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# VECTORIAL QUASILINEAR DIFFUSION EQUATION WITH DYNAMIC BOUNDARY CONDITION

RYOTA NAKAYASHIKI*

**Abstract.** In this paper, we consider a class of initial-boundary value problems for quasilinear PDEs, subject to the dynamic boundary conditions. Each initial-boundary problem is denoted by $(\mathrm{S})_\varepsilon$ with a nonnegative constant $\varepsilon$, and for any $\varepsilon \geq 0$, $(\mathrm{S})_\varepsilon$ can be regarded as a vectorial transmission system between the quasilinear equation in the spatial domain $\Omega$, and the parabolic equation on the boundary $\Gamma := \partial\Omega$, having a sufficient smoothness. The objective of this study is to establish a mathematical method, which can enable us to handle the transmission systems of various vectorial mathematical models, such as the Bingham type flow equations, the Ginzburg–Landau type equations, and so on. On this basis, we set the goal of this paper to prove two Main Theorems, concerned with the well-posedness of $(\mathrm{S})_\varepsilon$ with the precise representation of solution, and $\varepsilon$-dependence of $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$.

**Key words.** vectorial parabolic equation, quasilinear diffusion, dynamic boundary condition

**AMS subject classifications.** 35K40, 35K59, 35R35.

**1. Introduction.** Let $0 < T < \infty$ and $\kappa > 0$ be fixed constants, and let $m \in \mathbb{N}$ and $1 < N \in \mathbb{N}$ be fixed constants of dimensions. Let $\Omega$ be a bounded spatial domain in $\mathbb{R}^N$ with a smooth boundary $\Gamma := \partial\Omega$, and let $n_\Gamma$ be the unit outer normal to $\Gamma$. Besides, we denote by $Q := (0,T) \times \Omega$ the product space of a time interval $(0,T)$ and the spatial domain $\Omega$, and we put $\Sigma := (0,T) \times \Gamma$.

In this paper, we take a constant $\varepsilon \geq 0$, and consider the following initial-boundary value problem:

$(\mathrm{S})_\varepsilon$:
$$\begin{cases} \partial_t u - \mathrm{div}\left(\dfrac{\nabla u}{\|\nabla u\|} + \kappa^2 \nabla u\right) \ni \theta \text{ in } Q, \\ \partial_t u_\Gamma - \varDelta_\Gamma(\varepsilon^2 u_\Gamma) + \left(\frac{\nabla u}{\|\nabla u\|} + \kappa^2 \nabla u\right)_{|_\Gamma} n_\Gamma \ni \theta_\Gamma \text{ and } u_{|_\Gamma} = u_\Gamma \text{ on } \Sigma, \\ u(0,\cdot) = u_0 \text{ in } \Omega \text{ and } u_\Gamma(0,\cdot) = u_{\Gamma,0} \text{ on } \Gamma, \end{cases}$$

for the vectorial unknowns $u : Q \to \mathbb{R}^m$ and $u_\Gamma : \Sigma \to \mathbb{R}^m$. In the context, $\theta : Q \to \mathbb{R}^m$ and $\theta_\Gamma : \Sigma \to \mathbb{R}^m$ are given forcing terms, and $u_0 : \Omega \to \mathbb{R}^m$ and $u_{\Gamma,0} : \Gamma \to \mathbb{R}^m$ are given initial data for $u$ and $u_\Gamma$, respectively. "${}_{|_\Gamma}$" denotes the trace of a Sobolev function on $\Omega$, and $\varDelta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$.

The boundary condition of $(\mathrm{S})_\varepsilon$ is given in the form of the so-called "*dynamic boundary condition*". In particular, since we can use the equation $u_{|_\Gamma} = u_\Gamma$ on $\Sigma$ to resemble a kind of the transmission condition, we can say that the problem $(\mathrm{S})_\varepsilon$ is a vectorial transmission system between the quasilinear equation in $\Omega$, and the parabolic equation on $\Gamma$.

The objective of this study is to establish a mathematical method, which enables us to handle various nonlinear phenomena described by vectorial unknowns. In this regard, the study on $(\mathrm{S})_\varepsilon$, for any $\varepsilon \geq 0$, is aimed at the mathematical analysis for quasilinear transmission systems, associated with the Bingham type flow equations, the Ginzburg–Landau type equations, and so on.

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In view of such backgrounds, we set the goal to obtain some generalized results for the previous works [4, 10], which dealt with the scalar-valued cases of quasilinear transmission systems. On this basis, our principal results will be stated in forms of the following two Main Theorems, which will be to verify the qualitative properties of the systems, for every $\varepsilon \geq 0$.

**Main Theorem 1** : the well-posedness for $(\mathrm{S})_\varepsilon$ with the precise expression of solutions, for any $\varepsilon \geq 0$.

**Main Theorem 2** : the continuous dependence of solutions to $(\mathrm{S})_\varepsilon$ with respect to $\varepsilon \geq 0$.

The content of this paper is as follows. Main Theorems are stated in Section 3 and these are discussed on the basis of the preliminaries prepared in Section 2. The keypoints of the proofs are specified in Section 4, and the proofs of the Main Theorems are provided in the last Section 5.

## 2. Preliminaries.

In this section, we outline some basic notations.

Abstract Notations. For an abstract Banach space $X$, we denote by $|\cdot|_X$ the norm of $X$, and denote by $\langle \cdot\,, \cdot \rangle_X$ the duality pairing between $X$ and the dual space $X^*$ of $X$. Let $\mathcal{I}_X : X \to X$ be the identity map from $X$ onto $X$. In particular, when $X$ is a Hilbert space, we denote by $(\,\cdot\,, \cdot\,)_X$ the inner product of $X$.

For any proper lower semi-continuous (l.s.c. from now on) and convex function $\Psi$ defined on a Hilbert space $X$, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of $\Psi$, and it turns out to be a maximal monotone graph in the product space $X^2 := X \times X$ (see [1–3, 7], for details). More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0), \text{ for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{z \in X \,|\, \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation "$[z_0, z_0^*] \in \partial\Psi$ in $X^2$", to mean that "$z_0^* \in \partial\Psi(z_0)$ in $X$ with $z_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in $X^2$.

