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A WELL-POSEDNESS RESULT FOR A STOCHASTIC MASS CONSERVED ALLEN-CAHN EQUATION WITH NONLINEAR DIFFUSION.*

PERLA EL KETTANI†, DANIELLE HILHORST‡, AND KAI LEE§

Abstract. In this paper, we prove the existence and uniqueness of the solution of the initial boundary value problem for a stochastic mass conserved Allen-Cahn equation with nonlinear diffusion together with a homogeneous Neumann boundary condition in an open bounded domain of $\mathbb{R}^n$ with a smooth boundary. We suppose that the additive noise is induced by a Q-Brownian motion.

Key words. stochastic nonlocal reaction-diffusion equation, monotonicity method, conservation of mass.

AMS subject classifications. 35K55, 35K57, 60H15, 60H30

1. Introduction. In this paper, we study the problem

$$(P)\begin{cases} \dfrac{\partial\varphi}{\partial t} = \operatorname{div}(A(\nabla\varphi)) + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx + \dfrac{\partial W}{\partial t}, & x\in D, \quad t\geq 0 \\ \quad A(\nabla\varphi).\nu = 0, & x\in\partial D, t\geq 0 \\ \quad \varphi(0,x) = \varphi_0(x), & x\in D \end{cases}$$

where:

- D is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂D;
- ν is the outer normal vector to ∂D;
- The function f is given by $f(s) = s - s^3$;
- We assume that $A = \nabla_v\Psi(v) : \mathbb{R}^n \to \mathbb{R}^n$ for some strictly convex function $\Psi \in C^{1,1}(\mathbb{R}^n)$ (i.e. $\Psi \in C^1(\mathbb{R}^n)$ and $\nabla\Psi$ is Lipschitz continuous) which satisfies

$$\begin{cases} A(0) = \nabla\Psi(0) = 0, \Psi(0) = 0 \\ \|D^2\Psi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq c_1, \end{cases} \tag{1.1}$$

for some constant $c_1 > 0$. We remark that (1.1) implies that

$$|A(a) - A(b)| \leq C|a-b| \tag{1.2}$$

for all $a, b \in \mathbb{R}^n$,where C is a positive constant, and that the strict convexity of Ψ implies that A is strictly monotone, namely there exists a positive constant C_0 such that

$$(A(a) - A(b))(a-b) \geq C_0|a-b|^2, \tag{1.3}$$

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†University of Paris-Sud.
‡CNRS and University of Paris-Sud.
§University of Tokyo, Japan.

for all $a, b \in \mathbb{R}^n$.

We remark that if A is the identity matrix, the nonlinear diffusion operator $-div(A(\nabla u))$ reduces to the linear operator $-\Delta u$.

- The function $W = W(x,t)$ is a Q-Brownian motion. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(D)$, $\{e_l\}_{l\geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q, and $\{\lambda_l\}_{l\geq 1}$ be the corresponding eigenvalues, so that

$$Qe_l = \lambda_l e_l, \quad \text{for all } l \geq 1.$$

We suppose that Q satisfies

$$\operatorname{Tr} Q = \sum_{l=1}^{\infty} \langle Qe_l, e_l\rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0. \tag{1.4}$$

for some positive constant Λ_0. We suppose furthermore that $e_l \in H^1(D) \cap L^\infty(D)$ for $l = 1, 2...$ and that there exist positive constants Λ_1 and Λ_2 such that

$$\sum_{l=1}^{\infty} \lambda_l \|e_l\|^2_{L^\infty(D)} \leq \Lambda_1, \tag{1.5}$$

$$\sum_{l=1}^{\infty} \lambda_l \|\nabla e_l\|^2_{L^2(D)} \leq \Lambda_2. \tag{1.6}$$

- Next we define the following spaces:

$$H = \left\{ v \in L^2(D), \int_D v = 0 \right\}, \quad V = H^1(D) \cap H \quad \text{and} \quad Z = V \cap L^4(D)$$

where $\|\cdot\|$ corresponding to the space H.
We also define $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{Z^*,Z}$ as the duality product between Z and its dual space $Z^* = V^* + L^{\frac{4}{3}}(D)$ ([3], p.175).

