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SINGULARLY PERTURBED SET OF PERIODIC FUNCTIONAL-DIFFERENTIAL EQUATIONS ARISING IN OPTIMAL CONTROL THEORY

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Abstract. We consider the singularly perturbed set of periodic functional-differential matrix Riccati equations, associated with a periodic linear-quadratic optimal control problem for a singularly perturbed delay system. The delay is small of order of a small positive multiplier for a part of the derivatives in the system. A zero-order asymptotic solution to this set of Riccati equations is constructed and justified.

Key words. periodic linear-quadratic optimal control problem, singularly perturbed delay system, small delay, periodic functional-differential matrix Riccati equations, asymptotic solution

AMS subject classifications. 34H05, 34K13, 34K26, 35F50

1. Introduction.

One of the fundamental results in control theory is the solution of the finite horizon linear-quadratic optimal control problem with fixed initial and free terminal states. Due to this result, the solution of the control problem is reduced to a terminal-value problem either for a matrix differential Riccati-type equation (finite dimensional case, [16]) or for an operator differential Riccati-type equation (infinite dimensional case, [2, 4, 6, 7, 8, 20, 23]). This result was extended to the finite horizon periodic linear-quadratic optimal control problem. Solution of this problem is reduced to a differential periodic matrix/operator Riccati-type equation (see e.g. [1, 3]).

If the controlled equation is a differential equation with a delay in the state, the operator Riccati-type equation is reduced to a set of matrix functional-differential equations with ordinary and partial derivatives (see e.g. [2, 6, 8, 18, 19, 23]).

If the controlled equation is singularly perturbed, the corresponding differential Riccati equation also is singularly perturbed. Singularly perturbed non-periodic matrix/operator Riccati equations were well studied in many works (see e.g. [11, 12, 14, 15, 17, 21, 24]). Singularly perturbed periodic matrix Riccati equations also were studied in the literature (see [9, 22]). However, to the best of our knowledge, singularly perturbed periodic operator Riccati equations have not yet been considered in the literature.

In this paper, we consider a finite horizon periodic linear-quadratic optimal control problem for a singularly perturbed system with small delays in the state. We construct an asymptotic solution to the set of periodic functional-differential matrix equations of Riccati type, associated with this problem by the control optimality conditions.

2. Problem statement.

2.1. Original optimal control problem.

Consider the following linear system with delays in state variables

$$dx(t)/dt = A_1(t)x(t) + A_2(t)y(t) + H_1(t)x(t-\varepsilon h) + H_2(t)y(t-\varepsilon h)$$

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$$+\int_{-h}^{0}\Big[G_1(t,\eta)x(t+\varepsilon\eta)+G_2(t,\eta)y(t+\varepsilon\eta)\Big]d\eta+B_1(t)u(t)+f_1(t), \quad (2.1)$$

$$\varepsilon dy(t)/dt = A_3(t)x(t)+A_4(t)y(t)+H_3(t)x(t-\varepsilon h)+H_4(t)y(t-\varepsilon h)$$
$$+\int_{-h}^{0}\Big[G_3(t,\eta)x(t+\varepsilon\eta)+G_4(t,\eta)y(t+\varepsilon\eta)\Big]d\eta+B_2(t)u(t)+f_2(t), \quad (2.2)$$

where $x(t)\in E^n$, $y(t)\in E^m$, $u(t)\in E^r$ (u is a control); $\varepsilon>0$ is a small parameter ($\varepsilon << 1$), $h>0$ is some constant independent of ε; the matrix-valued functions $A_i(t)$, $H_i(t)$, $B_j(t)$, $(i=1,...,4;\ j=1,2)$ and the vector-valued functions $f_j(t)$, $(j=1,2)$ are continuously differentiable in the interval $[0,T]$; the matrix-valued functions $G_i(t,\eta)$, $(i=1,...,4)$, are piece-wise continuous in $\eta\in[-h,0]$ for any $t\in[0,T]$, and these functions are continuously differentiable in $t\in[0,T]$ uniformly with respect to $\eta\in[-h,0]$; E^k is k-dimensional real Euclidean space.

In what follows, we assume that:

$$A_i(0)=A_i(T),\ H_i(0)=H_i(T),\ G_i(0,\eta)=G_i(T,\eta),\ \eta\in[-h,0],\ i=1,...,4,$$
$$B_j(0)=B_j(T),\quad f_j(0)=f_j(T),\quad j=1,2. \quad (2.3)$$

The conditions (2.3) are called the T-periodicity conditions or, simply, the periodicity conditions of the corresponding functions.

The cost functional, evaluating the controlled process (2.1)-(2.2), is

$$J=\int_0^T\Big[x'(t)D_1(t)x(t)+2x'(t)D_2(t)y(t)+y'(t)D_3(t)y(t)+u'(t)M(t)u(t)\Big]dt \quad (2.4)$$

where the prime denotes the transposition; the matrix-valued functions $D_k(t)$, $(k=1,2,3)$ and $M(t)$ are continuously differentiable for $t\in[0,T]$ and satisfy the conditions:

$$D_1'(t)=D_1(t),\ D_3'(t)=D_3(t),\ D(t)\triangleq\begin{pmatrix}D_1(t) & D_2(t)\\ D_2'(t) & D_3(t)\end{pmatrix}>0,\quad t\in[0,T],$$
$$D_k(0)=D_k(T),\quad k=1,2,3, \quad (2.5)$$

$$M'(t)=M(t),\quad M(t)>0,\quad t\in[0,T],\quad M(0)=M(T). \quad (2.6)$$

The optimal control problem is to choose a control $u(t)\in L^2[0,T;E^r]$, satisfying the periodicity condition $u(0)=u(T)$ and minimizing the cost functional (2.4) along trajectories of the system (2.1)-(2.2) subject to the periodicity condition $x(\tau)=x(T+\tau)$, $y(\tau)=y(T+\tau)$, $\tau\in[-\varepsilon h,0]$. We call this problem the Original Optimal Control Problem (OOCP).

