EQUADIFF 14

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In: Karol Mikula (ed.): Proceedings of Equadiff 14, Conference on Differential Equations and Their Applications, Bratislava, July 24-28, 2017. Slovak University of Technology in Bratislava, SPEKTRUM STU Publishing, Bratislava, 2017. pp. 117–126.

Persistent URL: http://dml.cz/dmlcz/703036

Proceedings of EQUADIFF 2017
pp. 117–126

EXPONENTIAL CONVERGENCE TO THE STATIONARY MEASURE AND HYPERBOLICITY OF THE MINIMISERS FOR RANDOM LAGRANGIAN SYSTEMS. *

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Abstract. We consider a class of 1d Lagrangian systems with random forcing in the space-periodic setting:

$$\phi_t + \phi_x^2/2 = F^\omega,\ x \in S^1 = \mathbf{R}/\mathbf{Z}.$$

These systems have been studied since the 1990s by Khanin, Sinai and their collaborators [7, 9, 11, 12, 15]. Here we give an overview of their results and then we expose our recent proof of the exponential convergence to the stationary measure [6]. This is the first such result in a classical setting, i.e. in the dual-Lipschitz metric with respect to the Lebesgue space L_p for finite p, partially answering the conjecture formulated in [11]. In the multidimensional setting, a more technically involved proof has been recently given by Iturriaga, Khanin and Zhang [13].

Key words. Lagrangian dynamics, Random dynamical systems, Invariant measure, Hyperbolicity

AMS subject classifications. 35Q53, 35R60, 35Q35, 37H10, 76M35.

1. Introduction and setting. We are concerned with 1d random Lagrangian systems of the mechanical type, i.e. of the form:

$$L^\omega(x, v, t) = v^2/2 + F^\omega(x, t),\ x \in S^1 = \mathbf{R}/\mathbf{Z},$$

where $F^\omega(x, t)$ is a smooth function in x and a stationary random process in t (of the kick or white force type: see Section 1.1). The Legendre-Fenchel transform gives us the corresponding Hamiltonian $H^\omega(x, p, t) = p^2/2 - F^\omega(x, t)$, and the Hamilton-Jacobi equation:

$$\phi_t + \phi_x^2/2 = F^\omega. \tag{1.1}$$

Here, we consider only 1-periodic solutions ϕ. In this case the function $u = \phi_x$ satisfies the randomly forced inviscid Burgers equation:

$$u_t + uu_x = (F^\omega)_x,\ x \in S^1 = \mathbf{R}/\mathbf{Z}. \tag{1.2}$$

Note that it is equivalent to consider a solution of (1.2) and a solution of (1.1) defined up to an additive constant. Under the assumptions which are specified below, both of these equations are well-posed and their solutions define Markov processes. The existence and uniqueness of a corresponding stationary measure has been proved by E, Khanin, Mazel and Sinai in the white force case in the seminal work [9]. For more general (multi-d) results, see papers by Khanin and his collaborators [7, 11, 12, 15]. Note that in these papers, there are no explicit estimates on the speed of convergence

*This work was supported by the grants ANR WKBHJ and ANR ISDEEC.
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to the stationary measure; nevertheless, an exponential bound locally in space away from the shocks has been obtained by Bec, Frisch and Khanin in [1]. All these papers use Lagrangian techniques; except in [11] the authors do not consider the equation (1.2) with an additional viscous term νu_{xx}. Note that for $\nu > 0$ there is exponential convergence to the stationary measure, but the speed of convergence is not a priori uniform in ν [16].

In [6], we prove an exponential bound for the speed of convergence to the stationary measure for solutions of (1.2) for $\nu = 0$ in the natural dual-Lipschitz metric with respect to L_p, $p \in [1, \infty)$. This gives a partial answer in the 1d case to the conjecture stated in [11, Section 4]. This bound is the natural SPDE analogue to the results on the exponential convergence of the minimising action curves [7, 9]. The part of the conjecture in [11] which remains open is proving that if we add a positive viscosity coefficient ν, this exponential bound still holds, uniformly in ν.

