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VISCO-ELASTO-PLASTIC MODELING *

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Abstract. In this paper we deal with the mathematical modelling of rheological models with applications in various engineering disciplines and industry. We study the mechanical response of visco-elasto-plastic materials. We describe the basic rheological elements and focus our attention to the specific model of concrete, for which we derive governing equations and discuss its solution. We provide an application of rheological model involving rigid-plastic element as well - mechanical and mathematical model of failure of one dimensional construction member, straight beam. Herein, the physical model is considered with a homogeneous isotropic material of the beam, quasi static regime is supposed.

Key words. rheological elements, constitutive equation, large deformations, hysteresis, dissipated energy

AMS subject classifications. 35J86, 00A79

Introduction. In mechanics, the constitutive relation between the stress σ and the strain ε, is essential. Rheology deals with problems concerning deformation processes of materials exhibiting different kinds of material response, e.g. elastic, viscous and plastic behavior. Time dependent mechanical behavior is governed by constitutive equations describing the relations between stress and strain variables and their time derivatives. There exist materials that behave in a different way during loading and unloading, some are and some are not able to recover. This phenomenon is called *hysteresis*. Herein, and this it is very well known fact [9, 7, 2], the potential energy plays important role. There are elementary matters, called also members or elements involved in each model. Very elegant presentation of basic rheological models can an interested reader find in the monograph [2]. There, the main focus is on elasto-plastic materials, hysteresis phenomena being the main area of interest. For the description of visco-plastic materials we refer to the book [1]. Corresponding models in electricity are studied in detail in [3].

There are specific tests executed on material models by prescribed stress or strain load action. The creep or relaxation of stress is recorded. Creep is a deformation change in time under constant stress load being maintained, relaxation is a stress change in time when a constant deformation is maintained. Boltzmann theory using hereditary integrals is exerted, as well, [5].

In the paper the basic phenomena of rheology models are introduced together with constitutive relation derivation techniques. Involving a viscous member in the model yields the presence of derivatives in physical equations, plastic element brings a variational inequality. Finally, the three of them - elastic, plastic and viscous members are involved in a very simple model of concrete. The constitutive relation is derived.

1. Fundamental elements, compositions, relations. In agreement with Definition 1.1 in [2] we call a rheological element a system consisting of a constitutive relation between stress σ and strain ε and a potential energy $U \geq 0$. Along this paper

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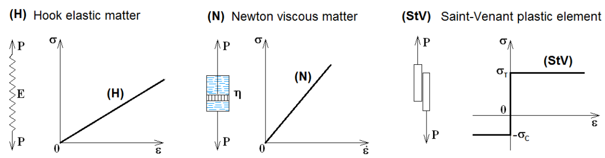

FIG. 1.1. *Stress - strain dependence in an a) elastic, b) viscous, c) rigid-plastic element.*

we will deal with uniaxial thermodynamically consistent rheological models, which means that the quantity called dissipation rate

$$\dot{q} = \langle \dot{\varepsilon}, \sigma \rangle - \dot{U} \tag{1.1}$$

will be supposed to be non-negative in sense of distributions for all ε, σ, [2].

1.1. Fundamental elements of a visco-elasto-plastic model, physical properties. [4, 2]

There are Newton's viscous (N), Hook's elastic (H) and Saint-Venant (StV) rigid-plastic elements involved in a visco-elasto-plastic model.

Elastic element (H) is represented by an ideally elastic spring, where the stress - strain relation is linear:

$$\sigma = \mathbf{A}\varepsilon, \tag{1.2}$$

with **A** an elastic modulus matrix, in the case of homogeneous isotropic material it is replaced by a real number E - Young elastic modulus. In more dimensions it includes both volumetric and deviatoric change. (H) is completely reversible, i.e. all inner potential energy U gathered in the loading process is conserved and no energy is dissipated. After loading stops all energy is used to reverse the previous position. Potential energy is given by $U = \frac{1}{2}E\varepsilon^2$ and it can be easily checked that the thermodynamical consistency of the model is fulfilled.