2.2. Control optimality conditions in the OOCP. Consider the following block-form matrices and vector

$$A(t,\varepsilon)=\begin{pmatrix}A_1(t) & A_2(t)\\ \varepsilon^{-1}A_3(t) & \varepsilon^{-1}A_4(t)\end{pmatrix},\quad H(t,\varepsilon)=\begin{pmatrix}H_1(t) & H_2(t)\\ \varepsilon^{-1}H_3(t) & \varepsilon^{-1}H_4(t)\end{pmatrix}, \quad (2.7)$$

$$G(t,\eta,\varepsilon)=\begin{pmatrix}G_1(\eta,t) & G_2(t,\eta)\\ \varepsilon^{-1}G_3(t,\eta) & \varepsilon^{-1}G_4(t,\eta)\end{pmatrix},\quad B(t,\varepsilon)=\begin{pmatrix}B_1(t)\\ \varepsilon^{-1}B_2(t)\end{pmatrix}, \quad (2.8)$$

$$S(t,\varepsilon)=B(t,\varepsilon)M^{-1}(t)B^{'}(t,\varepsilon)=\begin{pmatrix} S_1(t) & \varepsilon^{-1}S_2(t)\\ \varepsilon^{-1}S_2^{'}(t) & \varepsilon^{-2}S_3(t)\end{pmatrix},\ f(t,\varepsilon)=\begin{pmatrix} f_1(t)\\ \varepsilon^{-1}f_2(t)\end{pmatrix},$$
(2.9)
$$S_1(t)=B_1(t)M^{-1}(t)B_1^{'}(t),\ S_2(t)=B_1(t)M^{-1}(t)B_2^{'}(t),\ S_3(t)=B_2(t)M^{-1}(t)B_2^{'}(t).$$

Also, let us consider the following set of functional-differential equations (ordinary and partial) with respect to the matrix-valued functions $P(t)$, $Q(t,\tau)$, $R(t,\tau,\rho)$ in the domain $\Omega_\varepsilon=\big\{(t,\tau,\rho): t\in[0,T],\tau\in[-\varepsilon h,0],\rho\in[-\varepsilon h,0]\big\}$:

$$\begin{aligned} dP(t)/dt=-P(t)A(t,\varepsilon)-A^{'}(t,\varepsilon)P(t)+P(t)S(t,\varepsilon)P(t)\\ -Q(t,0)-Q^{'}(t,0)-D(t),\end{aligned}\tag{2.10}$$

$$\begin{aligned} (\partial/\partial t-\partial/\partial\tau)Q(t,\tau)=-\big[A(t,\varepsilon)-S(t,\varepsilon)P(t)\big]^{'}Q(t,\tau)\\ -\varepsilon^{-1}P(t)G(t,\tau/\varepsilon,\varepsilon)-R(t,0,\tau),\end{aligned}\tag{2.11}$$

$$\begin{aligned} (\partial/\partial t-\partial/\partial\tau-\partial/\partial\rho)R(t,\tau,\rho)=-\varepsilon^{-1}G^{'}(t,\tau/\varepsilon,\varepsilon)Q(t,\rho)\\ -\varepsilon^{-1}Q^{'}(t,\tau)G(t,\rho/\varepsilon,\varepsilon)+Q^{'}(t,\tau)S(t,\varepsilon)Q(t,\rho).\end{aligned}\tag{2.12}$$

The set (2.10)-(2.12) is subject to the boundary conditions

$$Q(t,-\varepsilon h)=P(t)H(t,\varepsilon),\tag{2.13}$$

$$R(t,-\varepsilon h,\tau)=H^{'}(t,\varepsilon)Q(t,\tau),\quad R(t,\tau,-\varepsilon h)=Q^{'}(t,\tau)H(t,\varepsilon).\tag{2.14}$$

Based on the results of the works [3, 5, 8, 23], we have the lemma.

LEMMA 2.1. *Let for a given $\varepsilon>0$, any $t\in[0,T]$ and any complex λ with* $\mathrm{Re}(\lambda)\geq 0$, *the following equality is valid:*

$$\begin{aligned}\mathrm{rank}\left[A(t,\varepsilon)+H(t,\varepsilon)\exp(-\lambda\varepsilon h)+\int_{-h}^{0}G(t,\eta,\varepsilon)\exp(\lambda\varepsilon\eta)d\eta-\lambda I_{n+m},B(t,\varepsilon)\right]\\ =n+m.\end{aligned}\tag{2.15}$$

Then, the optimal state-feedback control in the OOCP has the form

$$\begin{aligned}u^*[t,z_{\varepsilon h}(t)]=-M^{-1}(t)B^{'}(t,\varepsilon)\left[P(t,\varepsilon)z(t)+\textstyle\int_{-\varepsilon h}^{0}Q(t,\tau,\varepsilon)z(t+\tau)d\tau+\varphi(t,\varepsilon)\right],\\ z=\mathrm{col}(x,y),\quad z_{\varepsilon h}(t)=\{z(t+\tau),\ \tau\in[-\varepsilon h,0]\},\end{aligned}$$

where $P(t,\varepsilon)$ and $Q(t,\tau,\varepsilon)$ are the components of the unique solution $\big\{P(t,\varepsilon)$, $Q(t,\tau,\varepsilon),R(t,\tau,\rho,\varepsilon)\big\}$ of the problem (2.10)-(2.14) satisfying the periodicity condition

$$P(0,\varepsilon)=P(T,\varepsilon),\quad Q(0,\tau,\varepsilon)=Q(T,\tau,\varepsilon),\quad R(0,\tau,\rho,\varepsilon)=R(T,\tau,\rho,\varepsilon),\tag{2.16}$$

and such that for any $t\in[0,T]$ the matrix $\begin{pmatrix} P(t,\varepsilon) & Q(t,\rho,\varepsilon)\\ Q^{'}(t,\tau,\varepsilon) & R(t,\tau,\rho,\varepsilon)\end{pmatrix}$ *defines a linear bounded self-adjoint positive operator mapping the space $E^{n+m}\times L^2[-\varepsilon h,0;E^{n+m}]$ into itself. Moreover, the $(n+m)$-vector-valued function $\varphi(t,\varepsilon)$ is the unique periodic* *solution ($\varphi(0,\varepsilon)=\varphi(T,\varepsilon)$) of the equation*

