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Four functors into paved spaces

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by Zdeněk Frolík

Four refinements of uniform spaces are studied, each generated by a functor into paved spaces. The functors are closely related to cozero-sets, Baire sets, hyper-cozero-sets, and hyper-Baire-sets on uniform spaces. The results of "Three technical tools" are assumed.

§ 1. Cozero-sets and Baire sets on uniform spaces

1.1. A cozero set in a uniform space X is a set of the form $\operatorname{coz} f = \{X \mid fx \neq 0\}$ where $f \in U(X)$. The collection of all cozero sets in X is denoted by coz X, and a coz-mapping is defined to be a mapping $f: X \to Y$ such that

$$f^{-1}[\operatorname{coz} Y] \subset \operatorname{coz} X.$$

The class of all coz-mappings forms a category coz which is a refinement of U. Moreover, we have

$$U \hookrightarrow p \hookrightarrow \operatorname{coz} \hookrightarrow t \quad .$$

A Baire set in X is an element of the smallest σ-algebra which contains coz X. The set of all Baire sets in X is denoted by Ba X, and a mapping $f: X \to Y$ is called a Baire mapping if

$$f^{-1}[\operatorname{Ba} Y] \subset \operatorname{Ba} X.$$

We have (Ba is the class of all Baire mappings)

$$\operatorname{coz} \hookrightarrow \operatorname{Ba} \hookrightarrow \operatorname{Set}^U.$$

As in the case of t, one can show that coz-coarse, and hence Ba-coarse uniform spaces are just the set-coarse spaces.

Investigation of coz-fine spaces w.r.t. various classes of spaces leads to spaces useful in various applications.

If we assume that p is familiar, then by a general method the investigation of coz-fine spaces reduced to a study of objects X of p which are coz-fine (the last sentence means $p(X,Y) = \operatorname{coz}(X,Y)$ for each Y).

1.2. Coz-spaces. It is obvious from the definition that $\operatorname{coz} X = \operatorname{coz} p_c X$, that means, coz X is uniquely determined by the proximity structure. On the other hand, the proximity structure is not determined by the coz-structure; e.g. if X is a metric space then $\operatorname{coz} X = \operatorname{coz} t_f X$, however, usually $X \neq t_f X$, and hence $p_c X \neq p_c t_f X$ (metric spaces are proximally fine). Obviously:

$Y \in \operatorname{coz} X$ iff there exists a sequence $\{Y_n\}$ such that $\bigcup \{Y_n\} = Y$, and Y_n is distant to $X - Y$ for each n.

This condition is restated as follows:

Let α be a basis for uniform covers of X. Then $Y \in \operatorname{coz} X$ iff there exists a sequence $\{\mathcal{U}_n\}$ in α such that

$$Y = \bigcup_n \{ \bigcup \{ V \mid V \in \mathcal{U}_n, \ \operatorname{star}(V, \mathcal{U}_n) \subset Y \} \} .$$

The condition is not yet convenient, because $\operatorname{star}(V, \mathcal{U}_n)$ appears in the formula.

A collection $\mathfrak{X}$ is called completely $\mathcal{Y}$-additive, if the union of any sub-collection of $\mathfrak{X}$ belongs to $\mathcal{Y}$. If X is metric, then $\operatorname{coz} X$ is the topology, and hence $\operatorname{coz} X$ is completely additive (= $\operatorname{coz} X$-additive). It follows that, for every space X, the completely $\operatorname{coz} X$-additive uniform covers form a basis for all uniform covers; use the fact that every uniform space is projectively generated by mappings into metric spaces.

Since clearly $\operatorname{coz} X$ is always closed under countable unions we obtain:

Let α be a basis for uniform covers of X, and assume that each member in α is completely coz-additive. Then $Y \in \operatorname{coz} X$ iff there exists a sequence $\{U_n\}$ in α such that

$$Y = \bigcup \{ V \mid V \in \bigcup \{U_n\} , \ V \subset Y \} .$$

In conclusion we note an intrinsic characterization of $\langle U \rangle_{\operatorname{coz}}$.

Clearly $Y \in \langle X \rangle_{\operatorname{coz}}$ if $\operatorname{coz} X = \operatorname{coz} Y$.

A paved space is a pair $\langle X, \mathfrak{X} \rangle$ such that X is a set, and $\mathcal{X}$ is a cover of X. The category of all paved spaces has for its morphisms the mappings f: : $\langle X, \mathfrak{X} \rangle \to \langle Y, \mathcal{Y} \rangle$ such that $f^{-1}[\mathcal{Y}] \subset \mathfrak{X}$. So

$(f: \langle X \rangle_{coz} \to \langle Y \rangle_{coz}) \to (f: \langle \square X, \text{coz } Y \rangle \to \langle \square Y, \text{coz } Y \rangle)$

is a full embedding of $\langle U \rangle_{coz}$ into paved spaces. If there is no danger of confusion, the functor will be denoted by coz, hence coz X is a collection of sets, as well as the paved space $\langle \square X, \text{coz } X \rangle$.

Using the Urysohn Lemma for paved spaces (an obvious version of the usual Urysohn Lemma) we obtain that $\langle X, \mathfrak{X} \rangle$ = coz Y for some Y if and only if the following two conditions are fulfilled:

a) $\mathfrak{X}$ is finitely multiplicative, and countably additive.

b) $\mathfrak{X}$ is co-normal , that means: if $U_i \in \mathfrak{X}$ such that $U_1 \cup U_2 = X$, then there exists $V_i \in \mathfrak{X}$, $V_1 \cap V_2 = \emptyset$ such that $X - V_i \subset U_i$.

Since bounded uniformly continuous functions extend, it is easy to check that the functor coz preserves embeddings (this will be used without any reference).

In conclusion note that $X \in \langle Y \rangle_{Ba}$ iff Ba X = Ba Y. The paved spaces $\langle X, \text{Ba } X \rangle$ are distinguished among the paved spaces by the property that the pavements is a σ-algebra.

1.3. Products of coz-spaces. We shall prove that the category of coz-spaces (see 1.2) has products, and show exactly which products are preserved by the functor coz.

Lemma 1. coz X $\times$ coz Y = coz $(p^1 X \times Y)$.

Proof. Let $\mathfrak{X}$ be the structure of the space on the right-hand side. It is enough to show that $\mathfrak{X}$ is the smallest collection of sets such that

1. $\mathcal{Y}$ is closed under finite intersections and countable unions

2. $\mathcal{Y}$ contains the preimages under the projections of coz-set.

Then it will be obvious that $\mathfrak{X}$ is the structure of the space on the left-hand side.

Clearly $\mathfrak{X} \supset \mathcal{Y}$. Assume $G \in \mathfrak{X}$. Hence (see 1.2) there exists a sequence $\{\mathcal{V}_n\}$ of countable uniform coz-covers of p^1X, and a sequence $\{\mathcal{W}_n\}$ of completely coz-additive uniform covers of Y such that

$$G = \bigcup \bigcup \{ V \times W \mid \langle V, W \rangle \in \bigcup \{ \mathcal{V}_n \times \mathcal{W}_n \}, V \times W \subset G \}.$$

For each n, and each $V \in \mathcal{V}_n$, set

$$G_{n,V} = \bigcup \{ V \times W \mid W \in \mathcal{W}_n, V \times W \subset G \} = V \times \bigcup \{ W \mid W \in \mathcal{W}_n, V \times W \subset G \} .$$

Clearly (complete coz-additivity) $G_{n,V} \in \mathcal{Y}$, and since

$$G = \bigcup \{ G_{n,V} \mid V \in \mathcal{V}_n \} ,$$

G belongs to $\mathcal{Y}$ (the index set is countable).

By a routine argument we obtain from Lemma 1 the following result:

Theorem 1. The category of coz-spaces has products, and

$$\prod \{ \operatorname{coz} X_a \} = \operatorname{coz} \prod \{ p^0 X_a \} = \operatorname{coz} \prod \{ p^1 X_a \}$$

Lemma 2. If $\aleph_\alpha$ is sequentially regular and $p^\alpha X \neq X$, $p^\alpha Y \neq Y$, then

$$\operatorname{coz} (X \times Y) \neq \operatorname{coz}(p^\alpha X \times p^\alpha Y).$$

Corollary. If $p^{\alpha+1}X \neq X$, $p^{\alpha+1}Y \neq Y$ then

$$\operatorname{coz} (X \times Y) \neq \operatorname{coz} (p^{\alpha+1}X \times p^{\alpha+1} Y).$$

Proof of Lemma 2. We may and shall assume that X and Y are set-fine spaces of cardinal $\aleph_\alpha$, X = Y. The diagonal Δ is a cozero set in $X \times X$ (coz = exp), however, if we take a cover $\mathcal{U}$ of X, with cardinality less than $\aleph_\alpha$, then the set

$$\Delta(\mathcal{U}) = \bigcup \{U \times W \mid U \times W \subset \Delta \ , U,W \in \mathcal{U}\}$$

has cardinal less than $\aleph_\alpha$. Hence, if $\aleph_\alpha$ is sequentially regular then no countable union of $\Delta(\mathcal{U})$'s may be equal to Δ , and hence, by 2.1, Δ is no co-zero-set in $p^\alpha X \times p^\alpha X$.