Additionally, in this study, we use the following notion of convergence, called "Mosco-convergence", for sequences of convex functions.

Definition 2.1 (Mosco-convergence: cf. [9]). *Let $X$ be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on $X$, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled.*

**(M1) Lower-bound condition:** $\varliminf_{n\to\infty} \Psi_n(\check{z}_n) \geq \Psi(\check{z})$, *if* $\check{z} \in X$, $\{\check{z}_n\}_{n=1}^\infty \subset X$, *and* $\check{z}_n \to \check{z}$ *weakly in* $X$ *as* $n \to \infty$.

**(M2) Optimality condition:** *for any* $\hat{z} \in D(\Psi)$, *there exists a sequence* $\{\hat{z}_n\}_{n=1}^\infty \subset X$ *such that* $\hat{z}_n \to \hat{z}$ *in* $X$ *and* $\Psi_n(\hat{z}_n) \to \Psi(\hat{z})$, *as* $n \to \infty$.

Notations in real analysis. Let $d \in \mathbb{N}$ be any fixed dimension. Then, we simply denote by $a \cdot b$ and $|a|$ the standard scalar product of $a, b \in \mathbb{R}^d$ and the Euclidean norm of $a \in \mathbb{R}^d$, respectively. Besides, for arbitrary $d$-dimensional vectors $a = [a_i]$, $b = [b_i] \in \mathbb{R}^d$ with components $a_i, b_i \in \mathbb{R}$ $(i = 1, \ldots, d)$, we define:

$$a \otimes b := a^{\mathrm{t}} b = \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_d \\ \vdots & \ddots & \vdots \\ a_d b_1 & \cdots & a_d b_d \end{bmatrix} \in \mathbb{R}^{d \times d}.$$

For any $d \in \mathbb{N}$, the $d$-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$, and $d$-dimensional Hausdorff measure is denoted by $\mathcal{H}^d$. Unless otherwise specified, the measure theoretical phrases, such as "a.e.", "$dt$", "$dx$", and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observation on a smooth surface $S$, the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on $S$ is denoted by $dS$.

*Notations of surface-differentials.* Throughout this paper, let $1 < N \in \mathbb{N}$ be a fixed dimension, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^\infty$-boundary $\Gamma := \partial\Omega$, and let $n_\Gamma \in C^\infty(\Gamma; \mathbb{R}^N)$ be the unit outer normal on $\Gamma$. Besides, we suppose that the distance function $x \in \mathbb{R}^N \mapsto d_\Gamma(x) := \inf_{y \in \Gamma} |x - y| \in \mathbb{R}$ forms a $C^\infty$-function on a neighborhood of $\Gamma$. Based on these, we define:

$$L^2_{\mathrm{tan}}(\Gamma) := \left\{ \ \tilde{\omega} \in L^2(\Gamma; \mathbb{R}^N) \ \middle| \ \tilde{\omega} \cdot n_\Gamma = 0 \text{ on } \Gamma \ \right\},$$

and we define the so-called Laplace–Beltrami operator $\Delta_\Gamma$ as the composition $\Delta_\Gamma := \mathrm{div}_\Gamma \circ \nabla_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ of the *surface-gradient*:

$$\varphi \in C^1(\Gamma) \mapsto \nabla_\Gamma \varphi := \nabla \varphi^{\mathrm{ex}} - (\nabla d_\Gamma \otimes \nabla d_\Gamma) \nabla \varphi^{\mathrm{ex}} \in L^2_{\mathrm{tan}}(\Gamma) \cap C(\Gamma; \mathbb{R}^N),$$

and the *surface-divergence*:

$$\omega \in C^1(\Gamma; \mathbb{R}^N) \mapsto \mathrm{div}_\Gamma\, \omega := \mathrm{div}\, \omega^{\mathrm{ex}} - \nabla(\omega^{\mathrm{ex}} \cdot \nabla d_\Gamma) \cdot \nabla d_\Gamma \in C(\Gamma).$$

As is well-known (cf. [11]), the surface-gradient $\nabla_\Gamma$ can be extended to a linear operator from the Sobolev space $H^1(\Gamma)$ into $L^2_{\mathrm{tan}}(\Gamma)$, and the extension (also denoted by $\nabla_\Gamma$) define the inner product of the Hilbert space $H^1(\Gamma)$ as follows:

$$(\varphi, \psi)_{H^1(\Gamma)} := (\varphi, \psi)_{L^2(\Gamma)} + (\nabla_\Gamma \varphi, \nabla_\Gamma \psi)_{L^2(\Gamma; \mathbb{R}^N)}, \ \text{ for all } \varphi, \psi \in H^1(\Gamma).$$

Also, the surface-divergence $\mathrm{div}_\Gamma$ can be extended to an operator from $L^2(\Gamma; \mathbb{R}^N)$ into $H^{-1}(\Gamma)$, and as a consequence, the composition $-\mathrm{div}_\Gamma \circ \nabla_\Gamma = -\Delta_\Gamma : H^1(\Gamma) \to H^{-1}(\Gamma)$ provides a duality map, such that:

$$\langle -\Delta_\Gamma \varphi, \psi \rangle_{H^1(\Gamma)} = (\nabla_\Gamma \varphi, \nabla_\Gamma \psi)_{L^2(\Gamma; \mathbb{R}^N)}, \ \text{for all } \varphi, \psi \in H^1(\Gamma).$$

*Notations in tensor analysis.* Let $m \in \mathbb{N}$ be another dimension (besides $N$). For arbitrary $(m \times N)$-matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m \times N}$ with components $a_{ij}, b_{ij} \in \mathbb{R}$ ($i = 1, \dots m$, $j = 1, \dots, N$), we denote by $A : B$ and $\|A\|$ the scalar product of $A$ and $B$ and the Frobenius norm of $A$, respectively, i.e.:

$$A : B := \sum_{j=1}^{N} \sum_{i=1}^{m} a_{ij} b_{ij} \in \mathbb{R} \text{ and } \ \|A\| := \sqrt{A : A} \in \mathbb{R}, \ \text{for all } A, B \in \mathbb{R}^{m \times N}.$$