$$d\varphi(t,\varepsilon)/dt = -\big[A(t,\varepsilon)-S(t,\varepsilon)P(t,\varepsilon)\big]^{'}\varphi(t,\varepsilon) - \left\{\begin{array}{ll} H^{'}(t+\varepsilon h,\varepsilon)\varphi(t+\varepsilon h,\varepsilon), & t+\varepsilon h\le T \\ 0, & \text{otherwise}\end{array}\right\} - \int_{-h}^{0}\left\{\begin{array}{ll} \widetilde{G}(t,\eta,\varepsilon)\varphi(t-\varepsilon\eta,\varepsilon), & t-\varepsilon\eta\le T \\ 0, & \text{otherwise}\end{array}\right\} d\eta$$
$$-P(t,\varepsilon)f(t,\varepsilon) - \left\{\begin{array}{ll} \int_{t-T}^{0} Q(t-\tau,\tau,\varepsilon)f(t-\tau,\varepsilon)d\tau, & t\in(T-\varepsilon h,T] \\ \int_{-\varepsilon h}^{0} Q(t-\tau,\tau,\varepsilon)f(t-\tau,\varepsilon)d\tau, & t\in[0,T-\varepsilon h]\end{array}\right\},$$

where $\widetilde{G}(t,\eta,\varepsilon)=\big[G(t-\varepsilon\eta,\eta,\varepsilon)-\varepsilon S(t-\varepsilon\eta,\varepsilon)Q(t-\varepsilon\eta,\varepsilon\eta,\varepsilon)\big]^{'}$.

The objective of the present paper is to solve the set (2.10)-(2.12) subject to the conditions (2.13)-(2.14) and (2.16). The solution of this problem, mentioned in Lemma 2.1, satisfies the symmetry conditions $P^{'}(t,\varepsilon)=P(t,\varepsilon)$, $R^{'}(t,\tau,\rho,\varepsilon)=R(t,\rho,\tau,\varepsilon)$, $(t,\tau,\rho)\in\Omega_\varepsilon$. The system (2.10)-(2.12) consists of the three functional-differential Riccati-type matrix equations singularly depending on ε. One of these equations is ordinary, while the others are partial. The equations are with deviating arguments. All these features make the solving this set to be an extremely difficult task. An asymptotic approach turns out to be very helpful in solution of this set. This approach allows us to partition the original set of Riccati-type equations into several much simpler and ε-free subsets. Due to the latter circumstance, an approximate (asymptotic) solution to the original set of equations is derived once, while being valid for all sufficiently small values of ε.

3. Asymptotic solution of the problem (2.10)-(2.14),(2.16).

3.1. Equivalent transformation of (2.10)-(2.14),(2.16). To remove the singularities at $\varepsilon=0$ from the right-hand sides of (2.10)-(2.12), we represent the solution $\{P(t,\varepsilon),Q(t,\tau,\varepsilon),R(t,\tau,\rho,\varepsilon)\}$ to (2.10)-(2.14),(2.16) in the block form

$$P(t,\varepsilon)=\begin{pmatrix} P_1(t,\varepsilon) & \varepsilon P_2(t,\varepsilon)\\ \varepsilon P_2^{'}(t,\varepsilon) & \varepsilon P_3(t,\varepsilon)\end{pmatrix},\quad Q(t,\tau,\varepsilon)=\begin{pmatrix} Q_1(t,\tau,\varepsilon) & Q_2(t,\tau,\varepsilon)\\ Q_3(t,\tau,\varepsilon) & Q_4(t,\tau,\varepsilon)\end{pmatrix},$$
$$R(t,\tau,\rho,\varepsilon)=(1/\varepsilon)\begin{pmatrix} R_1(t,\tau,\rho,\varepsilon) & R_2(t,\tau,\rho,\varepsilon)\\ R_2^{'}(t,\rho,\tau,\varepsilon) & R_3(t,\tau,\rho,\varepsilon)\end{pmatrix}, \tag{3.1}$$

where $P_k(t,\varepsilon)$ and $R_k(t,\tau,\rho,\varepsilon)$, $(k=1,2,3)$, are matrices of dimensions $n\times n, n\times m, m\times m$, respectively; $Q_i(t,\tau,\varepsilon),(i=1,...,4)$, are matrices of dimensions $n\times n,n\times m,m\times n,m\times m$, respectively. Substitution of the block representations for the matrices $D(t)$, $A(t,\varepsilon)$, $H(t,\varepsilon)$, $G(t,\eta,\varepsilon)$, $S(t,\varepsilon)$, $P(t,\varepsilon)$, $Q(t,\tau,\varepsilon)$, $R(t,\tau,\rho,\varepsilon)$ (see (2.5),(2.7),(2.8),(2.9),(3.1)) into the problem (2.10)-(2.14),(2.16) yields after some rearrangement the following equivalent problem (in this problem, for simplicity, we omit the designation of the dependence of the unknown matrices on ε):

$$dP_1(t)/dt = -P_1(t)A_1(t)-A_1^{'}(t)P_1(t)-P_2(t)A_3(t)-A_3^{'}(t)P_2^{'}(t) +P_1(t)S_1(t)P_1(t)+P_1(t)S_2(t)P_2^{'}(t)+P_2(t)S_2^{'}(t)P_1(t) +P_2(t)S_3(t)P_2^{'}(t)-Q_1(t,0)-Q_1^{'}(t,0)-D_1(t), \tag{3.2}$$

$$\varepsilon dP_2(t)/dt = -P_1(t)A_2(t)-P_2(t)A_4(t)-\varepsilon A_1^{'}(t)P_2(t)-A_3^{'}(t)P_3(t) +\varepsilon P_1(t)S_1(t)P_2(t)+P_1(t)S_2(t)P_3(t)+\varepsilon P_2(t)S_2^{'}(t)P_2(t) +P_2(t)S_3(t)P_3(t)-Q_2(t,0)-Q_3^{'}(t,0)-D_2(t), \tag{3.3}$$