Remark. If $\aleph_\alpha$ is not sequentially regular, and if X is a set-fine space of cardinal $\aleph_\alpha$, then

$$\mathrm{coz}(X \times X) = \mathrm{coz}(p^\alpha X \times p^\alpha X).$$

In addition:

Theorem 2. For any spaces X and Y,

$$\mathrm{coz}\ (X \times Y) = \mathrm{coz}\ (DX \times DY).$$

A proof follows from:

Lemma 3. If X is a metrizable space, then there exists a homeomorphic uniformly continuous identity mapping of X onto a metrizable distal space.

Proof of Lemma 3 follows from A.H. Stone theorem which says that every open cover, and hence, every uniform cover of a metrizable uniform space has a σ -uniformly discrete open refinement. Indeed, for a uniformly discrete family $\alpha = \{U_a \mid a \in A\}$ take an equi-uniformly continuous family of functions $\{\varphi_a^\alpha\}$ such that $U_a = \mathrm{coz}\ \varphi_a$, and $|\varphi^\alpha_a| \leq 1$ and consider the uniformity projectively generated by all $\varphi^\alpha : X \longrightarrow \ell_\infty(A)$, with α's corresponding to a countable base for the uniformity of X.

§ 2. Inv(coz)

Recall that Inv(coz) is the class of all concrete functors F of uniform spaces such that

$$\mathrm{coz}\ (FX) = \mathrm{coz}\ X$$

for each X. Further, $\mathrm{Inv}_+(\mathrm{coz})$ consists of all positive elements of Inv(coz), and $\mathrm{Inv}_-(\mathrm{coz})$ consists of all negative functors in Inv(coz). The functors coz_+ and coz_- are defined as follows:

$$\mathrm{coz}_+X = \bigwedge \{ FX \mid F \in \mathrm{Inv}_+(\mathrm{coz}) \} ,$$

$$\mathrm{coz}_-X = \bigvee \{ FX \mid F \in \mathrm{Inv}_-(\mathrm{coz}) \} .$$

It was shown in "Tools" that $\mathrm{coz}_+ = p_c \in \mathrm{Inv}(\mathrm{coz})$, and that coz_+ is the coarsest functor in Inv(coz).

Here we show in 2.1 that coz_- is the finest functor in Inv(coz),

$$\mathrm{coz}_- = (\mathrm{coz}^{\times 2})_f$$

and

$$\mathrm{coz}_- \text{ is the metric-}t_f \text{ functor.}$$

One should try to understand that the concept of the minus functor is "globally" categorial, in comparison with fine functor (one cannot evaluate the value of the minus functor at one space disregarding the values of other spaces).

In 2.3 several characterizations of coz_- are given, indicating that coz_- may be a very useful functor.

In 2.3 the basic properties of Alexandrov spaces (coz-fine w.r.t. the closed unit interval) and inversion-closed spaces (coz-fine w.r.t. the space R_+ of positive reals) are recalled; for details we refer to "Three uniformities".

In 2.4 the basic results about coz-fine spaces are stated.

In conclusion (2.5) it is indicated that coz_+ is the distally coarse reflection.

2.1. coz_ . If $\mathcal{R}$ is a refinement of U, we define $\mathcal{R}^{\times 2}$ as follows:

$$\mathcal{R}^{\times 2}(X,Y) = \{f: X \longrightarrow Y \mid f \times f \in \mathcal{R}(X \times X,\ Y \times Y)\}$$

Certainly the products are taken in U. Clearly

$$U \rightarrow \mathcal{R}^{\times 2} \hookrightarrow \mathcal{R} .$$

Note that if $\mathcal{R}\_ \in \mathrm{Inv}(\mathcal{R})$, then necessarily

$$\mathcal{R}\_ \hookrightarrow \mathcal{R}^{\times 2} ,$$

that means

$$U(\mathcal{R}\_X, \mathcal{R}\_Y) \subset \mathcal{R}^{\times 2}(X,Y)$$

for each X and Y. Indeed, if f: $\mathcal{R}\_X \longrightarrow \mathcal{R}\_X$ is uniformly continuous, then

$$f \times f:\ \mathcal{R}\_X \times \mathcal{R}\_X \longrightarrow \mathcal{R}\_Y \times \mathcal{R}\_Y$$

is uniformly continuous, and since $\mathcal{R}\_$ is a negative functor, the maps

$$\mathcal{R}\_(X \times X) \longrightarrow \mathcal{R}\_X \times \mathcal{R}\_X \longrightarrow X \times X$$

are uniformly continuous, and similarly for Y, and hence

$$\mathcal{R}\_X \times \mathcal{R}\_Y \quad \text{and} \quad X \times X$$

are identity $\mathcal{R}$-isomorphic, and hence $f \times f \in \mathcal{R}(X \times Y,\ Y \times Y)$.

It follows that

$$\mathcal{R}\_ = (\mathcal{R}^{\times 2})_f$$

whenever $\mathcal{R}^{\times 2}$ is fine-maximal (i.e. $(\mathcal{R}^{\times 2})_f$ generates the refinement $\mathcal{R}^{\times 2}$). We shall show that this is the case for $\mathcal{R}$ = coz in this section, and for $\mathcal{R}$ = h coz in § 5.

This is formulated in Theorem 1 which is stated so that all constructs are explicitly described in the statement of Theorem 1, and hence the proof consists just of checking.

Theorem 1. For any X let mX be projectively generated by all f: mX $\longrightarrow$ Y such that $f \in \mathrm{coz}^{\times 2}(X,Y)$. Then:

(a) The collection of all cozero sets G in $X \times X$ which contains the diagonal is a basis for uniform vicinities of the diagonal of mX, and mX is projectively generated by all f: $mX \longrightarrow t_f S$ such that f: $X \longrightarrow S$ is uniformly continuous, and S is metric.

(b) $\operatorname{coz}(mX \times mX) = \operatorname{coz}(X \times X)$,
in particular,

$$\operatorname{coz} mX = \operatorname{coz} X.$$

(c) $f \in U(mX,Y)$ iff $f \in \operatorname{coz}^{\times 2}(X,Y)$.

(d) $f \in U(mX,Y)$ iff $f \in \operatorname{coz}^{\times 2}(mX,Y)$.

(e) m is the coreflection on $\operatorname{coz}^{\times 2}$-fine spaces, and $\operatorname{coz}^{\times 2}$ is fine-maximal.

(f) If $f \in U(mX,S)$, S metric then $f \in U(mX,t_f S)$, and m is the coreflection on the spaces with this property.

Statement (e) implies:

Corollary. $\operatorname{coz}^{\times 2}(X,Y) = U((\operatorname{coz}^{\times 2})_f X,Y)$

And Corollary implies immediately:

Theorem 2. $\operatorname{coz}_- = (\operatorname{coz}^{\times 2})_f \in \operatorname{Inv}(\operatorname{coz})$.

For the proof of Theorem 1 we need the following two obvious facts:

Lemma 1. Let $\{f_a : X \longrightarrow Y_a \mid a \in A\}$ be a projectively generating family of onto mappings which is "countably directed" (i.e. each countable $\{f_a \mid a \in B\}$ factorizes through some f_b). Then every uniformly continuous mapping of X into a metric space factorizes through some f_a, and hence each cozero-set in X is the preimage under some f_a of some cozero set in Y_a.

Remark. If $\{f_a\}$ is just finitely directed then each cozero-set in X is a countable union of preimages of cozero sets under f_a's.

Lemma 2. The family $\{f\times f: X\times X \longrightarrow S\times S \mid f \in U(X,S)$, S metric, f onto$\}$ is a countably directed projectively generating family.

Proof of Theorem 1 (Hint). To prove (a) one only needs to know that if S is metric (or more generally paracompact) then each neighborhood of the diagonal is a uniform vicinity of the diagonal of $t_f S$ (and this is proved in almost all textbooks in topology). For (b) one uses Lemma 1, and an obvious fact that

$$\text{coz}(t_f S \times t_f S) = \text{coz}(S\times S)$$

for S metric. The rest is easy, and if one wants to understand the material, he must do the proofs without any further comment.

Remarks. (1) The spaces in Theorem 1, Statement (f), called metric-fine or metric-t_f, were introduced by A. Hager, who studied them in the category of all 1-distal spaces (= subspaces of products of separable spaces) where everything is greatly simplified by the fact that for this category the coz refinement is fine-maximal. It was proved by M. Rice and the present author that the coreflection on metric-t_f spaces is obtained in one step. It is easy to see that if $\mathcal{K}$ is a class of spaces, and if c is a coreflection then the class of all spaces X with the property $f\in U(X,K)$ implies $f\in U(X,cK)$ for each K in $\mathcal{K}$, called $\mathcal{K}$-c spaces, form a coreflection class. This general situation has been studied by A. Hager, M. Rice, and in quite general categories, by J. Vilímovský.

(2) Actually we proved that coz_- is the finest functor in Inv(coz).

In conclusion we state a description of coz_- by means of uniform covers. (A note on metric-fine spaces):

Theorem 3. For any space X the σ-uniformly discrete completely coz-additive covers form a basis for uniform covers of coz_X.

Proof. By A.H. Stone theorem $t_f X$ has such a basis for X metric. Hence we should observe that each of the covers is uniform on coz_X.

This is easy: each of these covers can be realized by a uniformly continuous mapping into a metric space.

Corollary. coz_ D_c = coz_ (i.e. $D_c \in$ Inv(coz_)).

Proof: coz $D_c X$ = coz X, and the uniformly discrete families in the two spaces X and $D_c X$ coincide.