Viscous element (N) is symbolized graphically by a piston. Here we have is a linear relation between stress and strain rate, which can be expressed both in deviatoric and volumetric components $\sigma_{dev} = \eta\dot{\varepsilon}_{dev}, \quad \sigma_{vol} = \zeta\dot{\varepsilon}_{vol}$, with η and ζ being deviatoric and volumetric proportional coefficient respectively. For incompressible liquids only deviatoric component comes into play and the stress-strain relation can be expressed simply as

$$\sigma = \eta\, \dot{\varepsilon} \tag{1.3}$$

No potential energy is stored, i.e. $U = 0$, the deformation process is irreversible. Viscous elements act as dashpots.

Rigid plastic element (StV). Its graphical symbol is depicted as two touching plates with certain friction between. When a (StV) is exposed to a load, it remains rigid as long as the instantaneous stress does not reach the threshold. If so, material becomes plastic immediately.

Let Z be the space of all admissible stress values with all thresholds situated in its boundary ∂Z. The plasticity is governed by the following physical principles:

- $\sigma \in int(Z)$ ensures the rigidity persisting of the body
- $\sigma \in \partial Z$ (the plastic behavior is triggered)
- $\langle \dot{\varepsilon}, \sigma - \tilde{\sigma} \rangle \geq 0, \quad \forall \tilde{\sigma} \in Z$

The last principle, the variational inequality, is called the maximal dissipation rate principle with regard to admissible stress values. It states that while the threshold is not reached, the deformation does not change, i.e. $\forall \sigma \in int(Z) \implies \dot{\varepsilon} = 0$, [9, 2].

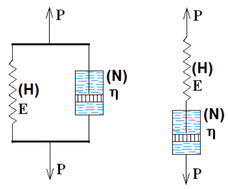

FIG. 1.2. *Parallel and serial combination of fundamental elements.*

In the uniaxial case $Z = \langle -\sigma_C, \sigma_T \rangle$ and $\partial Z = \{-\sigma_C, \sigma_T\}$, where we assume that σ_C, σ_T are two positive constants, so $0 \in Z$, which corresponds to the natural hypotheses that no deformation occurs for $\sigma = 0$. This condition is essential for the thermodynamic consistency of the model.

In Fig 1.1c) an uniaxial representation of rigid-plastic body is performed. The polygonal line graph is called three branch diagram. Herein, as Z is an interval, its boundary are the endpoints called compressive threshold $-\sigma_C$ and tension threshold σ_T. In general $\sigma_C \neq \sigma_T$. When a threshold is reached, plasticity proceeds and takes place until the magnitude decreases again and the rigidity comes back, permanent (plastic) deformation persists. No potential energy is stored, i.e. $U = 0$ and no recovery occurs. It has been observed that during plastic deformation the volume change is negligible. [6]

In Fig. 1.1 the graphical symbols representing particular elementary matters and graphical interpretation of the stress - strain relations are shown. Here P denotes the tension force.

1.2. Configuration, geometry and corresponding relations. There are two possible ways of connecting any couple of fundamental elements - either serially or in parallel, by using two auxiliary rigid slabs for this sake, as depicted in Fig. 1.2 a), the two slabs are represented by the upper and lower thick lines connecting (H) and (N).

- *Serial connection* of elementary members
  Under a load P, the resulting deformation of the system of serially connected elementary matters is the sum of the deformations of particular members, stress is distributed among the members equally:

$$\varepsilon = \varepsilon_H + \varepsilon_N, \quad \sigma = \sigma_H = \sigma_N. \tag{1.4}$$

  Sub-indexes H, N, then SvV indicate an incidence with Hook elastic, Newton viscous and Saint-Venant rigid-plastic matters.
- *Parallel connection* of elementary members
  As two linking slabs are shifting vertically up or down without any rotation, the deformation is the same, while the stress of the entire model is the sum of stresses of particular members:

$$\varepsilon = \varepsilon_H = \varepsilon_N, \quad \sigma = \sigma_H + \sigma_N. \tag{1.5}$$

Having the two or more elementary matters at hand and utilizing both parallel and serial connection, and considering the fundamental elements as simplest rheological models, we can proceed in composing visco-elasto-plastic models recursively. By connecting two simpler models serially or in parallel we compose the new, more complex one. When we couple the geometry equations yielded by configuration with the

fundamental element constitutive relations into account, we can derive the resulting constitutive equation of the entire model.
For the sake of clear notation it is worth utilizing an abbreviations of such models. Having, beside (N), (H) and (Stv) marks standing for the particular fundamental elements, the vertical line standing for parallel and the horizontal line standing for serial, we can assign a structural formula to each model. Accordingly, the structural formulas of the two-element models in Fig. 1.2 are: $(H)|(N)$ for the left one and $(H)-(N)$ for the right one respectively.