It is very likely that the estimate we obtain is sharp since it coincides with the optimal one obtained in the generic nonrandom case by Iturriaga and Sanchez-Morgado [14]. Note that the metrics are also optimal since it is impossible to obtain such an estimate in the Lipschitz-dual space corresponding to L_∞. Indeed, solutions of (1.2) are discontinuous with a positive probability.

Finally, we would like to emphasize that our work is part of a series of papers giving a stochastic version of the weak KAM theory developed by Fathi and Mather [10]. In particular, there is a striking correspondence between the scheme of our proof and the one in [14], which follows a general rule: the results which hold in the random case under fairly weak assumptions are similar to the results which hold in the nonrandom case under more stringent genericity assumptions. For more on this subject and the link with the Aubry-Mather theory, see [12].

Remark 1.1. *Our results extend to the case where ϕ, instead of being periodic in space, satisfies $\phi(x+1) = \phi(x) + b$, $x \in \mathbf{R}$. Indeed, we use the results of [7, 9], which hold for all values of b. Moreover, our results extend to a class of non-mechanical convex in p Hamiltonians of the type $H(p) + F^\omega(t,x)$ with F^ω as above, under assumptions of the Tonelli type [10].*

Remark 1.2. *After the manuscript [6] has been submitted, Iturriaga, Khanin and Zhang published a preprint containing more general results including also the multidimensional case [13]. Their methods are more technically involved.*

1.1. Random setting. We consider the mechanical Hamilton-Jacobi equation with two different types of additive forcing in the right-hand side and a continuous initial condition ϕ^0. We begin by formulating the assumptions on potentials, which are (except 1.1 (i) where we add an additional assumption for moments of the random variable) the same as in the paper [7]:

Assumption 1.1. *In the kicked case, we assume that:*

(i) The kicks at integer times j are of the form $F^\omega(j)(x) = \sum_{k=1}^K c_k^\omega(j) F^k(x)$, where F^k are C^∞-smooth potentials on $S^1 = \mathbf{R}/\mathbf{Z}$. The vectors $(c_k^\omega(j))_{1 \leq k \leq K}$ are independent identically distributed $\mathbf{R}^K$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Their distribution on $\mathbf{R}^K$, denoted by λ, is assumed to be absolutely continuous with respect to the Lebesgue measure, and all of its moments are assumed to be finite.

(ii) The potential 0 belongs to the support of λ.

(iii) The mapping from S^1 to $\mathbf{R}^K$ defined by $x \mapsto (F^1(x), ..., F^K(x))$ is an embedding.

ASSUMPTION 1.2. *In the case of the white force potential, we assume that:*

(i) The forcing has the form $F^\omega(x,t) = \sum_{k=1}^K (W_k^\omega)_t(t) F^k(x)$, where F^k are C^∞-smooth potentials on S^1, and $(W_k^\omega)_t$ are independent white noises defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. time derivatives of independent Wiener processes $W_k^\omega(t)$.

(ii) The mapping from S^1 to $\mathbf{R}^K$ defined by $x \mapsto (F^1(x), ..., F^K(x))$ is an embedding.

REMARK 1.3. *For both types of forcing, our results extend to the case of infinite-dimensional noise, as long as it remains smooth in space (for example independent white noises on each Fourier mode with the amplitude of the noise decreasing exponentially with the wavenumber).*

1.2. Functional spaces and Sobolev norms. Consider an integrable function v on S^1. For $p \in [1, \infty]$, we denote its L_p norm by $|v|_p$. The L_2 norm is denoted by $|v|$, and $\langle \cdot, \cdot \rangle$ stands for the L_2 scalar product. Subindices t and x, which can be repeated, denote partial differentiation with respect to the corresponding variables. We denote by $v^{(m)}$ the m-th derivative of v in the variable x. For brevity, the function $v(t, \cdot)$ is denoted by $v(t)$.

For a nonnegative integer m and $p \in [1, \infty]$, $W^{m,p}$ stands for the Sobolev space of zero mean value functions v on S^1 with finite homogeneous norm $|v|_{m,p} = \left|v^{(m)}\right|_p$. In particular, $W^{0,p} = L_p$ for $p \in [1, \infty]$. We will never use Sobolev norms with $m \geq 1$ for non-zero mean functions: in particular, for solutions of (1.1) we will only consider the Lebesgue norms. On the other hand, C^0 (resp. C^∞) will denote the space of C^0-smooth (resp. C^∞-smooth) (not necessarily zero mean value!) functions on S^1.