2.2. Alexandrov and Inversion closed spaces. Denote by U(X) the set of all uniformly continuous functions on X (i.e. U(X,R)), and denote by $U_b(X)$ the set of all bounded uniformly continuous functions on X. In this section we shall discuss two coreflections which are closely related to U(X) and $U_b(X)$. For most of the proofs we refer to the author's "Three uniformities associated with real valued functions" which will appear in the Proceedings of Conf. on algebras of continuous functions held in Rome in November of 1973, and which was the starting lecture presented in Seminar Uniform Spaces. On the other hand, the reader will profit from making the proofs independently of the paper referred to. Everything is fitting together so nicely that just the statements of the results suggest the proofs.

For convenience of the reader we shall prove the results which are not listed in the paper referred to above.

The spaces with the equivalent properties listed in the following Theorem 1 are called Alexandrov spaces (because A.D. Alexandrov studied extensively coz-spaces during early forties).

Theorem 1. The following properties on a space X are equivalent:

(1) $U_b(X)$ consists just of bounded coz-functions, i.e. X is coz-fine w.r.t. the closed unit interval of the space of reals.

(2) If Z_1 and Z_2 are two disjoint zero-sets in X, then there exists a uniformly continuous function which is 0 on Z_1, and 1 on Z_2.

(3) If $f_n \nearrow g \swarrow h_n$, f_n, $h_n \in U_b(X)$, then $f \in U_b(X)$.

(5) If $\mathcal{U}$ is a finite coz-cover of X, then there exists a partition of unity $\{f_U \mid U \in \mathcal{U}\}$ such that $f_U \in U(X)$, and $U = \text{coz}\ (f_U)$ for each U.

(6) Condition (4) with $\mathcal{U}$ countable.

(7) If $\{f_a\}$ is an ℓ_∞-partition of unity, then $\{f_a\}$ is uniformly continous.

(4) If f_n converges strictly continuously to f, f is bounded, and $f_n \in U(X)$, then $f \in U(X)$.

The spaces satisfying the equivalent condition in the following Theorem 2 are called inversion-closed.

Theorem 2. The following conditions on a space X are equivalent:

(1) $U(X) = \text{coz}(X,R)$, i.e. X is coz-fine w.r.t. the space of reals.

(2) X is R_+-t_f where R_+ is the space of positive reals, i.e. if $f: X \to R_+$ is uniformly continuous then so is $f: X \to t_f R_+$.

(3) If $f \in U(X)$, and coz f = X, then $1/f \in U(X)$.

(4) If $\{f_n\}$ converges strictly continuously to f, and $\{f_n\}$ ranges in U(X), then $f \in U(X)$.

(5) If $f \geqq 0$, $\min(f,n) \in U(X)$, then $f \in U(X)$.

(6) If $1 \geqq f_n \searrow 0$, then $\{f_n\}$ is equi-uniformly continuous.

The most interesting result is that (6) implies (1). This was proved first by D. Preiss, and a beautiful proof by M. Zahradník followed in a few hours.

The following result is useful:

Proposition 1. If X is inversion closed (or a subspace of an inversion closed space) then every uniformly continuous function on a subspace of X extends to a uniformly continuous function on X.

The proof follows immediately from the following

Lemma. If Z is a zero-set in X, then every coz-function on Z admits an extension to a coz-function on X.

The proof of Lemma is just a version of P. Urysohn's proof of the theorem that in a normal topological space continuous functions on closed subspaces extend continuously to the whole space.

Theorem 3. The class of Alexandrov spaces is coreflective, and the coreflection a is described as follows: aX is projectively generated by the identity aX $\longrightarrow$ X and all bounded coz-functions on X.

The class of all inversion-closed spaces is coreflective, and the coreflection H is projectively generated by the identity HX $\longrightarrow$ X, and all coz-functions.

Both coreflections preserve the cozero-sets (i.e. a, H $\in$ Inv(coz)).

Proof. Clearly the space HX projectively generated by HX $\longrightarrow$ X, and all coz-functions is coarser than coz_X (use Condition (2) in Theorem 2), and hence H $\in$ Inv(coz). Since a is coarser than H, necessarily a $\in$ Inv(coz). It follows that $H^2 = H$, $a^2 = a$, and obviously aX is Alexandrov, and H is inversion-closed.

In some considerations the spaces in the following theorem are useful:

Theorem 4. The following conditions on a space X are equivalent:

(a) X is coz-fine w.r.t. separable metric spaces.

(b) If $f: X \longrightarrow S$, S metric separable, is uniformly continuous, then so is $f: X \longrightarrow t_f X$.

The class of all spaces with these properties is coreflective, and the coreflection, designated by $m_{\aleph_0}$, is projectively generated by $m_{\aleph_0} X \longrightarrow X$, and all coz-mappings into separable metric spaces.

Proof. Obviously (a) implies (b). To prove (b) implies (a) it is enough to show that if $f: X \longrightarrow S$ is a coz-mapping onto a separable metric space, then there exists a uniformly continuous topological homeomorphisms g of S onto a separable metric space T such that $g \circ f: X \longrightarrow T$ is uniformly continuous. And this is routine. The rest is easy.

Remark. The equivalence of (a) and (b) in Theorem 4 is due to A. Hager (in fact, he assumed that X is 1-distal but this is irrelevant).

It should be noted that the following condition is equivalent to those in Theorem 4:

(c) if $f_n \in U(X)$, $f_n \searrow 0$ then $\{f_n\}$ is equi-uniformly continuous.

In conclusion note several easy relations:

(1) $p\, aX = a\, pX$

(2) $p^1 m = m\, p^1$

(3) $p\, aX = p\, HX = p\, mX = p\, coz_- X$.

(4) $p^1 m_{\aleph_0} = m_{\aleph_0} p^1$

(5) If X is projectively generated by U(X), then $HX = H\,PX = D_c\, coz\_X$.

2.3. More about metric-t_f spaces. Most of the results are taken from "Uniform maps into normed spaces", Ann. Inst. Fourier 1974, 43-55.

Theorem 1. The following properties of a space X are equivalent:

(1) $coz\_X = X$ (see 2.1)

(2) If S is a metric space, then U(X,S) is closed under taking strictly continuous limits of sequences.

(3) U(X) is inversion-closed, and U(X,B) is a U(X)-module for each normed (or Banach or $\ell_1(A)$) space B.

(4) If $\{f_a\} : X \longrightarrow \ell_1(A)$ is uniformly continuous and $\neq 0$ at each point, then $\{coz\ f_a\}$ is a uniform cover of X.

(5) Every ℓ_∞-partition of unity is ℓ_1 uniformly continuous.

(6) For any normed space B (or Banach space, or $\ell_1(A)$) U(X,B) is scalarly inversion-closed, i.e. if $f: X \longrightarrow B$ is uniformly continuous and $fx \neq 0$ for each x, then

$$\{x \longrightarrow 1/\|fx\|^2 \cdot fx\} : X \longrightarrow B$$

is uniformly continuous.

Proof. Just Condition (5) is new. Clearly (1) implies (5). We shall prove that (5) implies that any σ-uniformly discrete completely coz-additive cover is uniform. We need the following observation.

Lemma 1. If $\{f_n\}$ is a countable partition of unity of X ranging in U(X), then there exists an ℓ_∞-partition $\{g_k\}$ such that each g_k is a multiple of some f_n.

Proof. Replace each f_n by a finite sum of functions with norm at most $1/n$. The resulting sequence $\{g_n\}$ has the property that $\|g_k\|_\infty \to 0$, and certainly such a sequence in U(X) is equi-uniform.

From Lemma 1 we obtain (see 2.2, Theorem 2(b)) that (5) implies that X is inversion-closed, hence Alexandrov. By Theorem 1 each countable coz-cover of X is of the form $\{\text{coz } f_n\}$ with f_n in U(X). By Lemma 1, we may assume that $\{f_n\}$ is an ℓ_∞-partition, and by Condition 5, $\{f_n\}$ is ℓ_1 uniformly continuous. It follows that every countable coz-cover is uniform, and of the form $\{\text{coz } f_n\}$ with $\{f_n\}$ a ℓ_1 uniformly continuous partition of unity.

Finally, let $\mathcal{U} = \bigcup\{\mathcal{U}_n\}$ be a completely additive coz-cover, each $\mathcal{U}_n$ being uniformly discrete. Let U_n be the union of $\mathcal{U}_n$, and let $\{f_n\}$ be an ℓ_1-partition of unity such that $U_n = \text{coz } f_n$.

Choose a uniformly continuous pseudometric d such that each $\mathcal{U}_n$ is uniformly discrete w.r.t. d . Assume that the distances of sets in $\mathcal{U}_n$ are $\geqq 2 \ell_n \in \omega_0$. Then there exists an ℓ_n-Lipschitz family $\{g_U \mid U \in \mathcal{U}_n\}$ of non-negative functions such that $\|g_U\|_\infty \leqq 1$, and each g_U is 1 on U, and zero on the union of all remaining members of $\mathcal{U}_n$. Then replace each g_U by ℓ_n functions $1/\ell_n \cdot g_U$, and denote the resulting family $\{h_b \mid b \in B_n\}$. Now the partition of unity

$$\{f_n \cdot h_b \mid n \in \omega_0 \ , b \in B_n\}$$

is ℓ_∞ uniformly continuous, and is subordinated to $\mathcal{U}$. By Condition (5) this partition is ℓ_1-uniformly continuous, and hence the class of members of $\mathcal{U}$ in U forms a uniform cover.

