(N-A) \cap G \in N_{\mathfrak{m}}$ for these G for the property " $K \notin N_{\mathfrak{m}}$, $L \notin N_{\mathfrak{m}} \Longrightarrow$ $\Longrightarrow K \cup L \notin N_{\mathfrak{m}}$ " characterizes proximally discrete uniformities (here we put $K = A \cap G$, $L = (X - A) \cap G$ so that $K \cup L = G$). Now apply 1) changing only A to $X - A$ to obtain $X - A \in \mathcal{F}$.

To finish the proof, let us observe that $\mathfrak{M} \wedge \sigma_{\mathcal{F}}$ is uniformly non-discrete. As $\mathfrak{M}$ is an atom, $\mathfrak{M} \wedge \sigma_{\mathcal{F}} = \mathfrak{M}$, i.e. $\mathfrak{M} \prec \sigma_{\mathcal{F}}$.

Let us note that any atom $\mathfrak{M}$ which is an ultraproduct of another one cannot be of the form $\sigma_{\mathcal{F}}$, hence $\sigma_{\mathcal{F}}$ where $\mathfrak{M} \prec \sigma_{\mathcal{F}}$ is not an atom. However, we shall construct an ultrafilter $\mathcal{F}$ on a countable set N such that $\sigma_{\mathcal{F}}$ is an atom. Further we shall prove that such ultrafilters coincide with the selective ultrafilters. (The continuum hypothesis is assumed for the construction.)

<u>Construction</u>: Let $\{\rho_m, m < 2^{\aleph_0}\}$ be a well-ordered set of all pseudometrics on a countable set N; let ρ_0 be the 0-1-metric. We shall construct an increasing sequence $\mathcal{F}_m$ of filters ($m < 2^{\aleph_0}$) such that, for each m either $\mathcal{F}_m$ is ρ_m-Cauchy or $\mathcal{F}_m$ contains a ρ_m-uniformly discrete set. Then clearly $\mathcal{F} = \bigcup_m \mathcal{F}_m$ is an ultrafilter such that $\sigma_{\mathcal{F}}$ is an atom.

1) Put $\mathcal{F}_0 = \{A \subset N ; N - A$ is finite$\}$.

2) Let $\mathcal{F}_i$ have been defined for $i < n$ such that each $\mathcal{F}_i$ has a countable base. Then $\mathcal{F}'_n = \bigcup_{i<n} \mathcal{F}_i$ has also a countable base, say $\{A_i\}_{i=1}^{\infty}$. We may assume $A_1 \supsetneqq A_2 \supsetneqq$ $\supsetneqq A_3 \supsetneqq \dots$. Choosing $a_i \in A_i - A_{i+1}$, $i = 1, 2, 3, \dots$, we get a set $A = \{a_i\}_{i=1}^{\infty}$ such that each infinite subset B of A meets every member of $\mathcal{F}'_n$.

Lemma: If $(X, \wp)$ is an infinite pseudometric space then there is an infinite $B \subset X$, $B = \{x_i\}_{i=1}^{\infty}$ such that either $\{x_i\}_{i=1}^{\infty}$ is a $\wp$-Cauchy sequence or B is a $\wp$-uniformly discrete subset.

Proof is routine.

Now apply the preceding lemma to $(A, \wp/_A)$. Put $\mathcal{F}_n =$ $= \{F \subset X;$ there is $C \in \mathcal{F}'_n$ with $F \supset G \cap B\}$. Then $\mathcal{F}_n$ is either $\wp_n$-Cauchy or contains a $\wp_n$-uniformly discrete subset according as the former or the latter case in the lemma takes place. $\mathcal{F}_n$ has again a countable base and $i < n \Longrightarrow \mathcal{F}_i \subset \mathcal{F}_n$.

An ultrafilter $\mathcal{F}$ on N is said to be selective iff for any partition $\{U_i\}_{i \in I}$ of N there is either $i \in I$ with $U_i \in \mathcal{F}$ or there is $F \in \mathcal{F}$ meeting each U_i at most in one point. Equivalently, $\mathcal{F}$ is selective iff $\sigma_{\mathcal{F}}$ is an atom in the lattice of all zero-dimensional uniformities on N. Thus, if $\sigma_{\mathcal{F}}$ is an atom then $\mathcal{F}$ is a selective ultrafilter.

The existence of a selective ultrafilter on a countable set is provable under the assumption of the continuum hypothesis or Martin's axiom but there is a model of the set theory (ZFC) in which no selective ultrafilter on a countable set exists.

From certain point of view, the investigation of selective ultrafilters on a countable set is general enough because there is no uniform selective ultrafilter on any uncountable non-measureable cardinal (a filter $\mathcal{F}$ on a set X is said to be uniform if each member of has the cardinality of the set X).