Since the length of S^1 is 1, we have:

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \dots$$

We denote by $L_\infty/\mathbf{R}$ the space of functions in L_∞ defined modulo an additive constant endowed with the norm:

$$|u|_{L_\infty/\mathbf{R}} = \inf_{c \in \mathbf{R}} |u - c|_\infty$$

The quantities denoted by K, M or M' are positive constants which only depend on the general features of the system (i.e. the statistical distribution of the forcing): they are nonrandom and do not depend on the initial condition. Moreover the constants $K(p)$ depend on the Lebesgue exponent $p \in [1, \infty)$.

There are two quantities, denoted respectively by $\tilde{C}$ and $\tilde{C}_p$, which are time-independent random variables with all moments finite, which do not depend on the initial condition, but only "pathwise" on the forcing; moreover the quantity $\tilde{C}_p$ depends on the parameter p.

Quantities denoted by C are time-dependent random variables, which also have finite moments and do not depend on the initial condition, but only "pathwise" on the forcing ω. Moreover, these random variables are stationary in the sense that $C(s, \omega)$ coincides with $C(s+t, \theta^t \omega)$ for every t, where θ^t denotes the time shift [9].

We will always denote by $\phi(t,x)$ a solution of (1.1) and by $u(t,x)$ its derivative, which solves (1.2), respectively for initial conditions ϕ^0 and $u^0 = \phi_x^0$. We will denote accordingly the solutions for two initial conditions ϕ^0, $\overline{\phi^0}$.

2. Dynamical objects and stationary measure. Here we introduce the Lagrangian dynamical objects. Note that the results in Sections 2.2 hold under much more general assumptions; nevertheless these hypotheses will be extremely important for the results which will be given in Section 2.3. For more details see [11, 12].

2.1. Lagrangian formulation and minimisers. DEFINITION 2.1. *For a time interval $[s,t]$ and $x,y \in S^1$, we say that a curve $\gamma_{s,t}^{y,x}(\tau)$ is a* ***minimiser*** *if it minimises the action*

$$A(\gamma) = \frac{1}{2}\int_s^t \gamma_t(\tau)^2 d\tau + \sum_{n\in(s,t]} \Big(F(n)(\gamma(n))\Big)$$

in the "kicked" case and the action

$$\begin{aligned} A(\gamma) =& \frac{1}{2}\int_s^t \gamma_t(\tau)^2 d\tau + \int_s^t \left(\gamma_t(\tau)\Big(\frac{\partial G}{\partial x}(\gamma(\tau),s) - \frac{\partial G}{\partial x}(\gamma(\tau),\tau)\Big)\right) d\tau \\ &+ \Big(G(\gamma(t),t) - G(\gamma(t),s)\Big) \end{aligned}$$

in the white force case, respectively, over all absolutely continuous curves γ such that $\gamma(t) = x$ and $\gamma(s) = y$. Here G denotes a primitive in space of F. Note that in the kicked case, minimising curves are linear on intervals $[n, n+1]$ for integer values of n.

DEFINITION 2.2. *For a time interval $[s,t]$, $x \in S^1$ and a continuous function $\phi : S^1 \to \mathbf{R}$, we say that a curve $\gamma_{s,t,\phi}^x(\tau) : [s,t] \to S^1$ is a* ***ϕ-minimiser*** *if it minimises $A(\gamma) + \phi(\gamma(s))$ over all absolutely continuous curves on $[s,t]$ such that $\gamma(t) = x$. In particular, all ϕ-minimisers are minimisers.*

Now we can define the (pathwise) solution to (1.1) for a given $\omega \in \Omega$ and a given continuous initial condition.