$$\varepsilon dP_3(t)/dt = -\varepsilon P_2'(t)A_2(t) - \varepsilon A_2'(t)P_2(t) - P_3(t)A_4(t) - A_4'(t)P_3(t) + \varepsilon^2 P_2'(t)S_1(t)P_2(t) + \varepsilon P_2'(t)S_2(t)P_3(t) + \varepsilon P_3(t)S_2'(t)P_2(t) + P_3(t)S_3(t)P_3(t) - Q_4(t,0) - Q_4'(t,0) - D_3(t), \tag{3.4}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau)Q_1(t,\tau) = -\varepsilon\Big[A_1'(t) - P_1(t)S_1(t) - P_2(t)S_2'(t)\Big]Q_1(t,\tau) - \Big[A_3'(t) - P_1(t)S_2(t) - P_2(t)S_3(t)\Big]Q_3(t,\tau) - P_1(t)G_1(t,\tau/\varepsilon) - P_2(t)G_3(t,\tau/\varepsilon) - R_1(t,0,\tau), \tag{3.5}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau)Q_2(t,\tau) = -\varepsilon\Big[A_1'(t) - P_1(t)S_1(t) - P_2(t)S_2'(t)\Big]Q_2(t,\tau) - \Big[A_3'(t) - P_1(t)S_2(t) - P_2(t)S_3(t)\Big]Q_4(t,\tau) - P_1(t)G_2(t,\tau/\varepsilon) - P_2(t)G_4(t,\tau/\varepsilon) - R_2(t,0,\tau), \tag{3.6}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau)Q_3(t,\tau) = -\varepsilon\Big[A_2'(t) - \varepsilon P_2'(t)S_1(t) - P_3(t)S_2'(t)\Big]Q_1(t,\tau) - \Big[A_4'(t) - \varepsilon P_2'(t)S_2(t) - P_3(t)S_3(t)\Big]Q_3(t,\tau) - \varepsilon P_2'(t)G_1(t,\tau/\varepsilon) - P_3(t)G_3(t,\tau/\varepsilon) - R_2'(t,\tau,0), \tag{3.7}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau)Q_4(t,\tau) = -\varepsilon\Big[A_2'(t) - \varepsilon P_2'(t)S_1(t) - P_3(t)S_2'(t)\Big]Q_2(t,\tau) - \Big[A_4'(t) - \varepsilon P_2'(t)S_2(t) - P_3(t)S_3(t)\Big]Q_4(t,\tau) - \varepsilon P_2'(t)G_2(t,\tau/\varepsilon) - P_3(t)G_4(t,\tau/\varepsilon) - R_3(t,0,\tau), \tag{3.8}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_1(t,\tau,\rho) = -\varepsilon G_1'(t,\tau/\varepsilon)Q_1(t,\rho) - \varepsilon Q_1'(t,\tau)G_1(t,\rho/\varepsilon) - G_3'(t,\tau/\varepsilon)Q_3(t,\rho) - Q_3'(t,\tau)G_3(t,\rho/\varepsilon) + \varepsilon^2 Q_1'(t,\tau)S_1(t)Q_1(t,\rho) + \varepsilon Q_3'(t,\tau)S_2'(t)Q_1(t,\rho) + \varepsilon Q_1'(t,\tau)S_2(t)Q_3(t,\rho) + Q_3'(t,\tau)S_3(t)Q_3(t,\rho), \tag{3.9}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_2(t,\tau,\rho) = -\varepsilon G_1'(t,\tau/\varepsilon)Q_2(t,\rho) - \varepsilon Q_1'(t,\tau)G_2(t,\rho/\varepsilon) - G_3'(t,\tau/\varepsilon)Q_4(t,\rho) - Q_3'(t,\tau)G_4(t,\rho/\varepsilon) + \varepsilon^2 Q_1'(t,\tau)S_1(t)Q_2(t,\rho) + \varepsilon Q_3'(t,\tau)S_2'(t)Q_2(t,\rho) + \varepsilon Q_1'(t,\tau)S_2(t)Q_4(t,\rho) + Q_3'(t,\tau)S_3(t)Q_4(t,\rho), \tag{3.10}$$

$$\varepsilon(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_3(t,\tau,\rho) = -\varepsilon G_2'(t,\tau/\varepsilon)Q_2(t,\rho) - \varepsilon Q_2'(t,\tau)G_2(t,\rho/\varepsilon) - G_4'(t,\tau/\varepsilon)Q_4(t,\rho) - Q_4'(t,\tau)G_4(t,\rho/\varepsilon) + \varepsilon^2 Q_2'(t,\tau)S_1(t)Q_2(t,\rho) + \varepsilon Q_4'(t,\tau)S_2'(t)Q_2(t,\rho) + \varepsilon Q_2'(t,\tau)S_2(t)Q_4(t,\rho) + Q_4'(t,\tau)S_3(t)Q_4(t,\rho), \tag{3.11}$$

$$Q_j(t,-\varepsilon h) = P_1(t)H_j(t) + P_2(t)H_{j+2}(t), \quad j = 1,2, \qquad Q_l(t,-\varepsilon h) = \varepsilon P_2'(t)H_{l-2}(t) + P_3(t)H_l(t), \quad l = 3,4, \tag{3.12}$$

$$R_1(t,-\varepsilon h,\tau)=\varepsilon H_1^{'}Q_1(t,\tau)+H_3^{'}Q_3(t,\tau),$$
$$R_1(t,\tau,-\varepsilon h)=\varepsilon Q_1^{'}(t,\tau)H_1+Q_3^{'}(t,\tau)H_3,$$
$$R_2(t,-\varepsilon h,\tau)=\varepsilon H_1^{'}Q_2(t,\tau)+H_3^{'}Q_4(t,\tau)$$
$$R_2(t,\tau,-\varepsilon h)=\varepsilon Q_1^{'}(t,\tau)H_2+Q_3^{'}(t,\tau)H_4,$$
$$R_3(t,-\varepsilon h,\tau)=\varepsilon H_2^{'}Q_2(t,\tau)+H_4^{'}Q_4(t,\tau),$$
$$R_3(t,\tau,-\varepsilon h)=\varepsilon Q_2^{'}(t,\tau)H_2+Q_4^{'}(t,\tau)H_4. \tag{3.13}$$

$$P_k(0)=P_k(T),\quad Q_i(0,\tau)=Q_i(T,\tau),\quad R_k(0,\tau,\rho)=R_k(T,\tau,\rho), \tag{3.14}$$

where $k=1,2,3;\ i=1,...,4$. In the set (3.2)-(3.11), the equations (3.3)-(3.11) are with the small multiplier ε for the derivatives. Hence, (3.2)-(3.11) is singularly perturbed.

