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IMPLICIT CONSTITUTIVE SOLUTION SCHEME FOR MOHR-COULOMB PLASTICITY

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Abstract: This contribution summarizes an implicit constitutive solution scheme of the elastoplastic problem containing the Mohr-Coulomb yield criterion, a nonassociative flow rule, and a nonlinear isotropic hardening. The presented scheme builds upon the subdifferential formulation of the flow rule leading to several improvements. Mainly, it is possible to detect a position of the unknown stress tensor on the Mohr-Coulomb pyramid without blind guesswork. Further, a simplified construction of the consistent tangent operator is introduced. The presented results are important for an efficient solution of incremental boundary value elastoplastic problems.

Keywords: Mohr-Coulomb plasticity, implicit constitutive solution scheme, consistent tangent operator

MSC: 34L10, 74C05, 74D10, 74L10, 90C25

1. Introduction

We focus on a solution of an elastoplastic constitutive problem containing the *Mohr-Coulomb yield criterion* and a consequent construction of the *consistent tangent operator* which is important for Newton-like methods in elastoplasticity. This constitutive problem is broadly exploited in soil and rock mechanics and many various solution schemes were suggested. For their detailed overview and historical development, we refer the recent papers [1] and [3], respectively. Nevertheless, it is still a challenging problem due to its technical complexity. It follows from the fact that the Mohr-Coulomb yield surface is a *hexagonal pyramid* aligned with the hydrostatic axis in terms of *principal stresses*.

We consider the Mohr-Coulomb constitutive initial-value problem introduced in [2, Sections 6.3–6.6] which can optionally contain the *nonassociative flow rule* and

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the *nonlinear isotropic hardening*. The solution scheme mainly depends on a formulation of the problem and its discretization. For example, the scheme based on the multisurface representation of the flow rule and the implicit Euler discretization is derived in detail in [2, Section 8.2]. To improve this conventional scheme, we use the *subdifferential formulation* of the flow rule instead of the multisurface one. The subdifferential-based implicit solution concept was proposed in [5] for yield criteria containing 1 or 2 singular points on the yield surface. Then it was extended to the Mohr-Coulomb problem in [4]. Here, we summarize the main results from [4] and write the solution scheme in more readable form.

The rest of the contribution is organized as follows. In Section 2, the Mohr-Coulomb constitutive problem discretized by the implicit Euler method is introduced. Section 3 contains selected theoretical results characterizing the problem. Based on these results, the improved solution scheme is introduced, see Section 4. Finally, some concluding remarks are mentioned in Section 5.

Besides scalar variables, we work mainly with second and fourth order tensors. For easier orientation in the text, we denote the second order tensors by bold letters and the fourth order tensors by capital blackboard letters, e.g., $\mathbb{D}_e$ or $\mathbb{I}$. The symbols $\otimes$ and $:$ mean the tensor product and the biscalar product, respectively (see, e.g., [2]). We also use the following notation: $\mathbb{R}_+ := \{z \in \mathbb{R};\ z \geq 0\}$ and $\mathbb{R}^{3\times3}_{sym}$ for the space of symmetric, second order tensors.

2. Formulation of the discretized problem

Let $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p \in \mathbb{R}^{3\times3}_{sym}$, $\bar{\varepsilon}^p, \kappa, \lambda \in \mathbb{R}_+$ denote the stress tensor, the strain tensor, the plastic strain tensor, the hardening variable, the thermodynamical hardening force, and the plastic multiplier, respectively. The spectral decomposition of the stress tensor reads as:

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{e}_i \otimes \mathbf{e}_i, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3, \tag{1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the ordered eigenvalues (the principal stresses) of $\boldsymbol{\sigma}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the corresponding eigenvectors. Recall that $\sigma_1, \sigma_2, \sigma_3$ are uniquely defined with respect to the prescribed ordering. The Mohr-Coulomb yield function and the related plastic potential are defined as follows:

$$f(\boldsymbol{\sigma}, \kappa) = (1 + \sin\phi)\sigma_1 - (1 - \sin\phi)\sigma_3 - 2(c_0 + \kappa)\cos\phi, \tag{2}$$

$$g(\boldsymbol{\sigma}) = (1 + \sin\psi)\sigma_1 - (1 - \sin\psi)\sigma_3, \tag{3}$$

respectively. Here, the material parameters $c_0 > 0$, $\phi, \psi \in (0, \pi/2)$ represent the initial cohesion, the friction angle, and the dilatancy angle, respectively. It is worth mentioning that g is a convex function and thus one can use its subdifferential $\partial g(\boldsymbol{\sigma})$. Further, define the fourth order tensor

$$\mathbb{D}_e = \frac{1}{3}(3K - 2G)\boldsymbol{I} \otimes \boldsymbol{I} + 2G\mathbb{I}, \tag{4}$$

representing linear and isotropic elastic response, where $K, G > 0$ denote the bulk and shear moduli, respectively, $\boldsymbol{I}$ is the second order identity tensor ($[\boldsymbol{I}]_{ij} = \delta_{ij}$, $i,j = 1,2,3$), and $\mathbb{I}$ is the fourth order identity tensor ($[\mathbb{I}]_{ijkl} = \delta_{ik}\delta_{jl}$, $i,j,k,l = 1,2,3$). Finally, it holds that $\kappa = H(\bar{\varepsilon}^p)$, where H is a nondecreasing, continuous, and piecewise smooth function satisfying $H(0) = 0$. As in [2], we let this function in an abstract form.

The elastoplastic constitutive initial value problem is defined on a pseudo-time interval $[0,T]$. With respect to the implicit Euler discretization, we consider a partition $0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$, fix a step k and denote $\boldsymbol{\sigma} := \boldsymbol{\sigma}(t_k)$, $\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon}(t_k)$, $\boldsymbol{\varepsilon}^p := \boldsymbol{\varepsilon}^p(t_k)$, $\bar{\varepsilon}^p := \bar{\varepsilon}^p(t_k)$, $\triangle\lambda = \lambda(t_k) - \lambda(t_{k-1})$, $\bar{\varepsilon}^{p,tr} := \bar{\varepsilon}^p(t_{k-1})$, $\boldsymbol{\varepsilon}^{tr} := \boldsymbol{\varepsilon}(t_k) - \boldsymbol{\varepsilon}^p(t_{k-1})$, and $\boldsymbol{\sigma}^{tr} := \mathbb{D}_e : \boldsymbol{\varepsilon}^{tr}$. Here, the superscript tr is the standard notation for the so-called trial variables which are known (see, e.g., [2]). The k-th step problem reads as:

Given $\boldsymbol{\sigma}^{tr}$ and $\bar{\varepsilon}^{p,tr}$. Find $\boldsymbol{\sigma}$, $\bar{\varepsilon}^p$, and $\triangle\lambda$ satisfying:

$$\left.\begin{array}{c} \boldsymbol{\sigma} = \boldsymbol{\sigma}^{tr} - \triangle\lambda\mathbb{D}_e : \boldsymbol{\nu}, \quad \boldsymbol{\nu} \in \partial g(\boldsymbol{\sigma}), \\ \bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi), \\ \triangle\lambda \geq 0, \quad f(\boldsymbol{\sigma}, H(\bar{\varepsilon}^p)) \leq 0, \quad \triangle\lambda f(\boldsymbol{\sigma}, H(\bar{\varepsilon}^p)) = 0. \end{array}\right\} \tag{5}$$

Notice that the remaining unknown variables can be computed from the solution components $\boldsymbol{\sigma}$, $\bar{\varepsilon}^p$, and $\triangle\lambda$. For example, it holds that $\boldsymbol{\varepsilon}^p(t_k) = \boldsymbol{\varepsilon}(t_k) - \mathbb{D}_e^{-1} : \boldsymbol{\sigma}(t_k)$.