For any vectorial function $z = [z_i] \in L^2(\Omega; \mathbb{R}^m)$, we denote by $\nabla z$ the (distributional) gradient of $z$, defined as:

$$\nabla z := {}^{\mathrm{t}}[\nabla z_1, \dots, \nabla z_m] = \begin{bmatrix} \partial_1 z_1 & \cdots & \partial_N z_1 \\ \vdots & \ddots & \vdots \\ \partial_1 z_m & \cdots & \partial_N z_m \end{bmatrix} \in \mathcal{D}'(\Omega)^{m \times N},$$

and, for any matrix-valued function $Z = [z_{ij}] \in L^2(\Omega; \mathbb{R}^{m\times N})$, we denote by $\mathrm{div}Z$ the (distributional) divergence of $Z$, defined as:

$$\mathrm{div}Z := \left[\sum_{k=1}^{N} \partial_k z_{ik}\right] \in \mathcal{D}'(\Omega)^m.$$

Similarly, for any vectorial function $z = [z_i] \in H^1(\Gamma; \mathbb{R}^m)$, we define the surface-gradient $\nabla_\Gamma z$ of $z$ by $\nabla_\Gamma z := {}^{\mathrm{t}}[\nabla_\Gamma z_1, \dots, \nabla_\Gamma z_m] \in L^2_{\mathrm{tan}}(\Gamma)^m$, and we define $\Delta_\Gamma z := [\Delta_\Gamma z_i] \in H^{-1}(\Gamma; \mathbb{R}^m)$.

Remark 1 (cf. [8, Proposition 1.6]). The mapping $M \in H^1(\Omega; \mathbb{R}^{m\times N}) \mapsto M_{|\Gamma} n_\Gamma \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$ can be extended as a linear and continuous operator $[\,\cdot\,, n_\Gamma]_\Gamma$ from

$$L^2_{\mathrm{div}}(\Omega) := \left\{\ \tilde{M} \in L^2(\Omega; \mathbb{R}^{m\times N}) \ \middle|\ \mathrm{div}\tilde{M} \in L^2(\Omega; \mathbb{R}^m)\ \right\}$$

into $H^{-\frac{1}{2}}(\Gamma; \mathbb{R}^m)$, such that:

$$\begin{gathered}\left\langle [M, n_\Gamma]_\Gamma, z_{|\Gamma} \right\rangle_{H^{\frac{1}{2}}(\Gamma;\mathbb{R}^m)} := \int_\Omega \mathrm{div}M \cdot z\, dx + \int_\Omega M : \nabla z\, dx, \\ \text{for all } M \in L^2_{\mathrm{div}}(\Omega) \text{ and } z \in H^1(\Omega; \mathbb{R}^m).\end{gathered} \tag{2.1}$$

**3. Main Theorems.** Let us set

$$\mathscr{H} := L^2(\Omega; \mathbb{R}^m) \times L^2(\Gamma; \mathbb{R}^m),$$

and for any $\varepsilon \geq 0$, let us set:

$$\mathscr{V}_\varepsilon := \left\{\ [v, v_\Gamma] \in \mathscr{H}\ \middle|\ \begin{array}{l} v \in H^1(\Omega; \mathbb{R}^m),\ v_\Gamma \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m), \\ \varepsilon v_\Gamma \in H^1(\Gamma; \mathbb{R}^m), \text{ and } v_{|\Gamma} = v_\Gamma, \text{ a.e. on } \Gamma \end{array}\ \right\}.$$

Note that $\mathscr{H}$ is a Hilbert space endowed with the inner product:

$$\begin{gathered}([z_1, z_{\Gamma,1}], [z_2, z_{\Gamma,2}])_{\mathscr{H}} := (z_1, z_2)_{L^2(\Omega;\mathbb{R}^m)} + (z_{\Gamma,1}, z_{\Gamma,2})_{L^2(\Gamma;\mathbb{R}^m)}, \\ \text{for } [z_k, z_{\Gamma,k}] \in \mathscr{H},\ k = 1, 2.\end{gathered}$$

Also, if $\varepsilon > 0$ (resp. $\varepsilon = 0$), then the corresponding class $\mathscr{V}_\varepsilon$ (resp. $\mathscr{V}_0$) is a closed linear space in $H^1(\Omega; \mathbb{R}^m) \times H^1(\Gamma; \mathbb{R}^m)$ (resp. in $H^1(\Omega; \mathbb{R}^m) \times H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$), and hence, it is a Hilbert space endowed with the standard inner product of $H^1(\Omega; \mathbb{R}^m) \times H^1(\Gamma; \mathbb{R}^m)$ (resp. $H^1(\Omega; \mathbb{R}^m) \times H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$). Furthermore, for any $\varepsilon \geq 0$, $\mathscr{V}_\varepsilon$ is dense in $\mathscr{H}$, i.e. $\overline{\mathscr{V}_\varepsilon} = \mathscr{H}$, and the embedding $\mathscr{V}_\varepsilon \subset \mathscr{H}$ is compact.

By using the above notations, we define the solution to $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$, as follows.

Definition 3.1. *Let $\varepsilon \geq 0$ be a fixed constant. Then, a pair of functions $[u, u_\Gamma] \in L^2(0, T; \mathscr{H})$ is called a solution to $(S)_\varepsilon$, iff. the following conditions are fulfilled.*

**(S1)** $[u, u_\Gamma] \in C([0, T]; \mathscr{H}) \cap W^{1,2}_{\mathrm{loc}}((0, T]; \mathscr{H}) \cap L^2(0, T; \mathscr{V}_\varepsilon) \cap L^\infty_{\mathrm{loc}}((0, T]; \mathscr{V}_\varepsilon)$,
$[u(0), u_\Gamma(0)] = [u_0, u_{\Gamma,0}]$ *in* $\mathscr{H}$.