The corresponding deterministic equation with linear diffusion has been introduced by Rubinstein and Sternberg [10] as a model for phase separation in a binary mixture. The well-posedness and the stabilization of the solution for large times for the corresponding Neumann problem were proved by Boussaïd, Hilhorst and Nguyen [4]. They assumed that the initial function was bounded in $L^\infty(D)$ and proved the existence of the solution in an invariant set using the Galerkin method together with a compactness method.

A singular limit of a rescaled version of Problem (P) with linear diffusion has been studied by Antonopoulou, Bates, Blömker and Karali [1] to model the motion of a droplet. However, they left open the problem of proving the existence and uniqueness of the solution, which we address here. The proof of the existence of the solution of Problem (P) is based on a Galerkin method together with a monotonicity argument similar to that used in [9] for a deterministic reaction-diffusion equation, and that in [8] for a stochastic problem. We refer to the forthcoming article [6] for more details and for the proofs.

The organization of this paper is as follows. In section 2, we present regularity properties of the solution W_A of the nonlinear stochastic heat equation with a homogeneous Neumann boundary condition and initial condition zero. In section 3, we prove the existence of a solution of Problem (P). To that purpose we take the function $\varphi - W_A$ as the new unknown function. Finally, we prove the uniqueness of the solution in section 4.

2. An auxiliary problem. We consider the Neumann boundary value problem for the stochastic nonlinear heat equation

$$(P_1) \quad \begin{cases} \dfrac{\partial W_A}{\partial t} = \operatorname{div}(A(\nabla W_A)) + \dfrac{\partial W}{\partial t}, & x \in D,\ t \geq 0 \\ A(\nabla W_A).\nu = 0, & x \in \partial D,\ t \geq 0 \\ W_A(0,x) = 0, & x \in D. \end{cases}$$

First we define a solution of Problem (P_1).

Definition 2.1. *We say that W_A is a solution of Problem (P_1) if :*

1. $W_A \in L^\infty(0,T;L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T);H^1(D))$*;*
2. $div(A(\nabla W_A)) \in L^2(\Omega \times (0,T);(H^1(D))')$;
3. W_A *satisfies almost surely the problem*

$$\begin{cases} W_A(t) = \displaystyle\int_0^t \operatorname{div}(A(\nabla W_A(s)))ds + W(t), \ \textit{in the sense of distributions,} \\ A(\nabla W_A).\nu = 0, \quad \textit{in the sense of distributions on } \partial D \times \mathbb{R}^+. \end{cases} \tag{2.1}$$

Using ideas from Krylov & Rosovskii [8] we prove in [6] that this problem possesses a unique solution W_A. We are interested in further regularity properties of the solution W_A. A first step is to apply a result of Gess [7] who proves the existence and uniqueness of a solution in the sense of $L^2(D)$, namely almost everywhere in D. More precisely, he defines a strong solution as follows (cf. [7], Definition 1.3).

Definition 2.2. **(Strong solution)** *We say that W_A is a strong solution of Problem (P_1) if :*

1. W_A *is a solution in the sense of Krylov and Rosovskii;*
2. $W_A \in L^2(\Omega;C([0,T];L^2(D)))$*;*
3. $\operatorname{div}(A(\nabla W_A)) \in L^2(\Omega \times (0,T);L^2(D))$*;*
4. W_A *satisfies a.s. for all $t \in (0,T)$ the problem*

$$\begin{cases} W_A(t) = \displaystyle\int_0^t \operatorname{div}(A(\nabla W_A(s)))ds + W(t), \ \textit{in } L^2(D), \\ A(\nabla W_A(t)).\nu = 0, \quad \textit{in a suitable sense of trace on } \partial D. \end{cases} \tag{2.2}$$

We will show in [6] the boundedness of W_A in $L^\infty(0,T;L^q(\Omega \times D))$ for all $q \geq 2$. The proof of this result is based on an article by Bauzet, Vallet, Wittbold [2] where a similar result is proved for a convection-diffusion equation with a multiplicative noise involving a standard adapted one-dimensional Brownian motion.