Theorem: Let $\mathcal{F}$ be a selective ultrafilter on a countable set N . Then $(N, \mathcal{O}_{\mathcal{F}})$ is an atom.

Proof follows from two following lemmas.

Lemma 1: Let $\mathcal{F}$ be a selective ultrafilter on N . Let $\mathcal{V} = \{V_j\}_{j \in I}$ be a pointwise finite cover of N . Let $\mathcal{W}$ be a cover of N which star-refines $\mathcal{V}$ ($\mathcal{W} <^* \mathcal{V}$) . Then there is either $j \in J$ such that $V_j \in \mathcal{F}$ or there is $F \in \mathcal{F}$ such that $st(n, \mathcal{W}) \cap st(m, \mathcal{W}) = \emptyset$ for any two distinct points n, m of F .

Proof: $\mathcal{V} = \{V_j\}_{j \in J}$. Suppose $(J, \leq)$ is a well-ordered set. We define $R_j = \{x \in N \mid st(x, \mathcal{W}) \subset V_j$ & & $\forall j' < j : st(x, \mathcal{W}) \not\subset V_{j'}\}$. Clearly $\{R_j\}_{j \in J}$ is a partition of N . $\mathcal{F}$ is selective, hence either there is j such that $R_j \in \mathcal{F}$ or there is $F \in \mathcal{F}$ such that $card\, R_j \cap F = 1$ for each $j \in J$. The proof is finished in the former case. For the latter case, it is enough to consider that $st(x, \mathcal{W}) \cap F$ is finite for each $x \in F$ (because $\mathcal{V}$ is pointwise finite). It is clear now that the selectivity of $\mathcal{F}$ implies there is some $G \in \mathcal{F}$ such that $st(m, \mathcal{W}) \cap st(m, G) = \emptyset$ for each two distinct points of G.

Lemma 2 : Let (X, ρ) be a separable metric space. Then a metric uniformity of (X, ρ) has a basis formed by locally finite covers.

Proof: Let $N \subset X$ be a countable dense set. Choose $\varepsilon > 0$ and consider $\mathcal{B}_\rho(\varepsilon)$. Put $\delta = \frac{\varepsilon}{3}$. We show there is a locally finite open cover $\mathcal{V}$ of X such that $\mathcal{B}_\rho(\varepsilon) > \mathcal{V} > \mathcal{B}_\rho(\delta)$.

Suppose that N is the set of all positive integers. We define $\mathcal{V} = \{V_j\}_{j \in J}$ by the induction: 1) $V_1 = B_\rho(1, \varepsilon)$, $F_1 = \overline{B_\rho(1, \delta)}$.

Added in proof: Lemma 2 is well-known, see [1]

$j+1)$ $V_{j+1} = B_\rho(j+1, \varepsilon) - \bigcup_{i=1}^{j} F_i \qquad F_{j+1} = \overline{B_\rho(j+1, \delta)}$.

Evidently, $\mathcal{V}$ has all expected properties.

6. Remarks

1) Assumptions of Lemma 2 could be weakened: δ-network in (X, ρ) is a subset of X from which each point has the distance less than δ. The assertion of Lemma 2 is true for a metric space (X, ρ) satisfying: for each $\varepsilon > 0$, there is a sequence $\{D_i\}_{i=1}^{\infty}$ such that D_i is ε-discrete for each i and $\bigcup_{i=1}^{\infty} D_i$ is a δ-network where

$\delta < \frac{\varepsilon}{2} \{D_i \subset X$ for each i) .

2) We have fully described proximally non-discrete atoms. Assuming the continuum hypothesis, we have constructed a proximally discrete atom on a countable set. By means of ultraproducts, we get an example of a proximally discrete atom on every infinite cardinal κ so that any uniform cover contains a member of this cardinality κ .

We do not know any example of an atom which is not isomorphic to an ultraproduct of the $\mathcal{U}_{\mathcal{F}}$'s; but the existence of such an atom seems to be very probable. We even do not know whether every atom is zerodimensional.

3) As we have shown, for any uniformity $\mathcal{U}$ on a set X , there is an atom $\mathcal{M}$ with $\mathcal{M} \preceq \mathcal{U}$. However a uniformity is in general far from being a supremum of a set of atoms. Indeed, each atom $\mathcal{Q} \neq \mathcal{Q}_0$ refines $\mathcal{U}_{\mathcal{N}}$ where $\mathcal{N} = \{F \subset X ; \operatorname{card} X - F < \operatorname{card} X\}$. Thus any Hausdorff uniformity, which is a supremum of atoms, is topologically discrete. We do not know whether each topologically discrete uniformity is a supremum of atoms.

References: [1] G. Vidossich: Uniform spaces of countable type, Proc. AMS, 1970, v.25, pp. 551-553.