3. Useful theoretical results

In this section, we summarize some theoretical results concerning problem (5). This framework is important for understanding of the solution scheme introduced in Section 4.

The first result enables to write problem (5) only in terms of principal stresses. For its derivation, it was necessary to find the subdifferential $\partial g(\boldsymbol{\sigma})$ in closed form with respect to (3), see [4, Lemma 4.1].

Lemma 1. *Let $(\boldsymbol{\sigma}, \bar{\varepsilon}^p, \triangle\lambda)$ be a solution to (5) for given $\boldsymbol{\sigma}^{tr}$ and $\bar{\varepsilon}^{p,tr}$. Let σ_i, σ_i^{tr}, $i = 1,2,3$, be the ordered eigenvalues of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{tr}$, respectively. Then $(\sigma_1, \sigma_2, \sigma_3, \bar{\varepsilon}^p, \triangle\lambda)$ is a solution to:*

$$\left.\begin{array}{c} \sigma_i = \sigma_i^{tr} - \triangle\lambda\left[\frac{2}{3}(3K - 2G)\sin\psi + 2G\nu_i\right], \quad i = 1,2,3, \\ \bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi), \\ \triangle\lambda \geq 0, \;\; (1+\sin\phi)\sigma_1 - (1-\sin\phi)\sigma_3 - 2(c_0 + H(\bar{\varepsilon}^p))\cos\phi \leq 0, \\ \triangle\lambda\left[(1+\sin\phi)\sigma_1 - (1-\sin\phi)\sigma_3 - 2(c_0 + H(\bar{\varepsilon}^p))\cos\phi\right] = 0, \end{array}\right\} \tag{6}$$

where ν_1, ν_2, ν_3 are the eigenvalues of $\boldsymbol{\nu} \in \partial g(\boldsymbol{\sigma})$ satisfying

$$\left.\begin{array}{c} 1 + \sin\psi \geq \nu_1 \geq \nu_2 \geq \nu_3 \geq -1 + \sin\psi, \quad \nu_1 + \nu_2 + \nu_3 = 2\sin\psi, \\ (\nu_1 - 1 - \sin\psi)(\sigma_1 - \sigma_2) = 0, \quad (\nu_3 + 1 - \sin\psi)(\sigma_2 - \sigma_3) = 0. \end{array}\right\} \tag{7}$$

Conversely, if $(\sigma_1, \sigma_2, \sigma_3, \bar{\varepsilon}^p, \triangle\lambda)$ is a solution to (6) then $(\boldsymbol{\sigma}, \bar{\varepsilon}^p, \triangle\lambda)$ solves (5), where $\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{e}_i^{tr} \otimes \mathbf{e}_i^{tr}$ and $\mathbf{e}_1^{tr}, \mathbf{e}_2^{tr}, \mathbf{e}_3^{tr}$ are the eigenvectors of $\boldsymbol{\sigma}^{tr}$ with respect to the ordering $\sigma_1^{tr} \geq \sigma_2^{tr} \geq \sigma_3^{tr}$.

A further simplification of the problem is possible under additional assumptions on the solution to problem (6). First, assume $\triangle\lambda = 0$. Then the *elastic response* appears and it holds: $\sigma_i = \sigma_i^{tr}$, $i = 1, 2, 3$, $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr}$, and

$$f(\boldsymbol{\sigma}^{tr}, H(\bar{\varepsilon}^{p,tr})) = (1 + \sin\phi)\sigma_1^{tr} - (1 - \sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi \leq 0. \quad (8)$$