**(S2)** *There exists a function $M_u : Q \to \mathbb{R}^{m\times N}$, such that:*

$$M_u(t) \in L^2_{\mathrm{div}}(\Omega),\ \ \text{a.e. } t \in (0, T) \text{ and } [\nabla u, M_u] \in \partial\|\cdot\| \text{ in } [\mathbb{R}^{m\times N}]^2, \text{ a.e. in } Q,$$

*and*

$$\int_\Omega \partial_t u(t) \cdot z \, dx + \int_\Omega (M_u(t) + \kappa^2 \nabla u(t)) : \nabla z \, dx$$
$$+ \int_\Gamma \partial_t u_\Gamma(t) \cdot z_\Gamma \, d\Gamma + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma(t)) : \nabla_\Gamma(\varepsilon z_\Gamma) \, d\Gamma$$
$$= \int_\Omega \theta(t) \cdot z \, dx + \int_\Gamma \theta_\Gamma(t) \cdot z_\Gamma \, d\Gamma, \quad \textit{for any } [z, z_\Gamma] \in \mathscr{V}_\varepsilon, \textit{ and a.e. } t \in (0,T),$$

*where* $\partial \| \cdot \| \subset [\mathbb{R}^{m \times N}]^2$ *denotes the subdifferential of the Frobenius norm* $\| \cdot \|$ *on* $\mathbb{R}^{m \times N}$.

Based on this, the Main Theorems of this paper are stated as follows.

MAIN THEOREM 1 (Well-posedness). *Let* $\varepsilon \geq 0$ *be a fixed constant. Then, the following two items hold.*

**(I-1)** *(Solvability) For every* $[\theta, \theta_\Gamma] \in L^2(0,T;\mathscr{H})$ *and* $[u_0, u_{\Gamma,0}] \in \mathscr{H}$*, the system* (S)$_\varepsilon$ *admits a unique solution* $[u, u_\Gamma]$.

**(I-2)** *(Continuous dependence) For* $k = 1, 2$*, let* $[u_k, u_{\Gamma,k}]$ *be two solutions to* (S)$_\varepsilon$*, corresponding to the forcing pairs* $[\theta_k, \theta_{\Gamma,k}] \in L^2(0,T;\mathscr{H})$ *and the initial pairs* $[u_{0,k}, u_{\Gamma,0,k}] \in \mathscr{H}$*, respectively. Then, it follows that:*

$$|[u_1 - u_2, u_{\Gamma,1} - u_{\Gamma,2}]|^2_{C([0,T];\mathscr{H})}$$
$$\leq \mathrm{e}^T \big( |[\theta_1 - \theta_2, \theta_{\Gamma,1} - \theta_{\Gamma,2}]|^2_{L^2(0,T;\mathscr{H})} + |[u_{0,1} - u_{0,2}, u_{\Gamma,0,1} - u_{\Gamma,0,2}]|^2_{\mathscr{H}} \big).$$

MAIN THEOREM 2 (Continuous dependence with respect to $\varepsilon \geq 0$). *Let* $\varepsilon_0 \geq 0$ *be a fixed constant. Let* $\{[\theta_\varepsilon, \theta_{\Gamma,\varepsilon}]\}_{\varepsilon \geq 0} \subset L^2(0,T;\mathscr{H})$ *be a sequence of the forcing pair, let* $\{[u_{0,\varepsilon}, u_{\Gamma,0,\varepsilon}]\}_{\varepsilon \geq 0} \subset \mathscr{H}$ *be a sequence of the initial pair, and for any* $\varepsilon \geq 0$*, let* $[u_\varepsilon, u_{\Gamma,\varepsilon}]$ *be a solution to* (S)$_\varepsilon$ *corresponding to the forcing pair* $[\theta_\varepsilon, \theta_{\Gamma,\varepsilon}]$ *and the initial pair* $[u_{0,\varepsilon}, u_{\Gamma,0,\varepsilon}]$*. Here, if:*

$$\begin{cases} [\theta_\varepsilon, \theta_{\Gamma,\varepsilon}] \to [\theta_{\varepsilon_0}, \theta_{\Gamma,\varepsilon_0}] \textit{ weakly in } L^2(0,T;\mathscr{H}), \\ [u_{0,\varepsilon}, u_{\Gamma,0,\varepsilon}] \to [u_{0,\varepsilon_0}, u_{\Gamma,0,\varepsilon_0}] \textit{ in } \mathscr{H}, \end{cases} \qquad \textit{as } \varepsilon \to \varepsilon_0,$$

*then:*

$$[u_\varepsilon, u_{\Gamma,\varepsilon}] \to [u_{\varepsilon_0}, u_{\Gamma,\varepsilon_0}] \textit{ in } C([0,T];\mathscr{H}), \textit{ and in } L^2(0,T;\mathscr{V}_0) \textit{ as } \varepsilon \to \varepsilon_0, \tag{3.1}$$

*and in particular, if* $\varepsilon_0 > 0$*, then:*

$$u_{\Gamma,\varepsilon} \to u_{\Gamma,\varepsilon_0} \textit{ in } L^2(0,T;H^1(\Gamma;\mathbb{R}^m)), \textit{ as } \varepsilon \to \varepsilon_0. \tag{3.2}$$

**4. Keypoints of the proofs.** In this section, we specify the keypoints in the proofs of Main Theorems. Roughly summarized, we will prove the Main Theorems by reformulating our system (S)$_\varepsilon$ to the following Cauchy problem for an evolution equation, denoted by (CP)$_\varepsilon$:

$$(\mathrm{CP})_\varepsilon \quad \begin{cases} U'(t) + \partial \Phi_\varepsilon(U(t)) \ni \Theta(t) \text{ in } \mathscr{H}, \text{ a.e. } t \in (0,T), \\ U(0) = U_0 \text{ in } \mathscr{H}, \end{cases} \quad \text{for } \varepsilon \geq 0.$$

In the context, the unknown $U \in L^2(0,T;\mathscr{H})$ corresponds to the solution $[u, u_\Gamma]$ to the system (S)$_\varepsilon$, and $\Theta := [\theta, \theta_\Gamma]$ in $L^2(0,T;\mathscr{H})$ and $U_0 := [u_0, u_{\Gamma,0}]$ in $\mathscr{H}$ correspond to the pair of the forcing terms and the pair of the initial data, respectively. $\partial\Phi_\varepsilon$ denotes the subdifferential of a proper l.s.c. and convex function $\Phi_\varepsilon : \mathscr{H} \to [0, \infty]$, defined as:

$$U = [u, u_\Gamma] \in \mathscr{H} \mapsto \Phi_\varepsilon(U) = \Phi_\varepsilon(u, u_\Gamma)$$

$$:= \begin{cases} \displaystyle\int_\Omega \left( \|\nabla u\| + \frac{\kappa^2}{2}\|\nabla u\|^2 \right) dx + \frac{1}{2}\int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2 \, d\Gamma, \\ \qquad \text{if } U = [u, u_\Gamma] \in \mathscr{V}_\varepsilon, \\ \infty, \quad \text{otherwise,} \end{cases} \quad \text{for } \varepsilon \geq 0.$$

Now, the essential keypoint is to show the following Key-Lemma, which is to sustain a certain association between the system $(\mathrm{S})_\varepsilon$ and the Cauchy problem $(\mathrm{CP})_\varepsilon$, for any $\varepsilon \geq 0$.