DEFINITION 2.3. *For a time interval $[s,t]$ and a continuous initial condition $\phi(s) : S^1 \to \mathbf{R}$, for every ω by definition the (pathwise) solution $\phi : [s,t] \times S^1 \to \mathbf{R}$ of (1.1) is defined using the ω-dependent action A by the Hopf-Lax formula:*

$$\phi(\tau, x) = A(\gamma) + \phi(s, \gamma(s)),\ \tau \in [s,t],$$

where $\gamma = \gamma_{s,\tau,\phi(s)}^x$ is an ω-dependent $\phi(s)$-minimiser defined on $[s,\tau]$ satisfying $\gamma(\tau) = x$.

REMARK 2.4. *It is easy to check that the solution ϕ verifies the semigroup property: in other words, one can define a solution operator*

$$\Sigma_{t_1}^{t_2} :\ \phi(t_1) \mapsto \phi(t_2),\ s \le t_1 \le t_2 \le t,$$

such that for $t_1 \le t_2 \le t_3$, $\Sigma_{t_2}^{t_3} \circ \Sigma_{t_1}^{t_2} = \Sigma_{t_1}^{t_3}$. In particular, for any $\tau \in (s,t)$, the restriction of any $\phi(s)$-minimiser defined on $[s,t]$ to the time interval $[\tau,t]$ is a

$\Sigma_s^\tau\phi(s)$*-minimiser.*

REMARK 2.5. *Note that the solution* ϕ *is the limit in* C^0 *of the strong solutions to the equation obtained if we add a viscous term* $\nu\phi_{xx}$ *to (1.1) and then make* ν *tend to* 0 *(see [11]).*

DEFINITION 2.6. *For a time* t *and a point* $x \in S^1$*, we say that a curve* $\gamma_t^{x,+}(\tau):$ $[t,+\infty) \mapsto S^1$ *is a forward one-sided minimiser if it minimises* $A(\gamma)$ *over all absolutely continuous curves such that* $\gamma(t) = x$ *for compact in time perturbations.*

Namely, we require that if for a curve $\tilde{\gamma}$ *such that* $\tilde{\gamma}(t) = x$ *there exists* T *such that* $\tilde{\gamma}(s) \equiv \gamma(s)$ *for* $s \geq T$*, then* $A(\gamma) - A(\tilde{\gamma}) \leq 0$ *(this difference is well-defined since it is equal to the difference of the actions on the finite interval* $[t,T]$*).*

2.2. Stationary measure and related issues. Here we give a few results which hold under weak assumptions and are sufficient to ensure that the stationary measure corresponding to (1.2) exists and is unique. Up to some natural modifications due to the fact that the forcing is now discrete in time, the convergence estimates can be generalised to the kick force case in 1d [2] and to the multidimensional setting [5].

The flow corresponding to (1.2) induces a Markov process, and then we can define the corresponding semigroup denoted by S_t^*, acting on Borel measures on any L_p, $1 \leq p < \infty$. A *stationary measure* for (1.2) is a Borel probability measure defined on L_p, invariant with respect to S_t^* for every t. A *stationary solution* of (1.2) is a random process v defined for $(t,\omega) \in [0,+\infty)\times\Omega$, satisfying (1.2) and taking values in L_p, such that the distribution of $v(t)$ does not depend on t. This distribution is automatically a stationary measure. Existence of a stationary measure for (1.2) is obtained using uniform bounds for solutions in BV, which is compactly injected into $L_p,\ p \in [1,\infty)$, and the Bogolyubov-Krylov argument. It is more difficult to obtain uniqueness of a stationary measure, which implies uniqueness for the distribution of a stationary solution.

REMARK 2.7. *The most natural space for our model would be the space* $L_\infty/\mathbf{R}$ *(for the solutions to the equation (1.1)). Moreover, this is the space in which exponential convergence to the unique stationary solution is proved in the deterministic generic setting in [14]. However, this space is not separable, which makes dealing with the stationary measure a delicate issue.*

DEFINITION 2.8. *Fix* $p \in [1,\infty)$*. For a continuous function* $g:\ L_p \to \mathbf{R}$*, we define its Lipschitz norm as*

$$|g|_{L(p)} := |g|_{Lip} + \sup_{L_p} |g|,$$

where $|g|_{Lip}$ *is the Lipschitz constant of* g*. The set of continuous functions with finite Lipschitz norm will be denoted by* $L(p)$*.*