In fact, (8) is a necessary and sufficient condition for $\triangle\lambda = 0$. If $\triangle\lambda > 0$ then the unknown principal stresses lie on the yield surface of the Mohr-Coulomb pyramid as follows from $(6)_4$. We distinguish four possible positions on the yield surface: the smooth portion ($\sigma_1 > \sigma_2 > \sigma_3$), the left edge ($\sigma_1 = \sigma_2 > \sigma_3$), the right edge ($\sigma_1 > \sigma_2 = \sigma_3$), and the apex ($\sigma_1 = \sigma_2 = \sigma_3$). This terminology follows from [2]. For each position, one can introduce a special solution scheme, the so-called *return-mapping scheme*. These schemes are introduced in Sections 4.4-4.7. Briefly speaking, nonlinear equations $q_s^{tr}(\triangle\lambda) = 0$, $q_l^{tr}(\triangle\lambda) = 0$, $q_r^{tr}(\triangle\lambda) = 0$, and $q_a^{tr}(\triangle\lambda) = 0$ are derived within these schemes, respectively. After finding their solutions, one can easily compute the remaining unknowns. However, only one type of the return-mapping usually leads to the solution of problem (6) and the remaining schemes produce incorrect solutions. To find the correct scheme, we define the intervals C_s^{tr}, C_l^{tr}, C_r^{tr}, C_a^{tr} introduced in Section 4.1. These intervals are mutually disjoint, their union is equal to $\mathbb{R}_+$, and either $C_l^{tr} = \emptyset$ or $C_r^{tr} = \emptyset$. For example, the return to the smooth portion appears if the solution of $q_s^{tr}(\triangle\lambda) = 0$ belongs to C_s^{tr}. Analogous criteria hold for the remaining return types.

It seems that one must successively solve the nonlinear equations with q_s^{tr}, q_l^{tr}, q_r^{tr}, q_a^{tr} to find the correct scheme. Similar blind guesswork is also introduced, e.g., in [2, Section 8.2]. Nevertheless, the presented approach enables to derive a priori decision criteria to detect the stress position on the yield surface, without any blind guesswork. To this end, we introduce the following useful result [4, Lemma 4.2].

Lemma 2. *There exists a unique function $q^{tr} : \mathbb{R}_+ \to \mathbb{R}$ satisfying:*

(i) $q^{tr}|_{C_s^{tr}} = q_s^{tr}$, $q^{tr}|_{C_l^{tr}} = q_l^{tr}$, $q^{tr}|_{C_r^{tr}} = q_r^{tr}$, $q^{tr}|_{C_a^{tr}} = q_a^{tr}$.

(ii) q^{tr} *is continuous, piecewise smooth, and decreasing in* $\mathbb{R}_+$.

(iii) $q^{tr}(0) = f(\boldsymbol{\sigma}^{tr}, H(\bar{\varepsilon}^{p,tr}))$.

(iv) $q^{tr}(\gamma) \to -\infty$ *as* $\gamma \to +\infty$.

The properties of the function q^{tr} have many important consequences. First, they imply the main solvability result [4, Theorems 4.4–4.6].

Theorem 3. *Problems (5) and (6) have unique solutions and the solution component $\triangle\lambda$ satisfies $q^{tr}(\triangle\lambda) = 0$.*

Second, one can easily detect one of the intervals C_s^{tr}, C_l^{tr}, C_r^{tr}, C_a^{tr} where values of q^{tr} change the sign. It leads to the a priori decision criteria introduced in Section 4.2. Finally, by Lemma 2, one can easily investigate properties of the stress-strain constitutive operator: $\boldsymbol{\sigma} = \boldsymbol{T}(\boldsymbol{\varepsilon}^{tr}; \bar{\varepsilon}^{p,tr})$. It is expected that this mapping is Lipschitz continuous and semismooth as follows from the discussion in [4].

The generalized derivative (in Clark's sense) of $\boldsymbol{T}$ represents the *consistent tangent operator*. This derivative defines the fourth order tensor $\mathbb{T}$, i.e., if $\boldsymbol{T}$ is differentiable at $(\boldsymbol{\varepsilon}^{tr}; \bar{\varepsilon}^{p,tr})$ then $\mathbb{T} = \partial \boldsymbol{T}/\partial \boldsymbol{\varepsilon}^{tr}$. The formulas defining $\mathbb{T}$ for each stress position are introduced in Sections 4.3-4.7. In case of the associative plasticity, i.e., if $\psi = \phi$, the tangent stiffness matrix is symmetric, otherwise it is nonsymmetric.