Theorem 2.3. *Let W_A be a solution of Problem (P_1); then $W_A \in L^\infty(0,T;L^q(\Omega\times D))$, for all $q \geq 2$.*

3. Existence and uniqueness of the solution of Problem (P). To begin with, we perform the change of functions $u(t) := \varphi(t) - W_A(t)$; then φ is a solution of (P) if and only if u satisfies:

$$(P_2) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}[A(\nabla(u+W_A)) - A(\nabla W_A)] + f(u+W_A) \\ \qquad\qquad -\dfrac{1}{|D|}\displaystyle\int_D f(u+W_A)dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u+W_A)).\nu = 0, & x \in \partial D, t \geq 0, \\ u(0,x) = \varphi_0(x), & x \in D. \end{cases}$$

We remark that Problem (P_2) has the form of a deterministic problem; however it is stochastic since the random function W_A appears in the parabolic equation for u.

Definition 3.1. *We say that u is a solution of Problem (P_2) if :*
1. *$u \in L^\infty(0,T;L^2(\Omega\times D)) \cap L^2(\Omega\times(0,T);H^1(D)) \cap L^4(\Omega\times(0,T)\times D)$; $div[A(\nabla(u+W_A))] \in L^2(\Omega\times(0,T);(H^1(D))')$;*
2. *u satisfies almost surely the problem: for all $t \in [0,T]$*

$$\begin{cases} u(t) = \varphi_0 + \displaystyle\int_0^t \operatorname{div}[A(\nabla(u+W_A)) - A(\nabla W_A)]\, ds + \int_0^t f(u+W_A)ds \\ -\displaystyle\int_0^t \frac{1}{|D|}\int_D f(u+W_A)dxds, \textit{in the sense of distributions}, \\ A(\nabla(u+W_A)).\nu = 0, \quad \textit{in the sense of distributions on } \partial D\times\mathbb{R}^+. \end{cases} \tag{3.1}$$

The conservation of mass property holds, namely

$$\int_D u(x,t)dx = \int_D \varphi_0(x)dx, \quad \text{a.s. for a.e. } t \in \mathbb{R}^+.$$

The main result of this section is the following.

Theorem 3.2. *There exits a unique solution of Problem (P_2).*

Proof. In this section we apply the Galerkin method to prove the existence of solution of Problem (P_2). Denote by $0 < \gamma_1 < \gamma_2 \leq ... \leq \gamma_{\tilde{k}} \leq ...$ the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions, and by $w_{\tilde{k}}, \tilde{k} = 0, ...$ the corresponding unit eigenfunctions in $L^2(D)$. Note that they are smooth functions. We remark that the functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$ and satisfy

$$\int_D w_j w_0 = 0 \quad \text{for all } j \neq 0 \text{ and } w_0 = \frac{1}{\sqrt{|D|}}.$$

We look for an approximate solution of the form

$$u_m(x,t) - M = \sum_{i=1}^m u_{im}(t)w_i = \sum_{i=1}^m \langle u_m(t), w_i\rangle w_i,$$

where $M = \frac{1}{|D|}\int_D \varphi_0(x)dx$; the function u_m satisfies the equations

$$\begin{aligned}\int_D \frac{\partial}{\partial t}(u_m(x,t)-M)w_j dx = & -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)]\nabla w_j dx \\ & + \int_D f(u_m+W_A)w_j dx - \frac{1}{|D|}\int_D \Big(\int_D f(u_m+W_A)dx\Big)w_j dx,\end{aligned} \tag{3.2}$$

for all w_j, $j=1,...,m$. We remark that $u_m(x,0) = M + \sum_{i=1}^{m}(\varphi_0, w_i)w_i$ converges strongly to φ_0 in $L^2(D)$ as $m \to \infty$.