DEFINITION 2.9. *For two Borel probability measures* μ_1,μ_2 *on* L_p*, we denote by* $\|\mu_1-\mu_2\|_{L(p)}^*$ *the Lipschitz-dual distance:*

$$\|\mu_1-\mu_2\|_{L(p)}^* := \sup_{g\in L(p),\ |g|_{L(p)}\leq 1} \Big| \int_{S^1} g d\mu_1 - \int_{S^1} g d\mu_2 \Big|.$$

The following result proved in [2, 3, 5] is, as far as we are aware, the first explicit estimate for the speed of convergence to the stationary measure of the equation (1.2) with an additional viscous term νu_{xx} which is uniform with respect to the viscosity coefficient ν and is formulated in terms of Lebesgue spaces only.

THEOREM 2.10. *There exists $\delta > 0$ such that for every $p \in [1, \infty)$, we have:*

$$\|S_t^* \mu_1 - S_t^* \mu_2\|_{L(p)}^* \leq K(p) t^{-\delta/p}, \qquad t \geq 1,$$

for any probability measures μ_1, μ_2 on L_p.

2.3. Main results and scheme of the proof. Now we are ready to state the main result of the paper.

THEOREM 2.11. *For every $p \in [1, \infty)$, we have:*

$$\|S_t^* \mu_1 - S_t^* \mu_2\|_{L(p)}^* \leq K(p) exp(-M' t/p), \qquad t \geq 0, \tag{2.1}$$

for any probability measures μ_1, μ_2 on L_p.

The proof is, in the spirit, similar to the proof of [14, Theorem 1]. In that paper the authors use the objects of the weak KAM theory which do not have any directly available counterparts in our setting. However, there is a straightforward dynamical interpretation of their method.

Namely, consider a mechanical Lagrangian $v^2/2 - V(x)$ such that the *deterministic* potential V is smooth and generic (i.e. it has a unique nongenerate maximum at a unique point y_0). An action-minimising curve on $[0, T]$ remains in a small neighbourhood of y_0 on $[C, T-C]$. We obtain by linearising the Euler-Lagrange equation that at the time $T/2$, all minimisers (independently of the initial condition) are $C \exp(-CT)$-close to y_0, and then we conclude that for any initial conditions ϕ^0, $\overline{\phi^0}$, the solutions of (1.1) at time T are $C \exp(-CT)$-close up to an additive constant.

There are two main ingredients in the proof. On one hand for a given initial condition ϕ^0, the ϕ^0-minimisers corresponding to different final points concentrate exponentially. On the other hand, one-sided minimisers, which are the limits of ϕ^0_T-minimisers on $[0, T]$ as $T \to +\infty$ for *any* set of initial conditions $\{\phi^0_T\}$, also concentrate exponentially.

Now we introduce some definitions.

The diameter of a closed set Z can be thought of as the minimal length of a closed interval on S^1 containing Z.

DEFINITION 2.12. *Consider a closed subset Z of S^1. Let $a(Z)$ denote the maximal length of a connected component of $S^1 - Z$. We define the diameter of Z as $d(Z) = 1 - a(Z)$.*

DEFINITION 2.13. *For $-\infty < r < s \leq t < +\infty$ and for a fixed function $\phi^0 : S^1 \to \mathbf{R}$, let Ω_{r,s,t,ϕ^0} be the set of points reached, at the time s, by ϕ^0-minimisers on*

$[r,t]$:

$$\Omega_{r,s,t,\phi^0} = \{\gamma^x_{r,t,\phi^0}(s),\ x \in S^1\}.$$

Now we give two key estimates. The first one is - up to notation - [7, Corollary 2.1]. The second one is a forward-in-time version of [9, Lemma 5.6 (a)].

LEMMA 2.14. *We have the inequality:*

$$\sup_{\phi^0 \in C^0} d(\Omega_{0,s,s+s',\phi^0}) \leq C(s') \exp(-Ks').$$

LEMMA 2.15. *We have:*

$$\sup_{\tilde{\gamma}_1,\tilde{\gamma}_2 \in \Gamma} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)| \leq \tilde{C} \exp(-Kt),\ t \geq 0. \tag{2.1}$$

where Γ is the set of all forward one-sided minimisers defined on the time interval $[0,+\infty)$.