Since each cover of the form $\mathcal{U}$ is star-refined by a cover of this form, we get that each cover of that form is uniform. This concludes the proof.

2.4. Coz-fine spaces. Recall that X is coz-fine if coz (X,Y) = = U(X,Y) for each Y. The first general result says (A note on metric-fine spaces)

Theorem 1. The following conditions on a space X are equivalent:

(1) X is coz-fine.

(2) X is proximally fine, and Alexandrov.

(3) X is proximally fine and $coz_{-}X = X$.

Note that $coz_f \notin$ Inv (coz). Indeed, if $coz_f \in$ Inv (coz) then necessarily $coz_f = coz_{-}$ and this is absurd:

if X has an uncountable discrete family, then $coz_{-}\, p_c X$ is not finer than X by 2.1, Theorem 3, and of course, coz $p_c X$ = coz X. A straightforward example: Take an uncountable set-fine (= uniformly discrete) space X and consider the space $Y = p_c X \times p_c X$. Then $coz_f Y = X \times X$ (because the two projections $Y \longrightarrow X$ are coz-mappings), but coz $Y \neq$ coz $X \times X$.

In general, if a refinement $\mathcal{R}$ is not fine-maximal (i.e., $\mathcal{R}_f$ does not generate $\mathcal{R}$) the following two questions are basic

A. For which spaces X, $\mathcal{R}_f X = \mathcal{R}_{-}X$?

B. For which spaces X, $\mathcal{R}_f X \in \sim_{\mathcal{R}} X$

(i.e. $\mathcal{R}_f X$ is isomorphic to X in $\mathcal{R}$ under the identity mapping).

For $\mathcal{R}$ = coz these two questions were answered by the present author as follows (A remark on coz-fine spaces, Seminar Uniform Spaces, March 1975).

Theorem 2. Each of the following conditions is necessary and sufficient for $coz_{-}X = coz_f X$

(1) If $\{G_a \mid a \in A\}$ is a completely coz-additive disjoint fa-

mily in X, then the union of $\{G_a \times G_a\}$ is a cozero-set in $X \times X$.

(2) If $\{G_a \mid a \in A\}$ is a completely coz-additive disjoint family in X, then there exists a completely coz-additive family $\{G_{an}\}$ such that

$$G_a = \bigcup \{G_{an} \mid n \in \omega_0\} ,$$

and each $\{G_{an} \mid a \in A\}$ is uniformly discrete.

(3) If $f: X \longrightarrow Y$ is a coz-mapping, and if Y is a hedgehog, then $f: coz\_X \longrightarrow X$ is uniformly continuous.

Theorem 3. $coz_f X \sim_{coz} X$ (i.e. the coz-fine coreflection of X is coz-equivalent to X, i.e. coz $(coz_f X)$ = coz X) if and only if for each two completely coz-additive disjoint families $\{G_a\}$ and $\{H_b\}$, the family $\{G_a \cap H_b\}$ is completely coz-additive.

For the proof the following result is needed.

Lemma 1. If $\{G_a \mid a \in A\}$ is a completely coz-additive family in X, then there exists a coz-mapping $f: X \longrightarrow H(A)$ (the hedgehog over A) such that each set G_a is the preimage of the a-th spine, i.e.

$$G_a = f^{-1}[\{a\sigma \mid 0 < \sigma \leqq 1\}] .$$

For the proof of Lemma 1 we need the following property of hedgehogs.

Lemma 2. The following three proximities on H(A) coincide:

a) the proximity inherited from ℓ_1 ,

b) the proximity inherited from ℓ_1 with the weak topology,

c) the proximity generated by all functions

$$\Sigma \{ m_a \sigma_a \mid a \in A\}$$

with m_a attaining just values 0 and 1.

2.5. coz_{-+} is D_c. For the proof of the result in the title we need to know more about the cozero-sets in products. Probably the result will find further applications.

Theorem 1. The following three conditions on a disjoint family $\{V_a \mid a \in A\}$ in X are equivalent:

(1) $W = \bigcup \{ V_a \times V_a \}$ is a cozero-set in $X \times X$.

(2) For each n in ω_0 there exists a uniformly discrete family $\{ V_{an} \mid a \in A \}$ such that

$$V_a = \bigcup \{ V_{an} \mid n \in \omega_0 \}$$

and

$$V_n = \bigcup \{ V_{an} \mid a \in A \}$$

is a cozero-set in X for each n.

(3) There exists a uniformly continuous mapping of X into a metric space S (which can be taken distally coarse) and a disjoint family $\{G_a \mid a \in A\}$ of cozero-sets in S such that $V_a = f^{-1}[G_a]$ for each a.

We are ready to indicate the proof of the main result:

Theorem 6. $(coz^{\times 2})_+ = D_c$.

Proof. We already know that $D_c \in \mathrm{Inv}\,(coz^{\times 2})$. Assume that $F \in \mathrm{Inv}_+(coz^{\times 2})$. Then $D_c \circ F$ is also in $\mathrm{Inv}_+(coz^{\times 2})$ and so we may and shall assume that $D_c \circ F = F$. Since $F \in \mathrm{Inv}_+$ it is enough to show that FX is finer than X for each distally coarse space X, and since every distally coarse space is projectively generated by uniformly continuous mappings into hedgehogs, it is enough to prove that FX is finer than X for each hedgehog. The proof of this fact is just technically more involved that the proof of the following statement which is needed in the proof:

if D is a uniformly discrete space (i.e. set-fine in the terminology of refinements) then $FD = D$.

The first observation is that it is enough to prove the statement just for some D's of arbitrary large cardinal, because F preserves subspaces (being in Inv_+).

Let D be of a sequentially regular cardinal. Since

$$FD \times FD \text{ and } D \times D$$

have the same cozero-sets, the diagonal Δ of $D \times D$ is a cozero-set in $FD \times FD$, and hence, by Theorem 1 the set D is uniformly σ-discrete, and hence $D = \bigcup \{D_n\}$ with each D_n uniformly discrete. Clearly $FD_n = D$ for each n (again $F \in Inv_+$). Since the cardinal of D is sequentially regular, one of the sets D_n is equipollent to D, and hence uniformly isomorphic to D. It follows that $FD = D$.

The rest of the proof can be found in Cozero-sets on uniform spaces, Seminar Uniform Spaces, March 1975.

§ 3. Her and sub functors

If $\mathcal{C}$ is a class of spaces we denote by sub $\mathcal{C}$ the class of all subspaces of spaces in $\mathcal{C}$, and by her $\mathcal{C}$ the class of all spaces X such that each subspace of X belongs to $\mathcal{C}$. The class sub $\mathcal{C}$ seems to be quite useful (if $\mathcal{C}$ is coreflective then so is sub $\mathcal{C}$, and even if $\mathcal{C}$ is not understood well the class sub $\mathcal{C}$ may be quite reasonable, and in some theorems if $\mathcal{C}$ suffices then so does sub $\mathcal{C}$), on the other hand her $\mathcal{C}$ seems to be quite "bad" even if $\mathcal{C}$ is very reasonable. It should be remarked that $\mathcal{C}$ coreflective does not imply that her $\mathcal{C}$ is coreflective (e.g., hereditarily Alexandrov spaces in 3.3).

3.1. Hereditarily inversion-closed spaces. Here we present just two theorems, and two lemmas which might be more useful than the theorems.

Theorem 1. The following conditions on a space X are equivalent:

(1) Each subspace of X is inversion-closed (see 2.3), i.e. X is hereditarily inversion-closed.

(2) Each coz-function on each subspace of X is the restriction of a uniformly continuous function on X.

(3) X is inversion-closed, and coz X = Ba X.

(4) Each countable partition of X ranging in Ba X is a uniform cover of X.

(5) X is Baire-fine w.r.t. separable metric-spaces.

(6) X is Baire-fine w.r.t. the space R of reals.

Proof. Clearly (2) implies (1), and by Lemma 1 below (1) implies (2). So (1) and (2) are equivalent. By basic Lemma 2 below (1) implies (3). It is quite routine that (3) implies (4), and (4) implies (5), and it is obvious that (5) implies (6). Finally Condition (6) implies (3); it is enough to show that coz X = Ba X, and this is easy (for a cozero-set G consider uniformly continuous function f such that G = coz f).

Lemma 1. If Y is a subspace of an inversion-closed space X, then each uniformly continuous function on Y extends to a uniformly continuous function on X.

Proof. Let f be a uniformly continuous function on a subspace Y of X. First observe that f always extends to a uniformly continuous function on a subspace Z of X, which is a zero-set in X. Indeed, choose a uniformly continuous pseudometric d on X such that f is uniformly continuous on the subspace Y of $\langle X,d \rangle$. Since the space R of reals is complete, f extends to a uniformly continuous function defined on the closure Z of Y in $\langle X,d \rangle$. Certainly Z is a zero-set in X. The proof is concluded by the following version of the Tietze-Urysohn extension theorem:

Lemma 1'. If Y is a subspace of X, and if Y is a zero-set in X, then each coz-function on Y extends to a coz-function on X.

Proof. Routine.

Lemma 2 (the author's Measurable uniform spaces, TAMS 1974). If a cozero-set G in X is an inversion-closed subspace of X, then G is a zero-set in X.