3.2. Formal construction of the zero-order asymptotic solution to the problem (3.2)-(3.14). In the sequel we assume:

(A1) $\operatorname{rank}\Big[A_4(t)+H_4(t)\exp(-h\lambda)+\int_{-h}^{0}G_4(t,\eta)\exp(\eta\lambda)d\eta-\lambda I_m,\ B_2(t)\Big]=m$ for any $t\in[0,T]$ and any complex number λ with $\mathrm{Re}\lambda\geq 0$.

We seek the zero-order asymptotic solution $\{P_{k0}(t,\varepsilon),Q_{i0}(t,\tau,\varepsilon),R_{k0}(t,\tau,\rho,\varepsilon),$ $(k=1,2,3;\ i=1,...,4)\}$ of (3.2)-(3.14) in the form

$$P_{k0}(t,\varepsilon)=\bar{P}_{k0}(t),\quad Q_{i0}(t,\tau,\varepsilon)=Q_{i0}^{\tau}(t,\eta),\quad R_{k0}(t,\tau,\rho,\varepsilon)=R_{k0}^{\tau,\rho}(t,\eta,\chi),$$
$$\eta=\tau/\varepsilon,\quad \chi=\rho/\varepsilon\quad k=1,2,3,\quad i=1,...,4. \tag{3.15}$$

Equations and conditions for (3.15) are obtained by its substitution into (3.2)-(3.14) instead of $P_k(t)$, $Q_i(t,\tau)$, $R_k(t,\tau,\rho)$, $(k=1,2,3;\ i=1,...,4)$, and equating coefficients for ε^0 on both sides of the resulting equations. Thus, for the terms of the asymptotic solution, we obtain the set of 10 equations (8 differential and 2 algebraic ones) in the domain $\bar{\Omega}=\big\{(t,\eta,\chi):t\in[0,T],\eta\in[-h,0],\chi\in[-h,0]\big\}$, and 11 boundary conditions. It is remarkable that this set of the equations and the conditions can be partitioned into four simpler problems solved successively. Since the problem (3.2)-(3.14) is t-periodic, its asymptotic solution consists only of the outer solution.

3.2.1. The first problem. This problem has the form

$$\bar{P}_{30}(t)A_4(t)+A_4^{'}(t)\bar{P}_{30}(t)-\bar{P}_{30}(t)S_3(t)\bar{P}_{30}(t)+Q_{40}^{\tau}(t,0)+[Q_{40}^{\tau}(t,0)]^{'}+D_3(t)=0,$$
$$\partial Q_{40}^{\tau}(t,\eta)/\partial\eta=\big[A_4^{'}(t)-\bar{P}_{30}(t)S_3(t)]Q_{40}^{\tau}(t,\eta)+\bar{P}_{30}(t)G_4(t,\eta)+R_{30}^{\tau,\rho}(t,0,\eta),$$
$$(\partial/\partial\eta+\partial/\partial\chi)R_{30}^{\tau,\rho}(t,\eta,\chi)=G_4^{'}(t,\eta)Q_{40}^{\tau}(t,\chi)+[Q_{40}^{\tau}(t,\eta)]^{'}G_4(t,\chi)$$
$$-[Q_{40}^{\tau}(t,\eta)]^{'}S_3(t)Q_{40}^{\tau}(t,\chi),$$
$$Q_{40}^{\tau}(t,-h)=\bar{P}_{30}(t)H_4(t),$$
$$R_{30}^{\tau,\rho}(t,-h,\eta)=H_4^{'}(t)Q_{40}^{\tau}(t,\eta),\qquad R_{30}^{\tau,\rho}(t,\eta,-h)=[Q_{40}^{\tau}(t,\eta)]^{'}H_4(t). \tag{3.16}$$

Remark 1. *In the problem (3.16), $\eta\in[h,0]$, $\chi\in[h,0]$ are independent variables, while $t\in[0,T]$ is a parameter. Since the coefficients of this problem are T-periodic, then its solution (if it exists and is unique) also is T-periodic with respect to t.*

Based on the results of [7, 23] and using Remark 1, we have the lemma.

Lemma 3.1. *Let the assumption A1 be satisfied. Then for any $t\in[0,T]$:*

(i) *the First Problem has a solution $\{\bar{P}_{30}(t),Q_{40}^{\tau}(t,\eta),R_{30}^{\tau,\rho}(t,\eta,\chi),\ (\eta,\chi)\in[-h,0]\times$*

$[-h,0]\}$ *such that* $\bar{P}_{30}(t) \geq 0$ *and the matrix* $\begin{pmatrix} \bar{P}_{30}(t) & Q^{\tau}_{40}(t,\chi) \\ \big(Q^{\tau}_{40}(t,\eta)\big)' & R^{\tau,\rho}_{30}(t,\eta,\chi) \end{pmatrix}$ *defines a linear bounded self-adjoint positive operator mapping the space* $E^m \times L^2[-h,0;E^m]$ *into itself;*

(ii) *such a solution of the First Problem is unique;*

(iii) *all roots* λ *of the equation* $\det\Big[A_4(t) - S_3(t)\bar{P}_{30}(t) + H_4(t)\exp(-\lambda h)$ $+ \int_{-h}^{0}\Big(G_4(t,\eta) - S_3(t)Q^{\tau}_{40}(t,\eta)\Big)\exp(\lambda\eta)d\eta - \lambda I_m\Big] = 0$ *lie inside the left-hand half-plane;*

(vi) $\bar{P}_{30}(0) = \bar{P}_{30}(T)$, $Q^{\tau}_{40}(0,\eta) = Q^{\tau}_{40}(T,\eta)$, $R^{\tau,\rho}_{30}(0,\eta,\chi) = R^{\tau,\rho}_{30}(T,\eta,\chi)$, $(\eta,\chi) \in [-h,0]\times[-h,0]$.