COROLLARY 2.16. *Consider an initial condition ϕ^0 and a time $t > 0$. Then for any ϕ^0-minimiser $\gamma : [0,2t] \to S^1$ and any forward one-sided minimiser $\delta :\ [0,+\infty) \to S^1$ we have:*

$$|\gamma(t) - \delta(t)| \leq C(t) \exp(-Kt). \tag{2.2}$$

Proof of Corollary 2.16: Extracting a subsequence of minimisers (for example ϕ^0-minimisers) on $[0,s]$ and taking the limit while letting s go to $+\infty$ (which is possible because of the bounds on the velocity of the minimisers: see Lemma 3.1), one gets a forward one-sided minimiser. In particular, for every ϵ there exists $s(\epsilon) \geq 2t$, a ϕ^0-minimiser $\tilde{\gamma}$ defined on $[0,s]$ and a forward one-sided minimiser $\tilde{\delta}$ on $[0,+\infty)$ such that:

$$|\tilde{\gamma}(t) - \tilde{\delta}(t)| \leq \epsilon.$$

By Lemma 2.15 we have:

$$|\delta(t) - \tilde{\delta}(t)| \leq \tilde{C} \exp(-Kt),$$

and by Lemma 2.14, since the restriction $\tilde{\gamma}|_{[0,2t]}$ is a ϕ^0-minimiser, we have:

$$|\gamma(t) - \tilde{\gamma}(t)| \leq C \exp(-Kt).$$

Combining these three inequalities and then letting ϵ go to 0, we get (2.2).

3. Proof of Theorem 2.11. First we state some useful estimates. For the proof of the first lemma, see [12, Lemma 6].

LEMMA 3.1. *For $t \geq 1$, we have:*

$$\sup_{\phi^0 \in C^0} |\phi_x(t)|_{1,1} \leq C(t); \quad \sup_{s \in [t,t+1], \gamma \in \Gamma} |\gamma_t(s)| \leq C(t),$$

where Γ is the set of minimisers defined on $[0,t+1]$.

LEMMA 3.2. *Consider two minimisers γ_1, γ_2, both defined on $[t,T]$, $T \geq t+1$, and satisfying $\gamma_1(T) = \gamma_2(T)$. If for $\epsilon > 0$ we have $|\gamma_1(t) - \gamma_2(t)| \leq \epsilon$, then we have the following inequality for the actions of the minimisers:*

$$|A(\gamma_1) - A(\gamma_2)| \leq C(t)(\epsilon + \epsilon^2).$$

Proof: By symmetry, it suffices to prove that:

$$A(\gamma_2) \leq A(\gamma_1) + C(\epsilon + \epsilon^2). \tag{3.1}$$

We consider the curve $\tilde{\gamma}_1 : [t,T] \to S^1$ defined by:

$$\tilde{\gamma}_1(s) = \gamma_1(s) + (t+1-s)(\gamma_2(t) - \gamma_1(t)),\ s \in [t,t+1].$$
$$\tilde{\gamma}_1(s) = \gamma_1(s),\ s \in [t+1,T].$$

Using Definition 2.1 and Lemma 3.1, we get:

$$A(\tilde{\gamma}_1) \leq A(\gamma_1) + C(\epsilon + \epsilon^2).$$

On the other hand, since $\tilde{\gamma}_1$ has the same endpoints as the minimiser γ_2, we get $A(\gamma_2) \leq A(\tilde{\gamma}_1)$. Combining these two inequalities yields (3.1).

The proof of the following lemma follows the lines of [14].