Problem (3.2) is an initial value problem for a system of m ordinary differential equations, so that it has a unique solution u_m on some interval $(0,T_m)$, $T_m > 0$; in fact the following a priori estimates show that this solution is global in time.

First we remark that the contribution of the nonlocal term vanishes. Indeed for all $j = 1, ..., m$

$$\begin{aligned}-\frac{1}{|D|}\int_D \Big(\int_D f(u_m + W_A(t))dx\Big)w_j dx &= -\frac{1}{|D|}\Big(\int_D f(u_m+W_A(t))dx\Big) \times \int_D w_j dx \\ &= 0.\end{aligned} \tag{3.3}$$

We substitute (3.3) into (3.2), we multiply (3.2) by $u_{jm} = u_{jm}(t)$, sum on $j = 1, ..., m$ and use property (1.3) to deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_D (u_m - M)^2 dx + C_0\int_D |\nabla(u_m - M)|^2 dx + C_1\int_D (u_m-M)^4 dx \\ &\le C_2\int_D |W_A(t)|^4 dx + \tilde{C}_2(M)|D|.\end{aligned} \tag{3.4}$$

3.1. A priori estimates and passing to the limit. In what follows, we derive a priori estimates for the function u_m.

LEMMA 3.1. *There exists a positive constant $\mathcal{C}$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}\int_D (u_m - M)^2 dx \le \mathcal{C}, \quad \mathbb{E}\int_0^T \int_D |\nabla(u_m - M)|^2 dxdt \le \mathcal{C}, \tag{3.5}$$

$$\mathbb{E}\int_0^T \int_D (u_m-M)^4 dxdt \le \mathcal{C}, \tag{3.6}$$

$$\mathbb{E}\int_0^T \int_D (f(u_m+W_A))^{\frac{4}{3}} dxdt \le \mathcal{C}, \tag{3.7}$$

$$\mathbb{E}\int_0^T \|\operatorname{div}(A(\nabla(u_m+W_A)))\|^2_{(H^1(D))'}dt \le \mathcal{C}. \tag{3.8}$$

Proof. Integrating (3.4) from 0 to t and taking the expectation we deduce for all $t \in [0,T]$

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\int_D (u_m - M)^2(t)dx + C_0\mathbb{E}\int_0^t\int_D |\nabla(u_m - M)|^2 dxds + C_1\mathbb{E}\int_0^t\int_D (u_m - M)^4 dxds \\ &\leq \frac{1}{2}\int_D (u_m(0) - M)^2 dx + C_2\mathbb{E}\int_0^t\int_D |W_A(t)|^4 dxds + \tilde{C}_2(M)|D|T \\ &\leq K,\end{aligned}$$

where we have used Theorem 2.3 of section 2. Therefore u_m is bounded independently of m in $L^\infty(0,T,L^2(\Omega\times D)) \cap L^2(\Omega\times(0,T);H^1(D)) \cap L^4(\Omega\times(0,T)\times D))$.

Moreover we have that

$$\begin{aligned}\mathbb{E}\|f(u_m + W_A)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,T)\times D)} &\leq c_2\mathbb{E}\int_0^T\int_D |u_m - M|^4 dxdt + c_2\mathbb{E}\int_0^T\int_D |W_A|^4 dxdt \\ &\quad + C_5|D|T \\ &\leq K_1,\end{aligned}$$

by (3.6) and Theorem 2.3 in section 2.