By virtue of the results of [13], we have the corollary.

Corollary 3.2. *Let the assumption (A1) be satisfied. Then, the derivatives* $d\bar{P}_{30}(t)/dt$, $\partial Q^{\tau}_{40}(t,\eta)/\partial t$, $\partial R^{\tau,\rho}_{30}(t,\eta,\chi)/\partial t$ *exist and are continuous functions of* $t \in [0,T]$ *uniformly in* $(\eta,\chi) \in [h,0]\times[h,0]$.

3.2.2. The second problem. This problem has the form

$$\begin{aligned}
\partial Q^{\tau}_{30}(t,\eta)/\partial\eta &= \big[A_4'(t) - \bar{P}_{30}(t)S_3(t)\big]Q^{\tau}_{30}(t,\eta) + \bar{P}_{30}(t)G_3(t,\eta) + [R^{\tau,\rho}_{20}(t,\eta,0)]', \\
(\partial/\partial\eta + \partial/\partial\chi)R^{\tau,\rho}_{20}(t,\eta,\chi) &= G_3'(t,\eta)Q^{\tau}_{40}(t,\chi) + [Q^{\tau}_{30}(t,\eta)]'G_4(t,\chi) \\
&\quad - [Q^{\tau}_{30}(t,\eta)]'S_3(t)Q^{\tau}_{40}(t,\chi), \\
Q^{\tau}_{30}(t,-h) &= \bar{P}_{30}(t)H_3(t), \\
R^{\tau,\rho}_{20}(t,-h,\eta) = H_3'(t)Q^{\tau}_{40}(t,\eta), &\quad R^{\tau,\rho}_{20}(t,\eta,-h) = [Q^{\tau}_{30}(t,\eta)]'H_4(t).
\end{aligned} \tag{3.17}$$

Remark 2. *Like in the First Problem (3.16), in the Second problem (3.17)* $t \in [0,T]$ *is a parameter. Moreover, similarly to the First Problem, the solution of the Second Problem (if it exists and is unique) is* T*-periodic with respect to* t.

Based on Lemma 3.1, Corollary 3.2 and the results of [11], we obtain the lemma.

Lemma 3.3. *Under the assumption A1, for any* $t \in [0,T]$, *the Second Problem has the unique solution* $\{Q^{\tau}_{30}(t,\eta), R^{\tau,\rho}_{20}(t,\eta,\chi),\ (\eta,\chi) \in [-h,0]\times[-h,0]\}$, *where* $Q^{\tau}_{30}(t,\eta)$ *is the unique solution of the initial-value problem for the integral-differential equation*

$$\begin{aligned}
\partial Q^{\tau}_{30}(t,\eta)/\partial\eta &= \big[A_4'(t) - \bar{P}_{30}(t)S_3(t)\big]Q^{\tau}_{30}(t,\eta) \\
&+ \int_{-h}^{\eta}\Big[G_4(t,s-\eta) - S_3(t)Q^{\tau}_{40}(t,s-\eta)\Big]'Q^{\tau}_{30}(t,s)ds + [Q^{\tau}_{40}(t,-\eta-h)]'H_3(t) \\
&+ \int_{-h}^{\eta}[Q^{\tau}_{40}(t,s-\eta)]'G_3(t,s)ds, \qquad Q^{\tau}_{30}(t,-h) = \bar{P}_{30}(t)H_3(t).
\end{aligned} \tag{3.18}$$

The matrix-valued function $R^{\tau,\rho}_{20}(t,\eta,\chi)$ *has the explicit form*

$$\begin{aligned}
R^{\tau,\rho}_{20}(t,\eta,\chi) &= \Phi_{20}(t,\eta,\chi) + \int_{\max(\eta-\chi-h,-h)}^{\eta}\Big[G_3'(t,s)Q^{\tau}_{40}(t,s-\eta+\chi) \\
&\quad + [Q^{\tau}_{30}(t,s)]'G_4(t,s-\eta+\chi) - [Q^{\tau}_{30}(t,s)]'S_3(t)Q^{\tau}_{40}(t,s-\eta+\chi)\Big]ds \\
\Phi_{20}(t,\eta,\chi) &= \begin{cases} H_3'(t)Q^{\tau}_{40}(t,\chi-\eta-h), & -h \leq \eta-\chi \leq 0 \\ \big(Q^{\tau}_{30}(t,\eta-\chi-h)\big)'H_4(t), & 0 < \eta-\chi \leq h. \end{cases}
\end{aligned} \tag{3.19}$$

Moreover, $Q^{\tau}_{3}(0,\eta) = Q^{\tau}_{3}(T,\eta)$, $R^{\tau,\rho}_{20}(0,\eta,\chi) = R^{\tau,\rho}_{20}(T,\eta,\chi)$, $(\eta,\chi) \in [-h,0]\times[-h,0]$, *and the derivatives* $\partial Q^{\tau}_{3}(t,\eta)/\partial t$, $\partial R^{\tau,\rho}_{20}(t,\eta,\chi)/\partial t$ *exist and are continuous functions of* $t \in [0,T]$ *uniformly in* $(\eta,\chi) \in [-h,0]\times[-h,0]$.

3.2.3. The third problem. This problem has the form

$$(\partial/\partial\eta + \partial/\partial\chi)R_{10}^{\tau;\rho}(t,\eta,\chi) = G_3^{'}(t,\eta)Q_{30}^{\tau}(t,\chi) + [Q_{30}^{\tau}(t,\eta)]^{'}G_3(t,\chi) - [Q_{30}^{\tau}(t,\eta)]^{'}S_3(t)Q_{30}^{\tau}(t,\chi),$$

$$R_{10}^{\tau;\rho}(-h,\eta) = H_3^{'}Q_{30}^{\tau}(\eta), \quad R_{10}^{\tau;\rho}(\eta,-h) = [Q_{30}(\eta)]^{'}H_3. \tag{3.20}$$

REMARK 3. *Similarly to the First and Second Problems, the solution of the Third Problem (3.20) (if it exists and is unique) is T-periodic in the parameter t.*

Using Lemma 3.3 and the results of [11], we obtain the lemma.