Key-Lemma 1 (The representation of $\partial\Phi_\varepsilon$) *For any $\varepsilon \geq 0$, the following two items are equivalent.*

**(I)** $[u, u_\Gamma] \in D(\partial\Phi_\varepsilon)$ *and* $[u^*, u^*_\Gamma] \in \partial\Phi_\varepsilon(u, u_\Gamma)$ *in* $\mathscr{H}$*.*

**(II)** $[u, u_\Gamma] \in D(\Phi_\varepsilon)$ *and there exists* $M^*_u \in L^\infty(\Omega; \mathbb{R}^{m\times N})$*, such that:*

$$[\nabla u, M^*_u] \in \partial\|\cdot\| \text{ in } [\mathbb{R}^{m\times N}]^2, \text{ a.e. in } \Omega, \tag{4.1}$$

$$\begin{cases} M^*_u + \kappa^2 \nabla u \in L^2_{\mathrm{div}}(\Omega), \\ -\mathit{\Delta}_\Gamma(\varepsilon^2 u_\Gamma) + [(M^*_u + \kappa^2\nabla u), n_\Gamma]_\Gamma \in L^2(\Gamma; \mathbb{R}^m), \end{cases} \tag{4.2}$$

$$\begin{cases} u^* = -\mathrm{div}(M^*_u + \kappa^2\nabla u) \text{ in } L^2(\Omega; \mathbb{R}^m), \\ u^*_\Gamma = -\mathit{\Delta}_\Gamma(\varepsilon^2 u_\Gamma) + [(M^*_u + \kappa^2\nabla u), n_\Gamma]_\Gamma \text{ in } L^2(\Gamma; \mathbb{R}^m). \end{cases} \tag{4.3}$$

For the proof of the Key-Lemma, we prepare a class of relaxed convex functions $\{ \Phi^\delta_\varepsilon \,|\, \varepsilon \geq 0, \ 0 < \delta \leq 1\}$, defined as:

$$U = [u, u_\Gamma] \in \mathscr{H} \mapsto \Phi^\delta_\varepsilon(U) = \Phi^\delta_\varepsilon(u, u_\Gamma)$$

$$:= \begin{cases} \displaystyle\int_\Omega \left( \sqrt{\|\nabla u\|^2 + \delta^2} + \frac{\kappa^2}{2}\|\nabla u\|^2 \right) dx + \frac{1}{2}\int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2 \, d\Gamma, \\ \qquad \text{if } U = [u, u_\Gamma] \in \mathscr{V}_\varepsilon, \\ \infty, \quad \text{otherwise,} \end{cases}$$

for all $\varepsilon \geq 0$ and $0 < \delta \leq 1$.

Similar relaxation methods have been adopted in some previous results (e.g. [4, Key-Lemma 1-2 and Lemma 4.1]), and referring to some of these, we can verify the following facts.

**(Fact 1)** Let us fix all $\varepsilon > 0$, $0 < \delta \leq 1$, and let us define:

$$\mathscr{D}^\delta_\varepsilon := \left\{ [z, z_\Gamma] \in \mathscr{H} \left| \begin{array}{l} \frac{\nabla z}{\sqrt{\|\nabla z\|^2 + \delta^2}} + \kappa^2 \nabla z \in L^2_{\mathrm{div}}(\Omega), \\ -\mathit{\Delta}_\Gamma(\varepsilon^2 z_\Gamma) + [(\frac{\nabla z}{\sqrt{\|\nabla z\|^2+\delta^2}} + \kappa^2\nabla z), n_\Gamma]_\Gamma \in L^2(\Gamma; \mathbb{R}^m) \end{array} \right. \right\},$$

and let us define a single-valued operator $\mathcal{A}^\delta_\varepsilon : \mathscr{D}^\delta_\varepsilon \subset \mathscr{H} \to \mathscr{H}$, by letting:

$$[z, z_\Gamma] \in \mathscr{D}^\delta_\varepsilon \mapsto \mathcal{A}^\delta_\varepsilon[z, z_\Gamma]$$

$$:= {}^{\mathrm{t}}\begin{bmatrix} -\mathrm{div}(\frac{\nabla z}{\sqrt{\|\nabla z\|^2+\delta^2}} + \kappa^2 \nabla z) \\ -\mathit{\Delta}_\Gamma(\varepsilon^2 z_\Gamma) + [(\frac{\nabla z}{\sqrt{\|\nabla z\|^2+\delta^2}} + \kappa^2\nabla z), n_\Gamma]_\Gamma \end{bmatrix} \in \mathscr{H}.$$

Then, $\partial \Phi_\varepsilon^\delta \subset \mathscr{H}^2$ coincides with the (graph of) operator $\mathcal{A}_\varepsilon^\delta$, i.e.:

$$\partial \Phi_\varepsilon^\delta = \mathcal{A}_\varepsilon^\delta \ \text{ in } \mathscr{H}^2, \text{ for all } \varepsilon > 0 \text{ and } 0 < \delta \leq 1.$$

**(Fact 2)** Let $\varepsilon_0 \geq 0$, and let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$ and $\{\delta_n\}_{n=1}^\infty \subset (0, 1]$ be arbitrary sequences, which fulfill that $\varepsilon_n \to \varepsilon_0$ and $\delta_n \to 0$, as $n \to \infty$. Then, the sequence of convex functions $\{\Phi_{\varepsilon_n}^{\delta_n}\}_{n=1}^\infty$, converges to the convex function $\Phi_{\varepsilon_0}$ on $\mathscr{H}$, in the sense of Mosco, as $n \to \infty$.