Finally one can show that the elliptic term is bounded in the sense of distributions. ∎

Hence there exists a subsequence which we denote again by $\{u_m - M\}$ and a function $u - M \in L^2(\Omega\times(0,T);V) \cap L^4(\Omega\times(0,T)\times D) \cap L^\infty(0,T;L^2(\Omega\times D))$ such that

$$u_m - M \rightharpoonup u - M \quad \text{weakly in} \quad L^2(\Omega\times(0,T);V) \text{ and } L^4(\Omega\times(0,T)\times D) \tag{3.9}$$

$$u_m - M \rightharpoonup u - M \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega\times D)) \tag{3.10}$$

$$f(u_m + W_A) \rightharpoonup \chi \quad \text{weakly in} \quad L^{\frac{4}{3}}(\Omega\times(0,T)\times D) \tag{3.11}$$

$$\mathrm{div}(A(\nabla(u_m + W_A))) \rightharpoonup \Phi \quad \text{weakly in} \quad L^2(\Omega\times(0,T);(H^1(D))') \tag{3.12}$$

as $m \to \infty$.

Next, we pass to the limit as $m \to \infty$. To that purpose, we integrate in time the equation (3.2), we recall that the nonlocal term vanishes in (3.2) and multiply the equation by the product $y\psi$, where $y(\omega)$ is any a.s. bounded random variable and $\psi(t)$ is a bounded function on (0,T); we finally integrate between 0 and T and take the expectation, which yields for all $j = 1, .., m$

$$
\begin{aligned}
&\mathbb{E}\int_0^T \int_D y\psi(t)(u_m(t)-M)w_j dxdt \\
= &\mathbb{E}\int_0^T \int_D y\psi(t)(u_m(0)-M)w_j dxdt \\
&+\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \mathrm{div}[A(\nabla(u_m - M + W_A))], w_j\rangle ds\}dt \\
&-\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \mathrm{div}[A(\nabla W_A)], w_j\rangle ds\}dt \\
&+\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D f(u_m + W_A)w_j dxds\}dt. \qquad (3.13)
\end{aligned}
$$

Passing to the limit in (3.13) by using Lebesgue-dominated convergence theorem, we deduce that for a.e. $(t, \omega) \in (0, T) \times \Omega$ and for all $\tilde{w} \in V \cap L^4(D)$.

$$
\langle u(t) - M, \tilde{w}\rangle = \langle \varphi_0 - M, \tilde{w}\rangle + \int_0^t \langle \Phi + \chi - \mathrm{div}(A(\nabla W_A)), \tilde{w}\rangle ds. \qquad (3.14)
$$

LEMMA 3.2. *The function u is such that $u \in C([0, T]; L^2(D))$ a.s.*
Proof. Apply Lemma 1.2 p. 260 in [11]. □

It remains to prove that $\langle \Phi + \chi, \tilde{w}\rangle = \langle \mathrm{div}(A(\nabla(u - M + W_A))) + f(u + W_A(t)), \tilde{w}\rangle$ for all $\tilde{w} \in V \cap L^4(D)$.

3.2. Monotonicity argument. Let w be such that $w - M \in L^2(\Omega\times(0, T); V) \cap L^4(\Omega \times D \times (0, T))$ and let c be a constant such that $c \geq 2$. We define

$$
\begin{aligned}
\mathcal{O}_m = \mathbb{E}\big[\int_0^T e^{-cs}\{&2\langle \mathrm{div}\left(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)\right) \\
&- \mathrm{div}\left(A(\nabla(w - M + W_A)) - A(\nabla W_A)\right), u_m - M - (w - M)\rangle_{Z^*,Z} \\
&+2\langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M)\rangle_{Z^*,Z} \\
&-c\|u_m - M - (w - M)\|^2\}\big]ds.
\end{aligned}
$$

We have the following result
LEMMA 3.3. $O_m \leq 0$.