LEMMA 3.4. *Under the assumption A1, for any $t \in [0,T]$, the Third Problem has the unique solution $R_{10}^{\tau;\rho}(t,\eta,\chi)$, $(\eta,\chi) \in [-h,0]\times[-h,0]$:*

$$R_{10}^{\tau;\rho}(t,\eta,\chi) = \Phi_{10}(t,\eta,\chi) + \int_{\max(\eta-\chi-h,-h)}^{\eta}\Big[G_3^{'}(t,s)Q_{30}^{\tau}(t,s-\eta+\chi) + [Q_{30}^{\tau}(t,s)]^{'}G_3(t,s-\eta+\chi) - [Q_{30}^{\tau}(t,s)]^{'}S_3(t)Q_{30}^{\tau}(t,s-\eta+\chi)\Big]ds$$

$$\Phi_{10}(t,\eta,\chi) = \begin{cases} H_3^{'}(t)Q_{30}^{\tau}(t,\chi-\eta-h), & -h \le \eta-\chi \le 0 \\ \big(Q_{30}^{\tau}(t,\eta-\chi-h)\big)^{'}H_3(t), & 0 < \eta-\chi \le h. \end{cases} \tag{3.21}$$

Moreover, $R_{10}^{\tau;\rho}(0,\eta,\chi) = R_{10}^{\tau;\rho}(T,\eta,\chi)$, $(\eta,\chi)\in[-h,0]\times[-h,0]$, and the derivative $\partial R_{10}^{\tau;\rho}(t,\eta,\chi)/\partial t$ exists and is a continuous function of $t\in[0,T]$ uniformly in $(\eta,\chi)\in[-h,0]\times[-h,0]$.

3.2.4. The fourth problem. This problem has the form

$$d\bar{P}_{10}(t)/dt = -\bar{P}_{10}(t)A_1(t) - A_1^{'}(t)\bar{P}_{10}(t) - \bar{P}_{20}(t)A_3(t) - A_3^{'}(t)\bar{P}_{20}^{'}(t) + \bar{P}_{10}(t)S_1(t)\bar{P}_{10}(t) + \bar{P}_{10}(t)S_2(t)\bar{P}_{20}^{'}(t) + \bar{P}_{20}(t)S_2^{'}(t)\bar{P}_{10}(t) + \bar{P}_{20}(t)S_3(t)\bar{P}_{20}^{'}(t) - Q_{10}^{\tau}(t,0) - [Q_{10}^{\tau}(t,0)]^{'} - D_1(t),$$

$$\bar{P}_{10}(t)A_2(t) + \bar{P}_{20}(t)A_4(t) + A_3^{'}(t)\bar{P}_{30}(t) - \bar{P}_{10}(t)S_2(t)\bar{P}_{30}(t) - \bar{P}_{20}(t)S_3(t)\bar{P}_{30}(t) + Q_{20}^{\tau}(t,0) + [Q_{30}^{\tau}(t,0)]^{'} + D_2(t) = 0,$$

$$\partial Q_{10}^{\tau}(t,\eta)/\partial\eta = \Big[A_3^{'}(t) - \bar{P}_{10}(t)S_2(t) - \bar{P}_{20}(t)S_3(t)\Big]Q_{30}^{\tau}(t,\eta) + \bar{P}_{10}(t)G_1(t,\eta) + \bar{P}_{20}(t)G_3(t,\eta) + R_{10}^{\tau;\rho}(t,0,\eta),$$

$$\partial Q_{20}^{\tau}(t,\eta)/\partial\eta = \Big[A_3^{'}(t) - \bar{P}_{10}(t)S_2(t) - \bar{P}_{20}(t)S_3(t)\Big]Q_{40}^{\tau}(t,\eta) + \bar{P}_{10}(t)G_2(t,\eta) + \bar{P}_{20}(t)G_4(t,\eta) + R_{20}^{\tau;\rho}(t,0,\eta),$$

$$\bar{P}_{10}(0) = \bar{P}_{10}(T), \quad Q_{j0}^{\tau}(t,-h) = \bar{P}_{10}(t)H_j(t) + \bar{P}_{20}(t)H_{j+2}(t), \quad j=1,2. \tag{3.22}$$

REMARK 4. *In the differential equation with respect to $\bar{P}_{10}(t)$, $t\in[0,T]$ is an independent variable, while in the rest of the equations of the Fourth Problem (3.22) t is a parameter.*

Using the results of [11], we obtain the lemma.

LEMMA 3.5. *Under the assumption A1, the Fourth Problem is equivalent to the following set of equations:*

$$d\bar{P}_{10}(t)/dt = -\bar{P}_{10}(t)\bar{A}(t) - \bar{A}^{'}(t)\bar{P}_{10}(t) + \bar{P}_{10}(t)\bar{S}(t)\bar{P}_{10}(t) - \bar{D}(t), \ \bar{P}_{10}(0) = \bar{P}_{10}(T),$$

$$\bar{P}_{20}(t) = -\left(\bar{P}_{10}(t)L_1(t) + L_2(t) + \int_{-h}^{0}[Q_{30}^{\tau}(t,\eta)]^{'}d\eta\right),$$

$$Q^{\tau}_{j0}(t,\eta) = \bar{P}_{10}(t)H_j(t) + \bar{P}_{20}(t)H_{j+2}(t)$$
$$+[A^{'}_3(t) - \bar{P}_{10}(t)S_2(t) - \bar{P}_{20}(t)S_3(t)]\int_{-h}^{\eta} Q^{\tau}_{j+2,0}(t,\sigma)d\sigma$$
$$+\bar{P}_{10}(t)\int_{-h}^{\eta} G_j(t,\sigma)d\sigma + \bar{P}_{20}(t)\int_{-h}^{\eta} G_{j+2}(t,\sigma)d\sigma + \int_{-h}^{\eta} R^{\tau;\rho}_{j0}(t,0,\sigma)d\sigma, \tag{3.23}$$