Proof. Let G = coz f, where f is a uniformly continuous function on X, and choose a uniformly continuous pseudometric d on X such that f is uniformly continuous on $\langle X,d \rangle$, and $^1/f$ is uniformly continuous on the subspace G of $\langle X,d \rangle$. The function $^1/f$ extends to a uniformly continuous function g on the closure C of G in $\langle X,d \rangle$. If $x \in C - G$ then gx must be $^1/0$, and hence C = G, and so G is a zero-set in X.

Theorem 2. The class of all hereditarily inversion-closed spaces is coreflective, and the coreflection of X has all the covers of the following form for the basis of all uniform covers: $[\mathcal{U}] \cap \{B_n\}$ where $\mathcal{U}$ is a uniform cover of X and $\{B_n\}$ is a countable partition of X ranging in Ba X.

Corollary. The coreflection in Theorem 2 preserves the Baire-sets.

Remark. (a) Lemma 2 holds (trivially) for sub-inversion-closed spaces.

(b) Condition (2) in Theorem 1 suggests the following condition: each coz-function on a subspace of X extends to a coz-function on X. Clearly this condition is satisfied if and only if the inversion-closed coreflection of X is hereditarily inversion-closed, and this is equivalent to coz X = Ba X. These results may be worth of stating for further references.

Corollary. The following properties of a space X are equivalent

(1) the inversion-closed coreflection of X is hereditarily inversion-closed.

(2) Each coz-function on any subspace of X extends to a coz-function on X.

(3) coz X = Ba X.

3.2. Measurable uniform spaces. Following the paper "Measurable uniform spaces" the spaces satisfying the equivalent conditions in the following theorem are called measurable. We refer to 3.4 for further properties which may support the definition to use the term for this class.

Theorem 1. The following properties of a space X are equivalent:

(1) If S is a metric space then U(X,S) is sequentially closed in pointwise topology (in all mappings).

(2) coz_X = X, and coz X = Ba X (see 3.1 Corollary)

(3) For each subspace Y of X, coz_Y = Y

(4) Each σ-uniformly discrete completely Baire-additive partition of X is a uniform cover.

Here we shall recall just the proof of the equivalence of (2), (3), and (4), which is needed in the sequel. The reader should compare this section with 3.4 where Baire-fine and Ba_ functors will be mentioned. It should be noted that M. Rice proved (independently) the equivalence of (2), (3) and something similar to (4) (which is more convenient for the proof).

Proof. According to characterization of coz_X = X in the concluding theorem of 2.1, Condition 2 is equivalent to Condition 4. By Theorem 1 in 3.1 evidently (3) implies (2). The rest of the proof, e.g. (4) implies (3), is specific for this situation. Perhaps the simplest proof consists in showing that Condition 4 is hereditary, and in addition, each cover of the form in Condition

(4) of a subspace Y of X is the trace of a cover of this form of X. Assume that $\mathcal{B}$ is such a cover of Y, and $\mathcal{B} = \cup\{\mathcal{B}_n\}$ such that each $\mathcal{B}_n$ is uniformly discrete. Let B_n be the union of $\mathcal{B}_n$. Choose Baire sets B_n' in X such that $B_n' \cap Y = B_n$, and we may assume that the sequence $\{B_b'\}$ is disjoint. Finally, since $\mathcal{B}_n$ is uniformly discrete in Y, hence in X, there exists a uniformly continuous pseudometric d_n on X, such that

$$\{B(d,c,1) \mid C \in \mathcal{B}_n\}$$

is disjoint. For $C \in \mathcal{B}_n$ put

$$C' = B_n' \cap B(d_n,c,1).$$

Clearly $\mathcal{C} = \{C' \mid C \in \mathcal{B}\}$ is a σ-uniformly discrete completely Baire-additive disjoint family in X, and $\mathcal{B}$ is the trace of $\mathcal{C}$ on Y. Add $X - \cup\mathcal{C}$ to $\mathcal{C}$.

Theorem 2. The class of all measurable uniform spaces is coreflective, and the coreflection, designated by M, is described as follows: The covers in Condition (5) form a basis for all uniform covers of MX.

Proof. Coreflectivity is clear from Condition 1. It is very easy to check coreflectivity, and the description of the coreflection directly from Condition (4).

Corollary. $M \in \mathrm{Inv(Ba)}$ (i.e. M preserves the Baire sets), and M is metrically determined (i.e. MX is projectively generated by all f: MX $\longrightarrow$ MS, with f: M $\longrightarrow$ S uniformly continuous, and S metrizable).

3.3. Hereditarily Alexandrov spaces.

Theorem 1. The following conditions on a space X are equivalent:

1) Each subspace of X is Alexandrov, i.e. X is hereditarily Alexandrov.

2) Each bounded coz-function on any subspace of X is a restriction of a coz-function on X.

3) X is Alexandrov, and the p coz X is normal (it means, for each two disjoint cozero-sets G_1 and G_2 in X, there exist disjoint zero-sets Z_1 and Z_2 in X such that $Z_i \supset G_i$; see

4) X is Alexandrov, and if $Y \subset X$, Z_1, $Z_2 \in$ zero Y, and $Z_1 \cap$ $\cap Z_2 = \emptyset$, then there exist disjoint zero-sets Z_1' and Z_2' in X such that $Z_i' \supset Z_i$.

Proof. By definition of Alexandrov spaces Conditions 1 and 2 are equivalent (each bounded uniformly continuous function extends). Condition 2 implies Condition 3 because the characteristic function of G_1 on $Y = G_1 \cup G_2$ is a coz-function on Y. Condition 4 implies Condition 2 by Urysohn Extension Lemma. It remains to show 3) implies 4), and this follows from the following:

Lemma 1. If Z_1 and Z_2 are zero-sets in X, then there exist disjoint cozero-sets G_1 and G_2 such that $G_1 \supset Z_1 - Z_2$, $G_2 \supset Z_2 -$ $- Z_1$.

Proof. Choose non-negative uniformly continuous functions f_i such that Z_i = zero f_i, and define

$$G_1 = \{x \mid f_2 x > f_1 x\},$$

$$G_2 = \{x \mid f_1 x > f_2 x\}.$$

Corollary. Each of the following conditions is necessary and sufficient for X to be hereditarily Alexandrov:

a. The precompact reflection pX of X is hereditarily Alexandrov.

b. The Samual compactification $\check{X}$ of X (the coreflection of pX) is hereditarily Alexandrov.

Remarks. (a) The hereditarily Alexandrov compact spaces are usually called F-spaces. For example, extremally disconnected compact spaces are hereditarily Alexandrov, and so their subspaces; N - N is hereditarily Alexandrov but is not extremally disconnected.

(b) The class of all hereditarily Alexandrov spaces is not coreflective. For example for a compact space X let EX $\longrightarrow$ X be the projective progeny of X; i.e. EX is extremally disconnected space, and the mapping is perfect and, this is not important, irreducible. Then the mapping is a quotient mapping w.r.t. the unique uniformities inducing the topologies, EX is hereditarily Alexandrov by (a), but X does not need to be hereditarily Alexandrov (e.g. if we take an infinite compact metrizable space for X).

(c) The coreflective hall of hereditarily Alexandrov spaces consists exactly of spaces such that each finite partition by Baire sets is uniform. Hence the coreflection on the coreflective hall of hereditarily Alexandrov spaces has the following covers for a basis:

the meet of any uniform cover with a finite partition into Baire sets.

3.4. Some sub functors. We want to consider the subfunctors of functors we already studied. However, a general approach may be useful. Recall that a space X is injective if for any uniformly continuous mapping of a subspace Z of any space Y into X extends to a uniformly continuous mapping of Y into X. Every uniform space can be embedded into an injective space (Isbell).

Theorem 1(Vilímovský, partly Rice, idea of the proof Isbell). If $\mathcal{C}$ is coreflective then so is the class sub $\mathcal{C}$ of subspaces of spaces in $\mathcal{C}$. If c is the coreflection on $\mathcal{C}$, then the coreflection sub c on sub $\mathcal{C}$ is obtained as follows :

If Y is injective, and $X \hookrightarrow Y$ then

$$\text{sub } c\, X \hookrightarrow c\, Y,$$

particularly, sub c Y = c Y for injective spaces.

Theorem 2. If c is a metrically determined coreflection then so is sub c, and

sub c is (injective-metric) - c.

Theorem 3. Let c be metrically determined coreflection such that if $X \hookrightarrow Y$ with both X and Y complete metric implies $cX \hookrightarrow cY$. Then

sub c (complete metric) - c.

Theorem 3 is an immediate consequence of Theorem 2. On the other hand, a straghtforward proof, without any use of injections is based on the following modification of Lemma in A note on metric-fine spaces.

Lemma 1. Assume that c is a coreflection, and $\mathcal{M}$ is a countably productive class of metric spaces such that $\mathcal{M}$ is hereditary, or $\mathcal{M}$ is closed hereditary and all spaces in $\mathcal{M}$ are complete.

For any X let X' be projectively generated by all f: : $X' \longrightarrow cS$, $S \in \mathcal{M}$, such that $f \in U(X,S)$. Then $X'' = X'$.

Corollary. (a) sub coz_- = (complete metric) - t_f

(b) sub H = (complete metric) - H

Remark. Sub-inversion closed spaces seem to be quite useful. Recently J. Pachl showed that for these spaces one gets perhaps the most natural statement of Shirota - Katětov theorem and theorems connected with "completeness" and "measures". The paper will appear in Stud. Math.