2. Creep and relaxation tests. Creep and relaxation tests are typical for testing materials with the aim of their mechanical response prediction and materials' mechanical behavior comparison. The special load is imposed to the material and the response is recorded and monitored. Roughly speaking, creep-deformation change in time under the constant stress load is maintained or relaxation-stress change in time when a constant deformation is maintained.

Creep test is executed by inflicting an instantaneous stress keeping it constant in a given time period. The immediate change is obviously followed by subsidiary one - the creep. Resulting deformation response is recorded.

Relaxation test is executed by carrying out an instantaneous strain, keeping it constant during a given time period. The immediate change of stress is obviously followed by subsidiary one - the relaxation. Changes of stress are recorded.

3. Elasto-plasticity resulting in Hysteresis. Let us examine what happens when we combine elastic and plastic element. First for the combination in paralel we get

$$\varepsilon = \varepsilon_H = \varepsilon_{StV}, \tag{3.1}$$

$$\sigma = \sigma_H + \sigma_{StV}, \tag{3.2}$$

$$\sigma_H = E\,\varepsilon_H. \tag{3.3}$$

Let us employ the Saint Venant variational inequality for the rigid-plastic matter

$$\dot{\varepsilon}_{StV}\big(\sigma_{StV} - \tilde{\sigma}\big) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle \tag{3.4}$$

and we have

$$\dot{\varepsilon}\,(\sigma - E\varepsilon - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \tag{3.5}$$

For the potential energy we get $U = \frac{1}{2}E\varepsilon_H^2$ (the only contribution comes from the elastic element) and the thermodynamical consistency of the model follows actually from the variational inequality (3.5) (with $\tilde{\sigma} = 0$).

For the combination in series we get similarly

$$\varepsilon = \varepsilon_H + \varepsilon_{StV}, \tag{3.6}$$

$$\sigma = \sigma_H = \sigma_{StV}, \tag{3.7}$$

$$\sigma_H = E\,\varepsilon_H. \tag{3.8}$$

Let us now employ again the Saint Venant variational inequality for the rigid-plastic matter

$$\dot{\varepsilon}_{StV}\big(\sigma_{StV} - \tilde{\sigma}\big) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle \tag{3.9}$$

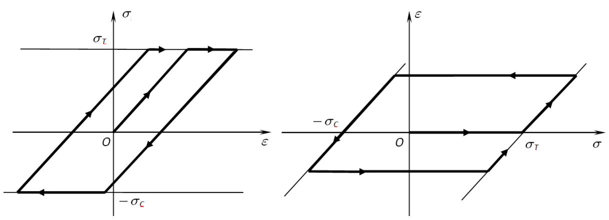

FIG. 3.1. *a) Stop and b) Play operators*

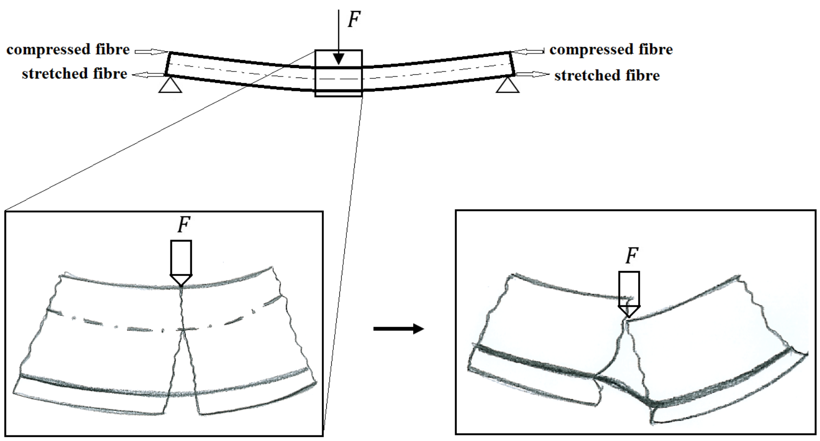

FIG. 4.1. *Concrete beam under heavy transversal load, compressed and stretched fibres, crack, rupture*

and we have as a consequence

$$\left(\dot{\varepsilon} - \frac{1}{E}\dot{\sigma}\right)(\sigma - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \tag{3.10}$$

The potential energy is again given by $U = \frac{1}{2}E\varepsilon_H^2$ and the thermodynamical consistency follows from (3.10) (again taking $\tilde{\sigma} = 0$).