where $j = 1,2$, $\bar{A}(t) = \hat{A}_1(t) - L_1(t)\hat{A}_3(t) + S_2(t)L^{'}_2(t) - L_1(t)S_3(t)L^{'}_2(t)$, $\hat{A}_i(t) = A_i(t) + H_i(t) + \int_{-h}^{0} G_i(t,\eta)d\eta$, $(i = 1, ..., 4)$, $\bar{S}(t) = \bar{B}(t)M^{-1}(t)\bar{B}^{'}(t)$, $\bar{B}(t) = B_1(t) - L_1(t)B_2(t)$, $\bar{D}(t) = D_1(t) - L_2(t)\hat{A}_3(t) - \hat{A}^{'}_3(t)L^{'}_2(t) - L_2(t)S_3(t)L^{'}_2(t)$, $L_1(t) = (\hat{A}_2(t) - S_2(t)N(t))K^{-1}(t)$, $L_2(t) = (\hat{A}^{'}_3(t)N(t) + D_2(t))K^{-1}(t)$, $K(t) = \hat{A}_4(t) - S_3(t)N(t)$, $N(t) = \bar{P}_{30}(t) + \int_{-h}^{0} Q^{\tau}_{40}(t,\eta)d\eta$.

In what follows, we assume:

(A2) $\operatorname{rank}\big[\bar{A}(t) - \lambda I_n, \bar{B}(t)\big] = n$ for any $t \in [0,T]$ and any complex λ with $\mathrm{Re}\lambda \geq 0$;
(A3) $\bar{D}(t) > 0$ for any $t \in [0,T]$.

Corollary 3.6. *Under the assumptions A1-A3, the Fourth Problem has the unique solution* $\big\{\bar{P}_{10}(t), \bar{P}_{20}(t), Q^{\tau}_{10}(t,\eta), Q^{\tau}_{20}(t,\eta), t \in [0,T], \eta \in [-h,0]\big\}$ *such that* $\bar{P}_{10}(t) > 0$, $t \in [0,T]$. *Moreover,* $\bar{P}_{20}(0) = \bar{P}_{20}(T)$, $Q^{\tau}_{10}(0,\eta) = Q^{\tau}_{10}(T,\eta)$, $Q^{\tau}_{20}(0,\eta) = Q^{\tau}_{20}(T,\eta)$, $\eta \in [-h,0]$, *and the derivatives* $d\bar{P}_{10}(t)/dt$, $d\bar{P}_{20}(t)/dt$, $\partial Q^{\tau}_{10}(t,\eta)/\partial t$, $\partial Q^{\tau}_{20}(t,\eta)/\partial t$ *exist and are continuous functions of* $t \in [0,T]$ *uniformly in* $\eta \in [-h,0]$.

Thus, the formal construction of the zero-order asymptotic solution to the problem (3.2)-(3.14) is completed.

3.3. Justification of the zero-order asymptotic solution to the problem (3.2)-(3.14). Consider the matrix

$$\begin{pmatrix} \bar{P}_{30}(t) & Q^{\tau}_{30}(\chi) & Q^{\tau}_{40}(\chi) \\ \big(Q^{\tau}_{30}(\eta)\big)^{'} & R^{\tau;\rho}_{10}(\eta,\chi) & R^{\tau;\rho}_{20}(\eta,\chi) \\ \big(Q^{\tau}_{40}(\eta)\big)^{'} & \big(R^{\tau;\rho}_{20}(\chi,\eta)\big)^{'} & R^{\tau;\rho}_{30}(\eta,\chi) \end{pmatrix}.$$

For any $t \in [0,T]$, this matrix defines a linear bounded self-adjoint operator $\mathcal{F}_t$ mapping the space $E^m \times L^2[-h,0;E^{n+m}]$ into itself. In what follows, we assume:
(A4) For any $t \in [0,T]$, the operator $\mathcal{F}_t$ is uniformly positive.

Using Lemmas 3.1, 3.3, 3.4, Corollaries 3.2, 3.6 and the results of [10, 11], we obtain the theorem.

Theorem 3.7. *Let the assumptions A1-A4 be valid. Then, there exists a number* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0,\varepsilon^*]$:
(I) the problem (3.2)-(3.14) has the unique solution $\big\{P_k(t,\varepsilon), Q_i(t,\tau,\varepsilon), R_k(t,\tau,\rho,\varepsilon), (k = 1,2,3; i = 1, ..., 4)\big\}$ *in the domain* Ω_ε *such that for any* $t \in [0,T]$ *the matrix* $\begin{pmatrix} P(t,\varepsilon) & Q(t,\rho,\varepsilon) \\ Q^{'}(t,\tau,\varepsilon) & R(t,\tau,\rho,\varepsilon) \end{pmatrix}$, *where* $P(t,\varepsilon)$, $Q(t,\tau,\varepsilon)$, $R(t,\tau,\rho,\varepsilon)$ *are given by (3.1), defines a linear bounded self-adjoint positive operator mapping the space* $E^{n+m} \times L^2[-\varepsilon h,0;E^{n+m}]$ *into itself;*
(II) this solution satisfies the inequalities $\big\|P_k(t,\varepsilon) - \bar{P}_{k0}(t)\big\| \leq a\varepsilon$, $\big\|Q_{i0}(t,\tau,\varepsilon) - Q^{\tau}_{i0}(t,\tau/\varepsilon)\big\| \leq a\varepsilon$, $\big\|R_k(t,\tau,\rho,\varepsilon) - R^{\tau;\rho}_{k0}(t,\tau/\varepsilon,\rho/\varepsilon)\big\| \leq a\varepsilon$, $(k = 1,2,3;\ i = 1, ..., 4)$, $(t,\tau,\rho) \in \Omega_\varepsilon$, *where* $a > 0$ *is some constant independent of* ε.

Remark 5. *Note, that the* ε*-free assumptions A1-A2 yield the fulfilment of the equality (2.15) providing the existence and uniqueness of the corresponding solution to the problem (2.10)-(2.14),(2.16) for all* $\varepsilon \in (0,\varepsilon^*]$. *Moreover, these conditions, along with A3-A4, guarantee the validity of the inequalities presented in Theorem 3.7.*

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