We turn to the refinement Ba. As far as I know it is not known any description of Ba. On the other hand, it is quite easy to show that $Ba_- \in Inv(Ba)$ (i.e. Ba_- preserves Baire sets). In this situation the following result (highly non-trivial) seems to be basic

Theorem 4. sub Ba_f = sub Ba_- = M.

Theorem 4 follows from the following result:

Theorem 5. If X is the product of a family of complete metric spaces then

$$Ba_f X = Ba_- X = MX.$$

This is a corollary of the fact that Theorem 5 is true for complete metric spaces (Baire sets are complete metric spaces, the proof was connected by a lemma by D. Preiss in Comment. Math. Univ. Carolinae 1974) and the following simple but extremely useful

Lemma (Tashjian) Every Baire mapping from a product into a metric space factorizes through a countable subproduct.

Remark. The formula in Theorem 5 is true for injective spaces.

§ 4. h coz-sets.

First there were hyper-Baire sets in uniform spaces (introduced for the purposes of non-separable descriptive theory of sets in uniform spaces in the present author's "Uniform and topological methods in measure theory and the theory of measurable spaces", Proc. 3rd Prague Symposium 1971; actually R. Hansell had already studied them in metric spaces). Then there were introduced hyper-coz sets, simply h coz-sets, to make the theory of hyper-Baire sets elegant, and perhaps to understand the subject (Interplay of measurable and uniform methods, Proc. 2nd Yugoslavian Int. Top. Symp.,Budva 1972, and Locally e-fine measurable spaces, Trans. Amer. Math. Soc., 196(1974),237-247). The reader is recommended to look first at these three papers. Nothing new has happened since then in the theory of hyper-Baire sets, and therefore we restrict our attention to h coz-sets, and moreover, we just try to expla in the main ideas of several new results. For details see the informal notes "Seminar Uniform Spaces 1974-5".

4.1. Generalities. Following "Basic refinements", the collection of all hyper-cozero-sets in a uniform space X, designated by h coz(X), is the smallest collection of sets which contains the collection coz(X) of all cozero-sets in X , and which is closed under uniformly σ-discrete unions. A hyper-coz-mapping of X into Y is a mapping of X into Y such that the preimages under f of sets in h coz(Y) are elements of h coz X. Clearly the class of all hyper-coz-mappings is a refinement h coz of uniform spaces.

We also denote by h coz the corresponding functor into paved spaces. Hence h coz X is the set X endowed by the collection h coz X of all h coz-sets in X .

Note:

$$\text{h coz}(X) = \text{h coz}(D_c X)$$

and hence

$$D_c \in \text{Inv}(\text{h coz}) .$$

It should be remarked that one can prove (and this is not easy) that $D_c = \text{h coz}_+$.

In what follows we shall need the following classification of h coz-sets. Put $h^0\text{coz}(X) = \text{coz } X$, and if $h^\beta\text{coz}(X)$ are defined for $\beta < \alpha$, then $h^\alpha\text{coz}(X)$ consists of uniformly σ-discrete unions of sets in the union of all $h^\beta\text{coz}(X)$, $\beta < \alpha$. The elements of $h^\alpha\text{coz}(X)$ are called the hyper-cozero-sets of X of the class at most α .

Clearly

$$\text{h coz } X = \bigcup \{ h^\alpha \text{coz } X \}.$$

4.2. (Compact $\times$ metric)-t_f = $(D^{\sigma r} \cap \text{h coz})_f$.

For a classification of hyper-cozero-sets we need a new characterization of metric-t_f locally p-fine spaces introduced by the author (Locally e-fine measurable spaces) for the studying of hyper-Baire sets. First we note the equivalence of the three new conditions.

Theorem 1. The following three conditions on a uniform space X are equivalent:

(1) If S is metric, and K is compact, and if $f: X \longrightarrow S \times K$ is uniformly continuous, then so is $f: X \longrightarrow t_f(S \times K)$.

(2) Condition (1) for $K = \check{X}$ (the Samuel compactification of X, i.e. the completion of PX).

(3) If S is metric, and if $f: X \longrightarrow S$ is uniformly continuous then the identity

$$X \longrightarrow t_f\ (\check{f}^{-1}[S])$$

is uniformly continuous (here $\check{f}: \check{X} \longrightarrow \check{S}$ is the continuous extension of f).

Proof. Clearly Condition (1) implies Condition (2), and the converse implication follows immediately from the elementary fact that every uniformly continuous mapping into a compact space factorizes through the Samuel compactification. Conditions (2) and (3) are equivalent because $\check{f}^{-1}[S]$ is homeomorphic with a closed subspace of $S \times \check{X}$, namely with the graph of the perfect mapping

$$\check{f}: \check{f}^{-1}[S] \longrightarrow S\ ,$$

and because $S \times \check{X}$ is paracompact (if P is paracompact, and if F is closed in P , then $t_f F$ is a subspace of $t_f P$).

Condition (1) suggests a name for these spaces: (metric $\times$ $\times$ compact)-t_f.

Theorem 2. Each of the conditions (1) - (3) in Theorem 2 is equivalent to each of the following conditions

(4) X is metric-t_f, and h coz X = coz X .

(5) X is metric-t_f, and each uniformly locally uniformly continuous function is uniformly continuous.

(6) X is metric-t_f, and locally p-fine.

(7) X is metric-t_f, and locally p^1-fine.

This is a theorem from the paper referred to above. The equivalence of (4) - (7) is proved from the characterization of metric-t_f spaces by means of the property that σ-uniformly discrete completely coz(X)-additive civers form a basis for uniform covers. Conditions (1) and (3) imply Condition (4) because, if (1) holds, then taking a singleton for K we get that X is metric-t_f, and if (3) holds then the uniformly discrete union of cozero sets in X is the intersection of X with a uniformly discrete union of cozero sets in $t_f \check{f}^{-1}[S]$ for some $f: X \longrightarrow S$, and this union is a cozero set, because if $t_f Y = Y$ then h coz Y = coz Y.

It remains to show that some of the conditions (4) - (7) implies one of the conditions (1) - (3). Condition (6) implies Condition (1) because for any compact space K, and any paracompact space Y

$$t_f (Y \times K)$$

is the coarsest uniform space finer than $t_f Y \times K$ with the property that it is uniformly locally p-fine. This follows from the following simple result:

Lemma 1. Let $f: Z \longrightarrow Y$ be a perfect mapping, and Y be paracompact. Let Z' be the set Z endowed with a uniformity such that $f: Z' \longrightarrow t_f Y$ is uniformly continuous and $t_f Z' = t_f Z$. Then $t_f Z$ is the coarsest uniformly finer than Z' and with the property: it is uniformly locally p-fine.

Proof. Let $\mathcal{V}$ be an open cover of Z. Let $\mathcal{W}$ consist of finite unions of elements of $\mathcal{V}$. By perfectness of f , $\mathcal{W}$ is refined by $f^{-1}[\mathcal{U}]$ where $\mathcal{U}$ is an open cover of Y , hence a uniform cover of $t_f Y$ by paracompactness. Hence $\mathcal{W}$ is a uniform cover of Z' . Now $\mathcal{V}$ is obtained by replacing each W by a finite number of elements of $\mathcal{V}$ (W is a finite union of elements of $\mathcal{V}$). Unfortunately, this finite union does not need to be a uniform cover of W . So we must replace each $W = \bigcup\{V \mid V \in \mathcal{F}\}$, $\mathcal{F}$ finite subset of $\mathcal{V}$, by all open $W' \subset W$ such that

$$\{V \cap W' \mid V \in \mathcal{F}\}$$

is a uniform cover of W' (as a subspace of Z'). This works, because of $K \subset Z'$ is compact, then every open (in Z') cover of K is a uniform cover of a neighborhood of K .

The class of all spaces satisfying the equivalent conditions in Theorems 1 and 2 is coreflective (this is obvious e.g. from Condition (1). The coreflection is constructed in the paper mentioned above: σ-uniformly discrete h coz(x)-covers of X form a basis for the uniform covers of the coreflection. Assume now that we want to get the coreflection by applying step by step Condition (1), and we want to describe the intermediate constructs). Then the following result is crucial (it follows also from Lemma 1).

Lemma 2. Assume that S is metric, and K is compact. If G is an open F_σ-set in $S \times K$ then G is a σ-uniformly discrete (in $S \times K$) union of cozero-sets in $S \times K$ (in addition, rectangular cozero-sets). Particularly,

coz $t_f(S \times K)$ consists of σ-uniformly discrete (in

$S \times K$) unions of elements of coz $(S \times K)$.

Proof. Let $\{F_n\}$ be a sequence of closed sets in $S \times K$ such that

$$G = \bigcup \{F_n\} .$$

We may and shall assume that $\{F_n\}$ is increasing.

Fix $n \in \omega_0$. We shall find a σ-uniformly discrete union G_n of cozero-sets in $S \times K$ such that

$$F_n \subset G_n \subset G .$$

To this end we need to show the crucial property:

if $x \in S$ then there exist an open neighborhood U_x of x, and a cozero-set V_x in K such that

$$F_n \cap (U_x \times V_x) = F_n \cap (U_x \times K) .$$

Once the property is verified, then the conclusion of the proof is routine: The collection of all $\{U_x\}$ is an open cover of S ; take a σ-uniformly discrete open refinement $\mathcal{W}$ of $\{U_x\}$, and for each W in $\mathcal{W}$ let V(W) be any V_x with $W \subset U_x$. Clearly the union G_n of all $W \times V(W)$, $W \in \mathcal{W}$, contains F_n and is contained in G_n by the crucial property.