The variational inequalities obtained in both cases (series or paralel) are both of the same type and it was shown e.g. in [2], Theorem 1.9, that there exists a unique solution of these variational inequalities, which is given by a hysteresis operator - the so-called play or stop operator respectively. Hysteresis operators exhibit memory effects (the current state depends on the previous history of the system) and they are rate independent (this property allows us to draw diagrams as on Fig.3.1. For more details in this direction we refer to [2] and the references therein.

4. Rheological model of concrete. There exists an exceptional group within the rheology of composite materials on a silicate basis, group of concretes and reinforced concretes. Due to mechanical, chemo-mechanical or thermo-mechanical load acting in concrete or steel-concrete constructions, some immediate, short-term and long-term deformations evolve, the change lasting up to several years. When the concrete mixture is poured into a form, it initiates the solidification together with a chemical processes resulting in a volume contraction regardless the load imposed. And, on the other hand, an imposed load activates a creep, hysteresis response in-

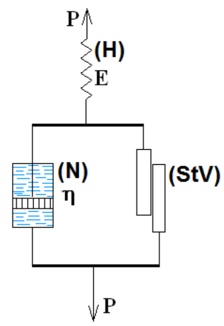

FIG. 4.2. *Parallel and serial connection of fundamental elements.*

volved. Creep is the essential phenomenon that has to be investigated carefully as the mechanical behavior of the designed structures and constructions can be predicted.

4.1. Concrete behavior under frequent heavy load. When the imposed load is of magnitude within the range obvious in concrete constructions, the resulting deformation as a consequence of creep will be several times greater than the initial (immediate) one. In this context, the notion "aging of concrete" is often used. [8] Nevertheless, the mechanical response of the concrete construction is proportional to the subjected load, accordingly the habitual operating load response of a construction is derived by using the superposition principle. However, once unloaded, a permanent deformation remains, [9].

Another essential fact concerning concrete has to be mentioned: Compressive strength of concrete is much higher then tensile strength. Hence, concrete mechanical response to tensile and to the compressive load of the same magnitude differs significantly. That is why the reinforcement with material strong in tension is placed where a tensile stress is supposed. In Fig. 4.1 the reinforcement is placed at the bottom of the beam. Namely, it is supposed to be doubly supported at the ends and loaded transversally by a pressure.

4.2. Simplified model of concrete - graphical representation structural form, geometrical relations. As proposed by [7], the simplest model of concrete can be set up by connecting viscous and rigid-plastic element in parallel and connect an elastic member with this couple in series. The structural form is $(H)-[(N)|(StV)]$. The physical relations are considered as given in Section 1.1, E being Young elastic modulus of (H), η the viscous coefficient of (N), σ_T and σ_C the stress tensile and compressive thresholds of the stress in (StV).

In the following considerations, the subindexes of stress and strain variables will be used to indicate the incidence with particular elementary members; e.g. ε_H will stand for partial deformation of Hook elastic matter, etc. Geometric equations of the $(H)-[(N)|(StV)]$ model are:

$$\varepsilon = \varepsilon_H + \varepsilon_N \tag{4.1}$$

$$\varepsilon_N = \varepsilon_{StV} \tag{4.2}$$

$$\sigma = \sigma_H = \sigma_N + \sigma_{StV}, \tag{4.3}$$

where $\sigma_{StV} \in \langle -\sigma_C, \sigma_T \rangle$.

Energy audit yields that the energy of elastic member is the only nonzero part of the potential energy of the whole system

$$U = \frac{1}{2} E \varepsilon_H^2$$

and the thermodynamical consistency of the model

$$\dot{\varepsilon}\sigma - \frac{1}{2} E \dot{\varepsilon}_H^2 \geq 0$$

follows from the variational inequality (4.9) bellow, taking $\tilde{\sigma} = 0$.