**(Fact 3)** A sequence of convex functions:

$$W \in L^2(\Omega; \mathbb{R}^{m \times N}) \mapsto \int_\Omega \sqrt{\|W\|^2 + \delta^2} \, dx \in [0, \infty), \text{ for any } 0 < \delta \leq 1$$

converges to the convex function:

$$W \in L^2(\Omega; \mathbb{R}^{m \times N}) \mapsto \int_\Omega \|W\| \, dx \in [0, \infty)$$

on $L^2(\Omega; \mathbb{R}^{m \times N})$, in the sense of Mosco, as $\delta \to 0$.

Finally, in the rest of this section, we give the proof of the Key-Lemma.

*Proof of Key-Lemma 1.* Let us take a constant $\varepsilon \geq 0$, and let us set:

$$\mathscr{D}_\varepsilon := \left\{ \ [u, u_\Gamma] \in \mathscr{V}_\varepsilon \ \middle| \ \text{there exists } M_u^* \in L^\infty(\Omega; \mathbb{R}^{m \times N}), \text{ such that (4.1)–(4.2)} \ \right\},$$

and let us define a set-valued operator $\mathcal{A}_\varepsilon$, by putting:

$$\begin{aligned} [u, u_\Gamma] \quad &\in \quad \mathscr{D}_\varepsilon \mapsto \mathcal{A}_\varepsilon[u, u_\Gamma] \\ &:= \left\{ \ [u^*, u_\Gamma^*] \in \mathscr{H} \ \middle| \ \begin{array}{l} \text{(4.3) holds, for some } M_u^* \in L^\infty(\Omega; \mathbb{R}^{m \times N}), \\ \text{fulfilling (4.1)–(4.2)} \end{array} \right\}. \end{aligned}$$

Then, the assertion of Key-Lemma 1 can be rephrased as follows:

$$\partial \Phi_\varepsilon = \mathcal{A}_\varepsilon \ \text{ in } \mathscr{H}^2, \text{ for any } \varepsilon \geq 0. \tag{4.4}$$

We prove the above (4.4) via the following two Claims.

*Claim #1: $\mathcal{A}_\varepsilon \subset \partial \Phi_\varepsilon$ in $\mathscr{H}^2$, for any $\varepsilon \geq 0$.*

Let us assume that $[u, u_\Gamma] \in \mathscr{D}_\varepsilon$ and $[u^*, u_\Gamma^*] \in \mathcal{A}_\varepsilon[u, u_\Gamma]$ in $\mathscr{H}$. Then, by (2.1) and the definition of the subdifferential, we can verify that:

$$\begin{aligned} &([u^*, u_\Gamma^*], [z, z_\Gamma] - [u, u_\Gamma])_{\mathscr{H}} \\ &= (-\mathrm{div}(M_u^* + \kappa^2 \nabla u), z - u)_{L^2(\Omega; \mathbb{R}^m)} \\ &\qquad + (-\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(M_u^* + \kappa^2 \nabla u), n_\Gamma]_\Gamma, z_\Gamma - u_\Gamma)_{L^2(\Gamma; \mathbb{R}^m)} \\ &= \int_\Omega (M_u^* + \kappa^2 \nabla u) : \nabla(z - u) \, dx + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) : \nabla_\Gamma(\varepsilon(z_\Gamma - u_\Gamma)) \, d\Gamma \\ &\leq \int_\Omega \left( \|\nabla z\| + \frac{\kappa^2}{2} \|\nabla z\|^2 \right) dx - \int_\Omega \left( \|\nabla u\| + \frac{\kappa^2}{2} \|\nabla u\|^2 \right) dx \\ &\qquad + \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon z_\Gamma)\|^2 \, d\Gamma - \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2 \, d\Gamma \\ &= \Phi_\varepsilon(z, z_\Gamma) - \Phi_\varepsilon(u, u_\Gamma), \quad \text{for any } [z, z_\Gamma] \in \mathscr{V}_\varepsilon. \end{aligned}$$

*Claim #2:* $(\mathcal{A}_\varepsilon + \mathcal{I}_{\mathscr{H}})\mathscr{H} = \mathscr{H}$.

It is sufficient to show $(\mathcal{A}_\varepsilon + \mathcal{I}_{\mathscr{H}}) \supset \mathscr{H}$, because the other inclusion is trivial. Let $[w, w_\Gamma] \in \mathscr{H}$ be any pair of functions. Then, owing to (Fact 1) and Minty's Theorem, we can configure a class of functions $\{[u_\delta, u_{\Gamma,\delta}]\}_{0<\delta\leq 1} \subset \mathscr{H}$, such that:

$$[u_\delta, u_{\Gamma,\delta}] := (\mathcal{A}_\varepsilon^\delta + \mathcal{I}_{\mathscr{H}})^{-1}[w, w_\Gamma] \text{ in } \mathscr{H}, \text{ for any } 0 < \delta \leq 1,$$

and by taking any $[z, z_\Gamma] \in \mathscr{V}_\varepsilon$, we can see that:

$$\begin{aligned}\int_\Omega \Big(\frac{\nabla u_\delta}{\sqrt{\|\nabla u_\delta\|^2 + \delta^2}} + \kappa^2 \nabla u_\delta\Big) : \nabla z \, dx + \int_\Gamma \nabla_\Gamma(\varepsilon u_{\Gamma,\delta}) : \nabla_\Gamma(\varepsilon z_\Gamma) \, d\Gamma \\ = (w - u_\delta, z)_{L^2(\Omega;\mathbb{R}^m)} + (w_\Gamma - u_{\Gamma,\delta}, z_\Gamma)_{L^2(\Gamma;\mathbb{R}^m)}, \text{ for any } 0 < \delta \leq 1.\end{aligned} \tag{4.5}$$

Here, let us put $[z, z_\Gamma] = [u_\delta, u_{\Gamma,\delta}] \in \mathscr{V}_\varepsilon$ in (4.5). Then, by using Young's inequality, we deduce that:

$$\begin{gathered}|[u_\delta, u_{\Gamma,\delta}]|_{\mathscr{H}}^2 + 2\big(\kappa^2 |\nabla u_\delta|_{L^2(\Omega;\mathbb{R}^m)}^2 + |\nabla_\Gamma(\varepsilon u_{\Gamma,\delta})|_{L^2(\Gamma;\mathbb{R}^m)}^2\big) \leq |[w, w_\Gamma]|_{\mathscr{H}}^2 + \delta \mathcal{L}^N(\Omega), \\ \text{for any } 0 < \delta \leq 1.\end{gathered}$$