It remains to check the crucial property. Firstly by compactness of K , for each $m \geq n$ there exist and open neighborhood U_m of x , and a cozero-set V_m in K such that

$$U_m \times V_m \subset G ,$$

and

$$((x) \times K) \cap F_m \subset V_m .$$

We may choose U_n such that the dia meters converge to 0 . Assume that no $U_m \times V_n$ has the property

$$F_n \cap (U_m \times V_m) = F_n \cap (U_m \times K) .$$

Then we can choose a sequence $\{\langle x_m, y_m \rangle\}$ in F_n such that

$$x_m \in U_m , \quad y_m \notin V_m$$

for each m . Since $\{x_m\}$ converges to x , and since $\{y_m\}$ has a cluster point y in the compact space K , the point $\langle x,y \rangle$ is a cluster point of $\{\langle x_m,y_m \rangle\}$ in $S \times K$. Since the sequence ranges in F_n , and F_n is closed, we have

$$\langle x,y \rangle \in F_n .$$

On the other hand, y is in no V_m , hence $\langle x,y \rangle$ does not belong to G , and this contradicts to $F \subset G$.

Lemma 2 follows also from the following lemma.

Lemma 3. If S is metric, and if K is compact then $t_f(S \times K)$ has σ-uniformly discrete (in $S \times K$) coz $(S \times K)$-covers for a basis for all uniform covers.

Proof: follows from the proof of Lemma 1.

Recall that a mapping $f: X \to Y$ belongs to $\mathcal{D}^{\sigma r}(X,Y)$ if for every uniformly discrete family $\{Y_a\}$ in Y the family $\{f^{-1}[Y_a]\}$ is σ-uniformly discretely refinable, shortly σ dr , i.e. there is a σ-uniformly discrete family $\{X_b\}$ in X which refines $\{f^{-1}[Y_a]\}$, and

$$\cup\{X_b\} = \cup\{f^{-1}[Y_a]\} ,$$

in other words, there exists a family $\{Z_{an}\}$ such that each $\{Z_{an} \mid a\}$ is uniformly discrete in X for each n , and

$$f^{-1}[Y_a] = \cup\{Z_{an} \mid n \in \omega_0\} .$$

Clearly $\mathcal{D}^{\sigma r}$ is a refinement of the category of uniform spaces.

Theorem 3. Let $\mathcal{R} = \mathcal{D}^{\sigma r} \cap$ h coz . Then $\mathcal{R}$-fine spaces are just the spaces in Theorems 1 and 2.

Proof. Assume that X is $\mathcal{R}$-fine. If f is any uniformly continuous mapping into a product $K \times S$ with K compact, and S metric, then

$$f: X \longrightarrow t_f(K \times S) \in \mathcal{R}$$

which follows from Lemma 3. Since X is $\mathcal{R}$-fine, the mapping is uniformly continuous.

Now assume that X has the properties in Theorems 1 and 2. Then each σ-uniformly discrete h coz-cover is uniform, and hence every $\mathcal{R}$-mapping into a metric space is uniformly continuous (even with the topological fine uniformity on the range).

Problem. Is $coz_{-} = (D^{\sigma r} \cap coz)_f$

In conclusion we shall try to express $h^{\alpha} coz\, X$ as co-zero sets in some coreflection. To this end we define for each space X two spaces: X^1 and $X^{(1)}$ as follows:

X^1 is projectively generated by all $f: X^1 \longrightarrow t_f(S \times K)$ such that $f: X \longrightarrow S \times K$ is uniformly continuous, and S is a complete metric space.

Similarly $X^{(1)}$ is projectively generated by all $f: X^{(1)} \longrightarrow t_f(S \times K)$ such that $f: X \longrightarrow S \times K$ is uniformly continuous, and S is metric (not necessarily complete).

By induction we define

$$X^{o} = X^{(o)} = X ,$$

and

$$X^{\alpha} = (\varprojlim \{X^{\beta} \mid \beta < \alpha\})^{1}$$
$$X^{(\alpha)} = (\varprojlim \{X^{\beta} \mid \beta < \alpha\})^{(1)} .$$

Clearly $\{X \longrightarrow X^{\alpha}\}$, and $\{X \longrightarrow X^{(\alpha)}\}$ are concrete negative functors of uniform spaces.

Theorem 4. For each space X ,

$$\operatorname{coz} X^{\alpha} = \operatorname{coz} X^{(\alpha)} = h^{\alpha} \operatorname{coz} X ,$$

all X^{α} and $X^{(\alpha)}$, $\alpha > 0$, are (complete-metric)-t_f , and

$$X^{(\alpha+1)} \longrightarrow \operatorname{coz}_{-} X^{(\alpha)} = \operatorname{coz}_{-} X^{\alpha} .$$

In consequence,

$\varprojlim X^{\alpha}$ is the (compact complete-metric)-t_f coreflection,

$\varprojlim X^{(\alpha)}$ is the (compact metric)-t_f coreflection,

and

$\varprojlim X^{(\alpha)}$ - $\operatorname{coz}_{-} \varprojlim X^{\alpha}$.

Proof. I. The first assertion follows from the case $\alpha = 1$ by transfinite induction. Clearly

$$\operatorname{coz} X^{(1)} \supset \operatorname{coz} X^{(1)} ,$$

and from Lemma 2 we get

$$\operatorname{coz} X^{(1)} \subset h^{1} \operatorname{coz} X .$$

On the other hand, if G is the union of a uniformly discrete family $\{G_a \mid a \in A\}$ in X , then we choose uniformly continuous mapping f of X into a complete metric space S such that $\{f[G_a]\}$ is uniformly discrete in S . Then we choose a uniformly discrete family $\{U_a\}$ in S such that $f[G_a] \subset U_a$ for each a .

Consider the reduced product mapping (of f and the identity)

(*) $$X \longrightarrow S \rtimes X .$$

If $\{H_a\}$ is a family of cozero sets in $\check{X}$ such that $H_a \cap \cap X = G_a$ for each a, then each G_a is the inverse image under ($*$) of $U_a \times H_a$, and G is the inverse image of the union V of all $U_a \times H_a$. But V is a cozero set in $t_f(S \times \check{X})$ as a uniformly discrete union of cozero-sets in $S \times \check{X}$.

II. All X^{α} and $X^{(\alpha)}$ are (complete metric)-t_f. Again we check just the case $\alpha = 1$. If S is a metric space, and if $f: X^{(1)} \longrightarrow S$ (or $f: X^1 \longrightarrow S$) is uniformly continuous, then there exists a uniformly continuous mapping $g: X \rightarrow T$, T metric (complete metric, resp.) such that f is uniformly continuous with respect to the uniformity projectively generated by the map

$$X \rightarrow t_f(T \times \check{X}) ,$$

and hence f factorizes through this map with the range restricted to the image of X. Since S is complete, the functor uniformly continuously extends to the closure of the image of X, and since the closure is topological fine, the mapping remains continuous if the uniform structure of S is replaced by $t_f S$. Hence $f: X^{(1)} \longrightarrow t_f S$ ($f: X^1 \longrightarrow t_f S$) is uniformly continuous.

Remark to II: Technically it may be easier to work with $t_f(g^{-1}[T])$.

III. $X^{(\alpha+1)} \longrightarrow coz\_ \; X^{(\alpha)}$ follows immediately from the definition by transfinite induction, and $coz\_X^{\alpha} = = coz\_ \; X^{(\alpha)}$ follows by induction using the first assertion in the theorem.

IV. The remaining assertion follow from the previous ones.

With every space there are associated the transfinite sequence $\{X^{\alpha}\}$ of (complete metric)-t_f spaces, and the sequence $\{ coz_{-} X^{(\alpha)} = coz_{-} X^{\alpha} \}$ of metric-t_f-spaces such that

$$h^{\alpha} coz\ X = coz\ X^{\alpha} = coz(coz_{-}X^{\alpha}) .$$

We know (3.4) that (complete metric)-t_f is just sub coz_{-} .

Theorem 5. The space $coz_{-}X^{\alpha}$ has σ-uniformly discrete (in X) completely $h^{\alpha} coz(X)$-additive covers for a basis for all uniform covers.

Proof of Theorem 5. It is easy to show that every cover in question is a uniform cover of $coz_{-}X^{\alpha}$ (by constructing a partition. On the other hand,

Corollary. $\underleftarrow{\lim}\ X^{(\alpha)} = \underleftarrow{\lim}\ coz_{-}X^{\alpha}$ has σ-uniformly discrete (in X or in itself) h coz X-covers for a basis of the uniform covers.

Problem. Is $\underleftarrow{\lim}\ X^{\alpha}$ the subfunctor of $\underleftarrow{\lim}\ coz_{-}X^{\alpha} = \underleftarrow{\lim}\ X^{(\alpha)}$?

4.3. $h\ coz_{-} = coz_{-} \circ \lambda = (h\ coz^2)_f$

The result in the title seems to be quite non-trivial. At least our proof is long and perhaps tricky. It is based on the concept of a hyper-distal space, and by controlling the step-by-step construction of the coreflection into hyper-distal spaces.