In the following we will deduce the constitutive relation of the model. We are looking for the $\sigma \sim \varepsilon$ equation, describing the dependence between global stress and global strain employing merely physical parameters of particular elementary members. It means we want to exclude the sub-indexed stress and strain variables from the dependence forms. Reminding elementary physical relations embedded in Section 1.1, we can proceed in the following way:

$$\sigma_H = E\,\varepsilon_H, \quad \sigma_N = \eta\,\dot{\varepsilon_N} \implies \varepsilon_N = \varepsilon - \varepsilon_H = \varepsilon - \frac{1}{E}\sigma_H, \tag{4.4}$$

$$\sigma = E\varepsilon_H = \eta\,\dot{\varepsilon}_N + \sigma_{StV}, \tag{4.5}$$

$$\sigma_{StV} = \sigma - \eta\,\dot{\varepsilon_N} = \sigma - \ \eta\dot{\varepsilon} + \frac{\eta}{E}\,\dot{\sigma}. \tag{4.6}$$

Let us employ the variational inequality of the Saint-Venant rigid-plastic matter

$$\dot{\varepsilon}_{StV}\,(\sigma_{StV} - \tilde{\sigma}) \geq 0, \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \tag{4.7}$$

Let us recall that

$$\varepsilon_{StV} = \varepsilon_N = \varepsilon - \varepsilon_H = \varepsilon - \frac{\sigma}{E}. \tag{4.8}$$

As a result we get the following variational inequality

$$\left(\dot{\varepsilon} - \frac{\dot{\sigma}}{E}\right)\left(\sigma - \eta\dot{\varepsilon} + \frac{\eta}{E}\dot{\sigma} - \tilde{\sigma}\right) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle \tag{4.9}$$

When we denote $\dot{v} = \dot{\varepsilon} - \frac{\dot{\sigma}}{E}$, we can rewrite (4.9) in the form

$$\dot{v}\big(\sigma - \eta\dot{v} - \tilde{\sigma}\big) \geq 0 \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle \tag{4.10}$$

Let us have a closer look at the variational inequality (4.10). First of all, if $\sigma - \eta\dot{v}$ is in the open interval $(-\sigma_C, \sigma_T)$, the second bracket can take positive or negative values as $\tilde{\sigma}$ changes. Therefore we must have $\dot{v} = 0$.

Let us now aim our attention to the compressive marginal value $\sigma - \eta\dot{v} = -\sigma_C$. In such a case, it is apparent for all $\tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle$ that the expression in brackets on the left hand side of (4.10) is non-positive. This implies that in order to hold the inequality (4.10) it must be hold $\dot{v} \leq 0$, so:

$$\dot{\varepsilon} - \frac{\dot{\sigma}}{E} \leq 0 \tag{4.11}$$

And accordingly, if $\sigma - \eta\dot{v} = -\sigma_T$ then it must be $\dot{v} \geq 0$ or

$$\dot{\varepsilon} - \frac{\dot{\sigma}}{E} \geq 0. \tag{4.12}$$

This can be described by a single relation in terms of v

$$\dot{v} = \frac{1}{\eta}(\sigma - \mathcal{P}(\sigma)), \tag{4.13}$$

where $\mathcal{P}$ denotes the projection on the interval $\langle -\sigma_C, \sigma_T \rangle$ (in the sense of convex analyses).

Alternatively in terms of σ and ε we have

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{1}{\eta}(\sigma - \mathcal{P}(\sigma)). \tag{4.14}$$

This is the constitutive relation we were looking for. It involves both stress on strain and strain on stress dependence. It means that with such a constitutive relation of the model at hand we can operate further and investigate and predict material behavior depending on the kind and magnitude of the load. Either we impose stress load, solving the corresponding linear non-homogenous differential equation in sense of deformation, or vice versa, i.e. we impose a strain and compute stress response. The initial condition have to be posed as well. The first equation is of course much simpler to solve, we can get the solution by simple integration. Alternatively the obtained differential equations can be solved easily e.g. numerically.

Creep and relaxation test are examples of material behavior investigation.

5. Conclusion. Nowadays, a lot of new material is developed and used in industry. Undoubtedly, the investigation prior to their usage is inevitable. Avoiding or predicting the failure due to heavy or repeating load is essential. For this sake the models with time dependent material behavior are utilized, each material matched with its appropriate models. Then by using mathematical tools various theoretical tests can be executed and response vs. load can be traced. Constitutive equations are essential, visco-elasto-plastic models being of great importance and interest within them.

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