Let us note that the stress-strain operator is substituted into the balance equation leading to the incremental boundary value elastoplastic problem. The consistent tangent operator is used for assembling of the tangent stiffness matrix which is important for solving this problem by Newton-like methods [2, 4, 5].

4. Solution scheme

This section is organized as follows. Section 4.1 contains an auxilliary notation. In Section 4.2, a priori decision criteria for the elastic response and the returns to the smooth portion, the left edge, the right edge, and the apex of the yield surface are summarized. The solution schemes for these cases are introduced in parallel Sections 4.3–4.7, respectively.

4.1. Auxilliary notation

Recall that $\boldsymbol{\varepsilon}^{tr}$ and $\bar{\varepsilon}^{p,tr}$ are known in (5). The ordered eigenvalues $\varepsilon_1^{tr} \geq \varepsilon_2^{tr} \geq \varepsilon_3^{tr}$ of $\boldsymbol{\varepsilon}^{tr}$ can be determined using the Haigh-Westergaard coordinates (see, e.g., [2]). Other auxilliary notation is summarized below:

- $\sigma_i^{tr} = \frac{1}{3}(3K - 2G)(\varepsilon_1^{tr} + \varepsilon_2^{tr} + \varepsilon_3^{tr}) + 2G\varepsilon_i^{tr},\ i = 1, 2, 3$ — trial principal stresses
- $\mathbb{E}^{tr,2}$, $[\mathbb{E}^{tr,2}]_{ijkl} = \delta_{ik}[\boldsymbol{\varepsilon}^{tr}]_{lj} + \delta_{jl}[\boldsymbol{\varepsilon}^{tr}]_{ik}$ — Fréchet derivative of $(\boldsymbol{\varepsilon}^{tr})^2$
- $\gamma_{s,l}^{tr} = \dfrac{\sigma_1^{tr} - \sigma_2^{tr}}{2G(1 + \sin\psi)}$, $\gamma_{s,r}^{tr} = \dfrac{\sigma_2^{tr} - \sigma_3^{tr}}{2G(1 - \sin\psi)}$,
- $\gamma_{l,a}^{tr} = \dfrac{\sigma_1^{tr} + \sigma_2^{tr} - 2\sigma_3^{tr}}{2G(3 - \sin\psi)}$, $\gamma_{r,a}^{tr} = \dfrac{2\sigma_1^{tr} - \sigma_2^{tr} - \sigma_3^{tr}}{2G(3 + \sin\psi)}$
- $C_s^{tr} = \big(0, \min\{\gamma_{s,l}^{tr}, \gamma_{s,r}^{tr}\}\big)$, $C_l^{tr} = \big[\gamma_{s,l}^{tr}, \gamma_{l,a}^{tr}\big)$,
- $C_r^{tr} = \big[\gamma_{s,r}^{tr}, \gamma_{r,a}^{tr}\big)$, $C_a^{tr} = \big[\max\{\gamma_{l,a}^{tr}, \gamma_{r,a}^{tr}\}, +\infty\big)$
- $S = \frac{4}{3}(3K - 2G)\sin\psi\sin\phi + 4G(1 + \sin\psi\sin\phi)$