We write $\mathcal{O}_m$ in the form $\mathcal{O}_m = \mathcal{O}^1_m + \mathcal{O}^2_m$ where

$$
\begin{aligned}
\mathcal{O}^1_m = \mathbb{E}\big[\int_0^T e^{-cs}\{&2\langle \mathrm{div}\left(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)\right), u_m - M\rangle_{Z^*,Z} \\
&+2\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} - c\|u_m - M\|^2\}\big]ds. \qquad (3.15)
\end{aligned}
$$

We integrate the equation (3.2) between 0 and T and recall that the nonlocal term in (3.2) vanishes, we apply a chain rule formula and take the expectation to obtain

$$\begin{aligned}
&\mathbb{E}[e^{-cT}\|u_m(T)-M\|^2]\\
&=\mathbb{E}[\|u_m(0)-M\|^2]-c\mathbb{E}[\int_0^T e^{-cs}\|u_m(s)-M\|^2ds]\\
&+2\mathbb{E}[\int_0^T e^{-cs}\langle \mathrm{div}[A(\nabla(u_m-M+W_A))-A(\nabla W_A)],u_m-M\rangle_{Z^*,Z}\\
&+2\mathbb{E}[\int_0^T e^{-cs}\langle f(u_m+W_A),u_m-M\rangle_{Z^*,Z}].
\end{aligned}\tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\lim_{m\to\infty}\sup\mathcal{O}_m^1=\mathbb{E}[e^{-cT}\|u(T)-M\|^2]-\mathbb{E}[\|u(0)-M\|^2]+\delta e^{-cT},\tag{3.17}$$

where $\delta=\lim_{m\to\infty}\sup\mathbb{E}[\|u_m(T)-M\|^2]-\mathbb{E}[\|u(T)-M\|^2]\geq 0$.

On the other hand, the equation (3.14) implies that a.s. in $Z^*=V^*+L^{\frac{4}{3}}(D)$:

$$u(t)-M=\varphi_0-M+\int_0^t\Phi-\mathrm{div}(A(\nabla W_A))+\int_0^t\chi,\quad\forall t\in[0,T].\tag{3.18}$$

Applying a chain rule formula we deduce that

$$\begin{aligned}
\mathbb{E}[e^{-cT}\|u(T)-M\|^2]=\mathbb{E}[\|u(0)-M\|^2]-c\mathbb{E}[\int_0^T e^{-cs}\|u(s)-M\|^2ds]\\
+2\mathbb{E}\int_0^T e^{-cs}\langle\Phi-\mathrm{div}(A(\nabla W_A)),u-M\rangle_{Z^*,Z}\\
+2\mathbb{E}[\int_0^T e^{-cs}\langle\chi,u-M\rangle_{Z^*,Z}].
\end{aligned}$$

which we combine with (3.17) to deduce that

$$\begin{aligned}
\lim_{m\to\infty}\sup O_m^1&=2\mathbb{E}[\int_0^T e^{-cs}\langle\Phi-\mathrm{div}(A(\nabla W_A)),u-M\rangle_{Z^*,Z}]\\
&+2\mathbb{E}\int_0^T e^{-cs}\langle\chi,u-M\rangle_{Z^*,Z}-c\mathbb{E}[\int_0^T e^{-cs}\|u(s)-M\|^2ds]+\delta e^{-cT}.
\end{aligned}\tag{3.19}$$

It remains to compute the limit of $\mathcal{O}_m^2$; in view of (3.9), (3.11) and (3.12), we deduce that

$$\begin{aligned}
&\lim_{m\to\infty}\mathcal{O}_m^2\\
&=\mathbb{E}\int_0^T e^{-cs}\{-2\langle\mathrm{div}[A(\nabla(w-M+W_A))-A(\nabla W_A)],u-M\rangle_{Z^*,Z}\\
&-2\langle\Phi-\mathrm{div}(A(\nabla W_A))-\mathrm{div}[A(\nabla(w-M+W_A))-A(\nabla W_A)],w-M\rangle_{Z^*,Z}\\
&-2\langle f(w+W_A),u-M\rangle_{Z^*,Z}-2\langle\chi-f(w+W_A),w-M\rangle_{Z^*,Z}\\
&-c\|w-M\|^2+2c\langle u-M,w-M\rangle\}ds.
\end{aligned}\tag{3.20}$$