Definition. A uniform space X is called hyper-distal (the term is bad) if the following condition is satisfied:

if $\{Y_a \mid a \in A\}$ is uniformly discrete,

and if for each a, $\{X_{ab} \mid \in B_a\}$ is uniformly discrete, and $Y_a = \bigcup \{X_{ab} \mid b \in B_a\}$ for each a, then

$$\{X_{ab} \mid \langle a,b \rangle \in \Sigma \{B_a \mid a \in A\}\}$$

is uniformly discrete.

The main result is coded in the proof of the following

Theorem 1. Assume that X is hyper-distal. Then each hyper-cozero-set in $X \times X$ containing the diagonal of X contains a cozero-set in $X \times X$ containing the diagonal.

The proof requires a more general statement:

Lemma 1. Under the assumptions on X in Theorem 1, for each $G \in h\,\mathrm{coz}\,(X \times X)$ there exists a cozero-set U in $X \times X$ such that

$$\Delta_X \cap G \subset U \subset G .$$

Proof. Fix X, and assume the statement is true for each $G \in h^{\beta}\mathrm{coz}(X)$, $\beta < \alpha$, $\alpha \geqq 1$. We shall prove that it holds for $G \in h^{\alpha}\mathrm{coz}(X)$. By definition, $G = \bigcup \{G_n \mid n \in \omega_0\}$, and each G_n is a uniformly discrete union of sets of class strictly less than α. We may and shall assume that $G = G_n$ for each n, and that

$$G = \bigcup \{G_a \mid a \in A\}$$

where $\{G_a\}$ is a uniformly discrete family of elements of $\bigcup \{h^{\beta}\mathrm{coz}(X) \mid \beta < \alpha\}$. By our assumption, we can take a family of cozero-sets $\{U_a \mid a \in A\}$ in $X \times X$ such that

$$\Delta_X \cap G_a \subset U_a \subset G_a$$

for each a. We may and shall assume that

$$U_a \subset (\Delta_X \cap G_a) \times (\Delta_X \cap G_a)$$

for each a. Now the family of all

$$U'_a = \{x \mid \langle x,x \rangle \in U_a\}$$

is a uniformly discrete family of cozero-sets in X. We have

$$U_a \subset U'_a \times U'_a$$

for each a.

We can express each cozero-set U_a as a union

$$\{V_{ab} \times W_{ab} \mid b \in \cup\{B_n(a)\}\}$$

such that each family

$$\{V_{ab} \mid b \in B_n(a)\} \quad \text{and} \quad \{W_{ab} \mid b \in B_n(a)\}$$

is uniformly discrete. Put

$$H_{ab} = \{x \mid \langle x,x \rangle \in V_{ab} \times W_{ab}\} .$$

Then the family

$$\{H_{ab} \mid b \in B_n(a)\}$$

is uniformly discrete in X for each n and a, and hence the family $\{H_{ab} \mid b \in \cup\{B_n(a) \mid a \in A\}\}$ is uniformly discrete, hence

$$G_n = \cup\{H_{ab} \times H_{ab} \mid a \in A , \ b \in B_n(a)\}$$

is a cozero-set. Clearly

$$G = \cup\{G_n \mid n \in \omega_0\}$$

has the required properties.

Proposition 1. If X is hyper-distal then

$$h \operatorname{coz} X = \operatorname{coz} X ,$$

and $\operatorname{coz}_{-} X$ is locally fine.

Proof. A. Let G be the union of a uniformly discrete family $\{G_a \mid a \in A\}$ of cozero-sets in X . We want to show that G is a cozero-set in X . First choose a uniformly discrete completely coz(X)-additive family $\{U_a\}$ such that $G_a \subset U_a$, and also a family $\{f_a\}$ of uniformly continuous functions such that $G_a = \operatorname{coz} f_a$, and $f_a \geqq 0$ for each a . Now given a positive real n consider the sets

$$G_{an} = \{x \mid f_a x \geqq 1/n\} ,$$

$$F_{an} = \{x \quad f_a x \leqq 1/n\} .$$

Since each pair G_{an} , F_{an} is uniformly discrete, necessarily the family of all G_{an} , and all $U_a \cap F_{an}$ is uniformly discrete, and hence the union of all G_{an} is distant to the union of all $U_a \cap F_{an}$, and hence there is a cozero-set G'_n which contains

$$\bigcup\{G_{an} \mid a \in A\}$$

and is disjoint to

$$\bigcup\{U_a \cap F_{an}\} ,$$

and hence

$$\bigcup\{G_{an}\} \subset G'_n \cap \{\bigcup\{U_a\} \subset G .$$

Since

$$\bigcup\{G_{an} \mid n \in \omega_0\} = G ,$$

G is a cozero-set in X .

B. Recall that coz_X has all uniformly σ-discrete completely coz(X)-additive covers for a basis of all uniform covers. Because of the first part of the proof it is easy to check that these covers are stable under taking the Ginsburg-Isbell-derivative [3] which will be recalled for the purpose of the construction of hyper-distal coreflection.

Definition. For each space X define

$$X^{*} = D(DX)'$$

where Y' denotes the Ginsburg-Isbell derivative, i.e. Y' has for a basis of uniform covers the covers of the form

$$\{U_a \cap V_{ab} \mid a \in A \ , \ B \in B_a\}$$

where $\{V_a\}$, and all $\{V_{ab} \mid b \in B_a\}$ are uniform covers of Y .

By the basic lemma, if Y has point-finite uniform covers for a basis then Y' is actually a uniform space, hence X^{*} is a uniform space for each X .

Define $X^{*\alpha}$ as follows: $X^{*0} = DX$,

and

$$X^{*\alpha} = \left(\varinjlim_{\beta < \alpha} X^{*\beta} \right)^{*} .$$

Proposition 2. For each α :

$$\operatorname{coz} X^{*\alpha} \subset h \operatorname{coz} X^{*\alpha} = h \operatorname{coz} X \ .$$

Proof. Assume that the relations hold for $\beta < \alpha$. Clearly

$$h \operatorname{coz} \left(\varinjlim_{\beta < \alpha} X^{*\beta} \right) \subset h \operatorname{coz} X$$

(any uniformly discrete family in $\varinjlim_{\beta<\alpha} X^{*\beta}$ is uniformly discrete in some $X^{*\beta}$, $\beta < \alpha$), and hence it is enough to show that

$$h \operatorname{coz} (Y^{*}) \subset h \operatorname{coz} (DY) \ (= h \operatorname{coz} Y) \ ,$$

and this follows from the fact that any uniformly discrete union in Y^{*} can be written as a uniformly discrete union of uniformly discrete unions in Y .

Theorem 2 . Denote by X^{∞} the projective limit of $\{X^{*\alpha} \mid \alpha\}$. Then

1) $DX^{\infty} = X^{\infty}$,

2) h coz X^{∞} = coz X^{∞} = h coz X .

3) X^{∞} is hyper-distal.

4) coz_X^{∞} is locally fine, and coz_X^{∞} is finer than λX , and λX is finer than X^{∞} .

Corollary: h coz(λX) = h coz X .

Proof. We only need to show that:

Lemma 2 . h coz X = h coz (coz_X) .

Proof. One shows that uniformly discrete union of cozero-sets in coz_X is a uniformly σ-discrete union of cozero-sets in X (recall that

$$\text{coz}\,(X) = \text{coz}\,(\text{coz\_}X)) .$$

This is all we need to know for the proof of the main results.

Remark. $X \vee X^{*\infty}$ is the coreflection on the hyper-distal spaces. The functionally locally fine coreflection is obviously finer than this one, and coarser than λ .

Theorem 3 . A. (h coz)_X = coz_($X^{*\infty}$) = coz_ λX .

B. The collection of all hyper-cozero-sets in $X \times X$ which contain the diagonal is a basis for the filter of the uniform vicinities of h coz_X .

C. h coz_ = $((\text{h coz})^2)_f$.

Proof. By Corollary to Theorem 2 we have

$$\mathrm{coz}\_(X^{*\infty}) = \mathrm{coz}\_\lambda X$$

for each X . Next by Theorem 2 ,

$$F = \{X \longrightarrow \mathrm{coz}\_X^{*\infty}\}$$

preserves the hyper-cozero-sets. It follows that

$$h\ \mathrm{coz}\ F\ (X \times X) = h\ \mathrm{coz}\ (X \times X)$$

and hence

$$h\ \mathrm{coz}\ (X \times X) = h\ \mathrm{coz}\ (FX \times FX) \quad .$$

Now it follows from Theorem 1 that FX satisfies Statement B. Hence

$$f\colon FX \longrightarrow FY$$

is uniformly continuous if and only if

$$f \times f\colon X \times X \longrightarrow Y \times Y \in h\ \mathrm{coz}\ (X \times X,\ Y \times Y) \ .$$

The proof is concluded by showing $F = h\ \mathrm{coz}\_$. Assume that G is any functor such that

$$h\ \mathrm{coz}\ (GX) = h\ \mathrm{coz}\ X \ ,$$

and show that G is coarser than F . By our assumption

$$h\ \mathrm{coz}\ G(X \times X) = h\ \mathrm{coz}\ (X \times X)$$

and hence

$$h\ \mathrm{coz}\ (GX \times GX) \subset h\ \mathrm{coz}\ (X \times X) \ .$$

It follows that GX has a basis for uniform vicinities consisting of hyper-cozero-sets in $X \times X$, and by what we have already proved, FX must be finer than G .

Remark. We have proved that if G is any functor such that $h\ \mathrm{coz}\ GX \subset h\ \mathrm{coz}\ X$ for each X , then G is coarser than $h\ \mathrm{coz}\_$.