LEMMA 3.3. *Consider two solutions ϕ and $\overline{\phi}$ of (1.1) defined on the time interval $[0,+\infty)$. Then we have:*

$$|\phi(t) - \overline{\phi}(t)|_{L_\infty/\mathbf{R}} \leq C(t)\exp(-Mt),\ t \geq 0.$$

Proof of Lemma 3.3: Consider two solutions ϕ and $\overline{\phi}$ to (1.1) corresponding to the same forcing and different initial conditions at time 0. Using Definition 2.3, we get for any $t \geq 1$ and $x \in S^1$:

$$\phi(2t,x) - \overline{\phi}(2t,x) = \phi(t,\gamma_1(t)) + A(\gamma_1|_{[t,2t]}) - \overline{\phi}(t,\gamma_2(t)) - A(\gamma_2|_{[t,2t]}), \tag{3.2}$$

where γ_1 and γ_2 are respectively a ϕ^0- and a $\overline{\phi^0}$-minimiser on $[0,2t]$ ending at x. By Corollary 2.16, we have:

$$|\gamma_i(t) - y| \leq C\exp(-Kt),\ i = 1,2, \tag{3.3}$$

where we fix *any* point y such that $y = \delta(t)$ for a one-sided minimiser δ defined on $[0,\infty)$. By Lemma 3.1, this inequality yields that:

$$\begin{aligned} |\phi(t,\gamma_1(t)) - \overline{\phi}(t,\gamma_2(t)) - R| &\leq (|\phi_x(t)|_\infty |\gamma_1(t) - y| + |\overline{\phi}_x(t)|_\infty |\gamma_2(t) - y|) \\ &\leq 2C\exp(-Kt), \end{aligned}$$

where $R = \phi(t,y) - \overline{\phi}(t,y)$. On the other hand, using (3.3), by Lemma 3.2 we get that:

$$|A(\gamma_1|_{[t,2t]}) - A(\gamma_2|_{[t,2t]})| \leq C\exp(-Kt).$$

Therefore, by (3.2), we get:

$$|\phi(2t) - \overline{\phi}(2t)|_{L_\infty/\mathbf{R}} \leq \sup_{x \in S^1} |\phi(2t,x) - \overline{\phi}(2t,x) - R| \leq C \exp(-Kt).$$

This proves the lemma's statement.

COROLLARY 3.4. *Consider two solutions u and $\overline{u}$ of (1.2) defined on the time interval $[0,+\infty)$. Then for any $p>0$ we have:*

$$|u(t) - \overline{u}(t)|_p \leq \tilde{C}_p \exp(-Mt/2p),\ t \geq 0.$$

Proof: This result follows from Lemma 3.3 using the Gagliardo-Nirenberg inequality [8] and Lemma 3.1.

Proof of Theorem 2.11: By the Fubini theorem, it suffices to prove this result in the case when the measures μ_1 and μ_2 are two Dirac measures concentrated at the initial conditions $u^0, \overline{u^0} \in L_p$.

It follows from Corollary 3.4 that if we denote by B the event

$$B = \{\omega \in \Omega \mid |u(t) - \overline{u}(t)|_{L(p)} \geq \exp(-Mt/4p)\},$$

then we have:

$$\mathbf{P}(B) \leq \exp(-Mt/4p)\ \mathbf{E}\ \tilde{C}_p,\ t \geq 0.$$

Now consider a function g defined on L_p which satisfies $|g|_L \leq 1$. We have for $t \geq 0$:

$$\begin{aligned} &\mathbf{E}\ (|g(u(t)) - g(\overline{u}(t))|_p) \\ &\leq \mathbf{P}(B)\ \mathbf{E}\ (|g(u(t)) - g(\overline{u}(t))|_p \mid B) + \mathbf{P}(\Omega - B)\ \mathbf{E}\ (|g(u(t)) - g(\overline{u}(t))|_p \mid \Omega - B) \\ &\leq 2\mathbf{P}(B) + \mathbf{P}(\Omega - B) \exp(-Mt/4p) \leq (2\mathbf{E}\ \tilde{C}_p + 1) \exp(-Mt/4p). \end{aligned}$$

REMARK 3.5. *The estimate in Lemma 3.3 is uniform with respect to the initial conditions: in other words, we have*

$$\mathbf{E} \sup_{\phi^0, \overline{\phi^0} \in C^0} |\phi(t) - \overline{\phi}(t)|_{L_\infty/\mathbf{R}} \leq K \exp(-Mt),\ t \geq 0.$$

A similar statement holds for the estimate in Corollary 3.4.

Acknowledgments. I would like to thank P. Bernard, A. Davini, R. Iturriaga, K. Khanin and K. Zhang for helpful discussions.

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