The above estimation may suppose that $\{[u_\delta, u_{\Gamma,\delta}]\}_{0<\delta\leq 1}$ is bounded in $\mathscr{V}_\varepsilon$, and is compact in $\mathscr{H}$. Therefore, we can find a sequence $\{\delta_n\}_{n=1}^\infty \subset \{\delta\}$ and a pair of functions $[u, u_\Gamma] \in \mathscr{V}_\varepsilon$, such that:

$$[u_n, u_{\Gamma,n}] := [u_{\delta_n}, u_{\Gamma,\delta_n}] \to [u, u_\Gamma] \text{ in } \mathscr{H} \text{ and weakly in } \mathscr{V}_\varepsilon, \text{ as } n \to \infty. \tag{4.6}$$

Additionally, since

$$\left| \frac{\nabla u_n}{\sqrt{\|\nabla u_n\|^2 + \delta_n^2}} \right| \leq 1, \text{ a.e. in } \Omega, \text{ for any } n \in \mathbb{N},$$

there exists a function $M_u^* \in L^\infty(\Omega; \mathbb{R}^{m\times N})$, such that:

$$\frac{\nabla u_n}{\sqrt{\|\nabla u_n\|^2 + \delta_n^2}} \to M_u^*, \text{ weakly-}* \text{ in } L^\infty(\Omega; \mathbb{R}^{m\times N}), \text{ as } n \to \infty, \tag{4.7}$$

by taking more one subsequence if necessary.

Now, with (4.6)–(4.7) in mind, let us take any function $\varphi_0 \in H_0^1(\Omega; \mathbb{R}^m)$, and let us put $[z, z_\Gamma] = [\varphi_0, 0] \in \mathscr{V}_\varepsilon$ in (4.5). Then, putting $\delta = \delta_n$ with $n \in \mathbb{N}$, and letting $n \to \infty$ in (4.5) yield that:

$$\int_\Omega (M_u^* + \kappa^2 \nabla u) : \nabla \varphi_0 \, dx = (w - u, \varphi_0)_{L^2(\Omega;\mathbb{R}^m)}.$$

It implies that:

$$-\mathrm{div}(M_u^* + \kappa^2 \nabla u) = w - u \in L^2(\Omega; \mathbb{R}^m) \text{ in } \mathcal{D}'(\Omega)^m. \tag{4.8}$$

As well as, putting $\delta = \delta_n$, letting $n \to \infty$ in (4.5) and applying (2.1) and (4.8) lead to:

$$\begin{gathered}(w_\Gamma - u_\Gamma, z_\Gamma)_{L^2(\Gamma;\mathbb{R}^m)} = \big\langle -\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(M_u^* + \kappa^2 \nabla u), n_\Gamma]_\Gamma, z_\Gamma \big\rangle_{H^1(\Gamma;\mathbb{R}^m)}, \\ \text{for any } z_\Gamma \in H^1(\Gamma; \mathbb{R}^m).\end{gathered}$$

Therefore, we can observe that:

$$-\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(M_u^* + \kappa^2 \nabla u), n_\Gamma]_\Gamma = w_\Gamma - u_\Gamma \in L^2(\Gamma; \mathbb{R}^m) \text{ in } H^{-1}(\Gamma; \mathbb{R}^m). \tag{4.9}$$

Finally, from (Fact 2)–(Fact 3), it is immediately seen that:

$$\begin{cases} \varliminf_{n \to \infty} \int_\Omega \sqrt{\|\nabla u_n\|^2 + \delta_n^2}\, dx \geq \int_\Omega \|\nabla u\|\, dx, \\ \varliminf_{n \to \infty} \left( \frac{\kappa^2}{2} \int_\Omega \|\nabla u_n\|^2\, dx \right) \geq \frac{\kappa^2}{2} \int_\Omega \|\nabla u\|^2\, dx, \\ \varliminf_{n \to \infty} \left( \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_{\Gamma,n})\|^2\, d\Gamma \right) \geq \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2\, d\Gamma. \end{cases} \tag{4.10}$$

Then, by putting $[z, z_\Gamma] = [u_n - u, u_{\Gamma,n} - u_\Gamma] \in \mathscr{V}_\varepsilon$ in (4.5), we can compute that:

$$\begin{aligned} &\int_\Omega \left( \sqrt{\|\nabla u_n\|^2 + \delta_n^2} + \frac{\kappa^2}{2} \|\nabla u_n\|^2 \right) dx + \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_{\Gamma,n})\|^2\, d\Gamma \\ &\leq \int_\Omega \left( \sqrt{\|\nabla u\|^2 + \delta_n^2} + \frac{\kappa^2}{2} \|\nabla u\|^2 \right) dx + \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2\, d\Gamma \\ &\qquad + (w - u_n, u_n - u)_{L^2(\Omega; \mathbb{R}^m)} + (w_\Gamma - u_{\Gamma,n}, u_{\Gamma,n} - u_\Gamma)_{L^2(\Gamma; \mathbb{R}^m)}. \end{aligned}$$

Based on these, we take the limit of the above inequality, and infer that:

$$\begin{aligned} &\varlimsup_{n \to \infty} \left( \int_\Omega \left( \sqrt{\|\nabla u_n\|^2 + \delta_n^2} + \frac{\kappa^2}{2} \|\nabla u_n\|^2 \right) dx + \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_{\Gamma,n})\|^2\, d\Gamma \right) \\ &\leq \int_\Omega \left( \|\nabla u\| + \frac{\kappa^2}{2} \|\nabla u\|^2 \right) dx + \frac{1}{2} \int_\Gamma \|\nabla_\Gamma(\varepsilon u_\Gamma)\|^2\, d\Gamma. \end{aligned} \tag{4.11}$$

By virtue of (4.6)–(4.7), (4.10)–(4.11) and the uniform convexity of the $L^2$-based topologies, it is further deduced that:

$$\nabla u_n \to \nabla u \text{ in } L^2(\Omega; \mathbb{R}^{m \times N}), \text{ as } n \to \infty. \tag{4.12}$$

On account of (4.12), (Fact 3), [1, Proposition 3.59 and Theorem 3.66], [3, Proposition 2.16] and [5, Appendix], we can obtain that:

$$M_u^* \in \left\{ \tilde{M} \in L^2(\Omega; \mathbb{R}^{m \times N}) \;\middle|\; [\nabla u, \tilde{M}] \in \partial \|\cdot\| \text{ in } [\mathbb{R}^{m \times N}]^2, \text{ a.e. in } \Omega \right\}. \tag{4.13}$$

As a consequence of (4.8)–(4.9) and (4.13), we verify Claim #2.