Combining (3.19) and (3.20), and remembering that $O_m \leq 0$, yields

$$\mathbb{E}\int_0^T e^{-cs}\{2\langle \Phi - \operatorname{div}\big(A\nabla(w-M+W_A)\big), u-M-(w-M)\rangle_{Z^*,Z}$$
$$+2\langle \chi - f(w+W_A), u-M-(w-M)\rangle_{Z^*,Z} - c\|u-M-(w-M)\|^2\}ds + \delta e^{-cT} \leq 0.$$

Let $v \in L^2(\Omega\times(0,T);V)\cap L^4(\Omega\times(0,T)\times D)$ be arbitrary and set

$$w-M = u-M-\lambda v, \text{ with } \lambda \in \mathbb{R}_+.$$

Dividing by λ and letting $\lambda \to 0$, we find that for all $v \in L^2(\Omega\times(0,T);V)\cap L^4(\Omega\times(0,T)\times D)$

$$\mathbb{E}\int_0^T \langle \Phi+\chi, v\rangle_{Z^*,Z} = \mathbb{E}\int_0^T \langle \operatorname{div}[A(\nabla(u-M+W_A))] + f(u+W_A), v\rangle_{Z^*,Z}.$$

One finally concludes that u satisfies Definition 3.1.

4. Uniqueness of the solution of Problem (P_2)**.** Let ω be given such that two pathwise solutions of Problem (P_2), $u_1 = u_1(\omega,x,t)$ and $u_2 = u_2(\omega,x,t)$ satisfy

$$u_i(\cdot,\cdot,\omega) \in L^\infty(0,T;L^2(D))\cap L^2(0,T;H^1(D))\cap L^4((0,T)\times D),$$
$$f(u_i+W_A) \in L^{\frac{4}{3}}((0,T)\times D),$$
$$\operatorname{div}(A(\nabla(u_i+W_A)) \in L^2((0,T);(H^1(D))')$$

for $i=1,2$, and $u_1(\cdot,0) = u_2(\cdot,0) = \varphi_0$. The difference u_1-u_2 satisfies the equation

$$\begin{aligned}u_1(t)-u_2(t) = &\int_0^t \operatorname{div}(A(\nabla(u_1+W_A))) - div(A(\nabla(u_2+W_A)))\\ &+\int_0^t [f(u_1+W_A) - f(u_2+W_A)]\\ &-\frac{1}{|D|}\int_0^t [\int_D f(u_1+W_A) - \int_D f(u_2+W_A)dx],\end{aligned}$$

in $L^2((0,T);V^*) + L^{\frac{4}{3}}((0,T)\times D)$.

We take the duality product of the equation for the difference u_1-u_2 with $u_1-u_2 \in L^2((0,T);V^*)\cap L^{\frac{4}{3}}((0,T)\times D)$, we use (1.3) to obtain

$$\begin{aligned}\|u_1-u_2\|^2_{L^2(D)} \leq &-C_0\int_0^t\int_D |\nabla(u_1-u_2)|^2\\ &+\int_0^t \langle f(u_1+W_A) - f(u_2+W_A)), u_1-u_2\rangle_{Z^*,Z}\\ &-\int_0^t \langle \frac{1}{|D|}\int_D (f(u_1+W_A) - f(u_2+W_A))dx, u_1-u_2\rangle_{Z^*,Z}.\end{aligned}$$

Since $\int_D u_1(x,t)dx = \int_D u_2(x,t)dx = \int_D \varphi_0(x)dx$, it follows that the nonlocal term vanishes. Using the fact that $f' \leq 1$ we obtain

$$\int_D (u_1-u_2)^2(x,t)dx \leq \int_0^t\int_D (u_1-u_2)^2(x,t)dxds, \quad \text{for all} \quad t \in (0,T).$$

which in turn implies by Gronwall's Lemma that $u_1 = u_2$ a.e. in $D\times(0,T)$.

□

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