Now, with Claims #1–#2 and the maximality of the subdifferential $\partial\Phi_\varepsilon \subset \mathscr{H}^2$ in mind, we can deduce the coincidence (4.4), and we conclude Key-Lemma 1. □

**5. Proofs of Main Theorems.** In this section, we will prove the Main Theorems 1–2 on the basis of Key-Lemma 1 and (Fact 1)–(Fact 3) as in the previous sections.

*Proof of Main Theorem 1.* First, we show the item (I-1). In the Cauchy problem $(\mathrm{CP})_\varepsilon$, let us first confirm that:

$$\Theta := [\theta, \theta_\Gamma] \in L^2(0, T; \mathscr{H}) \text{ and } U_0 := [u_0, u_{\Gamma,0}] \in \overline{D(\Phi_\varepsilon)} = \overline{\mathscr{V}_\varepsilon} = \mathscr{H}.$$

Then, by applying the general theories of evolution equations, e.g. [2, Theorem 4.1], [3, Proposition 3.2], [6, Section 2], and [7, Theorem 1.1.2], we immediately have the existence and uniqueness of the solution $U = [u, u_\Gamma] \in L^2(0,T;\mathscr{H})$ to $(\mathrm{CP})_\varepsilon$, such that:

$$U \in C([0,T];\mathscr{H}) \cap L^2(0,T;\mathscr{H}) \cap W^{1,2}_{\mathrm{loc}}((0,T];\mathscr{H}) \text{ and } \Phi_\varepsilon(U) \in L^1(0,T) \cap L^\infty_{\mathrm{loc}}((0,T]).$$

Now, by Key-Lemma 1, we observe that the solution $U = [u, u_\Gamma]$ to $(\mathrm{CP})_\varepsilon$ coincides with that to the system $(\mathrm{S})_\varepsilon$, and hence, we verify the item (I-1).

In the meantime, the equivalence between $(\mathrm{S})_\varepsilon$ and $(\mathrm{CP})_\varepsilon$ enables us to conclude the other item (I-2) by applying the standard methods for evolution equations: more precisely, by taking the difference between the two evolution equations, multiplying its both sides by the difference of solutions, and using Gronwall's lemma. □

*Proof of Main Theorem 2.* For any $\varepsilon \geq 0$, let us simply put $\Theta_\varepsilon := [\theta_\varepsilon, \theta_{\Gamma,\varepsilon}] \in L^2(0,T;\mathscr{H})$ and $U_{0,\varepsilon} := [u_{0,\varepsilon}, u_{\Gamma,0,\varepsilon}] \in \mathscr{H}$, and let us denote by $U_\varepsilon$ the solution $[u_\varepsilon, u_{\Gamma,\varepsilon}]$ to $(\mathrm{S})_\varepsilon$ corresponding to the forcing term $\Theta_\varepsilon = [\theta_\varepsilon, \theta_{\Gamma,\varepsilon}]$ and the initial data $U_{0,\varepsilon} = [u_{0,\varepsilon}, u_{\Gamma,0,\varepsilon}]$. Then, by the equivalence between $(\mathrm{S})_\varepsilon$ and $(\mathrm{CP})_\varepsilon$, we can apply some of analytic techniques for nonlinear evolution equations, e.g. [7, Theorem 2.7.1] and [4, Main Theorem 2], and we can derive the following convergences:

$$U_\varepsilon \to U_{\varepsilon_0} \text{ in } C([0,T];\mathscr{H}), \int_0^T \Phi_\varepsilon(U_\varepsilon(t))\,dt \to \int_0^T \Phi_{\varepsilon_0}(U_{\varepsilon_0}(t))\,dt, \text{ as } \varepsilon \to \varepsilon_0. \quad (5.1)$$

Now, the required convergences (3.1)–(3.2) will be obtained as straightforward convergences of (5.1) and the uniform convexity of $L^2$-based topologies. □

## REFERENCES


[1] Attouch, H. *Variational Convergence for Functions and Operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[2] Barbu, V. *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. Springer Monographs in Mathematics. Springer, New York, 2010.

[3] Brézis, H. *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[4] Colli, P.; Gilardi, G.; Nakayashiki, R.; Shirakawa, K. A class of quasi-linear Allen–Cahn type equations with dynamic boundary conditions. *Nonlinear Anal.*, **158**:32–59, 2017.

[5] Giga, Y.; Kashima, Y.; Yamazaki, N. Local solvability of a constrained gradient system of total variation. *Abstr. Appl. Anal.*, (8):651–682, 2004.

[6] Ito, A.; Yamazaki, N.; Kenmochi, N. Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces. *Discrete Contin. Dynam. Systems*, (Added Volume I):327–349, 1998. Dynamical systems and differential equations, Vol. I (Springfield, MO, 1996).

[7] Kenmochi, N. Solvability of nonlinear evolution equations with time-dependent constraints and applications. *Bull. Fac. Education, Chiba Univ. http://ci.nii.ac.jp/naid/110004715232*, **30**:1–87, 1981.

[8] Kenmochi, N. Pseudomonotone operators and nonlinear elliptic boundary value problems. *J. Math. Soc. Japan*, **27**:121–149, 1975.

[9] Mosco, U. Convergence of convex sets and of solutions of variational inequalities. *Advances in Math.*, **3**:510–585, 1969.

[10] Nakayashiki, R.; Shirakawa, K. Weak formulation for singular diffusion equations with dynamic boundary condition. Springer INdAM Series. to appear, 2017.

[11] Savaré, G.; Visintin, A. Variational convergence of nonlinear diffusion equations: applications to concentrated capacity problems with change of phase. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **8**(1):49–89, 1997.