- $L = \frac{4}{3}(3K - 2G)\sin\psi\sin\phi + G(1+\sin\psi)(1+\sin\phi) + 2G(1-\sin\psi)(1-\sin\phi)$
- $R = \frac{4}{3}(3K - 2G)\sin\psi\sin\phi + 2G(1+\sin\psi)(1+\sin\phi) + G(1-\sin\psi)(1-\sin\phi)$
- $A = 4K\sin\psi\sin\phi$
- $h(\gamma) = 2\left[c_0 + H\left(\bar{\varepsilon}^{p,tr} + \gamma(2\cos\phi)\right)\right]\cos\phi$
- $q_s^{tr}(\gamma) = (1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - h(\gamma) - S\gamma$
- $q_l^{tr}(\gamma) = \frac{1}{2}(1+\sin\phi)(\sigma_1^{tr} + \sigma_2^{tr}) - (1-\sin\phi)\sigma_3^{tr} - h(\gamma) - L\gamma$
- $q_r^{tr}(\gamma) = (1+\sin\phi)\sigma_1^{tr} - \frac{1}{2}(1-\sin\phi)(\sigma_2^{tr} + \sigma_3^{tr}) - h(\gamma) - R\gamma$
- $q_a^{tr}(\gamma) = \frac{2}{3}(\sigma_1^{tr} + \sigma_2^{tr} + \sigma_3^{tr})\sin\phi - h(\gamma) - A\gamma$
- $H_1 = h'(\triangle\lambda) = 4H'(\bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi))\cos^2\phi$ — possibly, we take the derivative from the left if $h'(\triangle\lambda)$ does not exist

4.2. A priori decision criteria

The criteria introduced below are mutually disjoint, i.e., for a given pair $(\boldsymbol{\varepsilon}^{tr}, \bar{\varepsilon}^{p,tr})$, only one possibility is realized.

The elastic response:

- $(1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi \leq 0$

The return to the smooth portion of the yield surface:

- $(1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi > 0$
- $q_s^{tr}(\min\{\gamma_{s,l}^{tr}, \gamma_{s,r}^{tr}\}) < 0$

The return to the left edge of the yield surface:

- $(1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi > 0$
- $\gamma_{s,l}^{tr} < \gamma_{l,a}^{tr}$, $q_l^{tr}(\gamma_{s,l}^{tr}) \geq 0$, $q_l^{tr}(\gamma_{l,a}^{tr}) < 0$

The return to the right edge of the yield surface:

- $(1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi > 0$
- $\gamma_{s,r}^{tr} < \gamma_{r,a}^{tr}$, $q_r^{tr}(\gamma_{s,r}^{tr}) \geq 0$, $q_r^{tr}(\gamma_{r,a}^{tr}) < 0$

The return to the apex of the yield surface:

- $(1+\sin\phi)\sigma_1^{tr} - (1-\sin\phi)\sigma_3^{tr} - 2(c_0 + H(\bar{\varepsilon}^{p,tr}))\cos\phi > 0$
- $q_a^{tr}(\max\{\gamma_{l,a}^{tr}, \gamma_{r,a}^{tr}\}) \geq 0$

Notice that other very useful necessary conditions for the returns to the smooth portion, the left and right edges are introduced in Sections 4.4–4.6, respectively. These conditions were derived in [4] and simplify the construction of the consistent tangent operator.

4.3. Solution scheme for the elastic response

- $\triangle\lambda = 0$
- $\sigma_i = \sigma_i^{tr},\ i = 1,2,3$
- $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr}$
- $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{tr}$
- $\mathbb{T} = \mathbb{D}_e$

4.4. Solution scheme for the return to the smooth portion

It is worth mentioning that $\varepsilon_1^{tr} > \varepsilon_2^{tr} > \varepsilon_3^{tr}$ is a necessary condition for this return. Therefore, the following auxilliary formulas are well-defined:

$$
\begin{aligned}
\boldsymbol{E}_i^{tr} &= \frac{(\boldsymbol{\varepsilon}^{tr} - \varepsilon_j^{tr}\boldsymbol{I})(\boldsymbol{\varepsilon}^{tr} - \varepsilon_k^{tr}\boldsymbol{I})}{(\varepsilon_i^{tr} - \varepsilon_j^{tr})(\varepsilon_i^{tr} - \varepsilon_k^{tr})}, \quad i \neq j \neq k \neq i, \ \ i = 1,2,3, \\
\boldsymbol{F}_{s,\phi}^{tr} &= 2G(1+\sin\phi)\boldsymbol{E}_1^{tr} - 2G(1-\sin\phi)\boldsymbol{E}_3^{tr} + \frac{2}{3}(3K-2G)\sin\phi\boldsymbol{I}, \\
\boldsymbol{F}_{s,\psi}^{tr} &= 2G(1+\sin\psi)\boldsymbol{E}_1^{tr} - 2G(1-\sin\psi)\boldsymbol{E}_3^{tr} + \frac{2}{3}(3K-2G)\sin\psi\boldsymbol{I}, \\
\mathbb{E}_i^{tr} &= \frac{\mathbb{E}^{tr,2} - (\varepsilon_j^{tr} + \varepsilon_k^{tr})\mathbb{I} - (2\varepsilon_i^{tr} - \varepsilon_j^{tr} - \varepsilon_k^{tr})\boldsymbol{E}_i^{tr} \otimes \boldsymbol{E}_i^{tr}}{(\varepsilon_i^{tr} - \varepsilon_j^{tr})(\varepsilon_i^{tr} - \varepsilon_k^{tr})} \\
&\quad - \frac{(\varepsilon_j^{tr} - \varepsilon_k^{tr})[\boldsymbol{E}_j^{tr} \otimes \boldsymbol{E}_j^{tr} - \boldsymbol{E}_k^{tr} \otimes \boldsymbol{E}_k^{tr}]}{(\varepsilon_i^{tr} - \varepsilon_j^{tr})(\varepsilon_i^{tr} - \varepsilon_k^{tr})}, \quad i \neq j \neq k \neq i, \ i = 1,2,3.
\end{aligned}
$$

It is well-known that $\boldsymbol{E}_1^{tr}, \boldsymbol{E}_2^{tr}, \boldsymbol{E}_3^{tr}$ define the eigenprojections of $\boldsymbol{\varepsilon}^{tr}$ [2]. Further, it holds: $\boldsymbol{E}_i^{tr} = \partial\varepsilon_i^{tr}/\partial\boldsymbol{\varepsilon}^{tr}$ and $\mathbb{E}_i^{tr} = \partial\boldsymbol{E}_i^{tr}/\partial\boldsymbol{\varepsilon}^{tr}$, $i = 1,2,3$. The solution scheme for the return to the smooth portion reads as:

- $\triangle\lambda \in C_s^{tr}$ and solves $q_s^{tr}(\triangle\lambda) = 0$
- $\sigma_1 = \sigma_1^{tr} - \triangle\lambda\left[\frac{2}{3}(3K-2G)\sin\psi + 2G(1+\sin\psi)\right]$
- $\sigma_2 = \sigma_2^{tr} - \triangle\lambda\left[\frac{2}{3}(3K-2G)\sin\psi\right]$
- $\sigma_3 = \sigma_3^{tr} - \triangle\lambda\left[\frac{2}{3}(3K-2G)\sin\psi - 2G(1-\sin\psi)\right]$
- $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi)$
- $\boldsymbol{\sigma} = \sigma_1\boldsymbol{E}_1^{tr} + \sigma_2\boldsymbol{E}_2^{tr} + \sigma_3\boldsymbol{E}_3^{tr}$
- $\mathbb{T} = \sum_{i=1}^{3}\left[\sigma_i\mathbb{E}_i^{tr} + 2G\boldsymbol{E}_i^{tr} \otimes \boldsymbol{E}_i^{tr}\right] + \frac{1}{3}(3K-2G)\boldsymbol{I} \otimes \boldsymbol{I} - \dfrac{1}{S+H_1}\boldsymbol{F}_{s,\psi}^{tr} \otimes \boldsymbol{F}_{s,\phi}^{tr}$

4.5. Solution scheme for the return to the left edge

For this return, the only one sharp inequality is guaranteed: $\varepsilon_2^{tr} > \varepsilon_3^{tr}$. We use the following auxilliary and well-defined formulas:

$$
\begin{aligned}
\boldsymbol{E}_3^{tr} &= \frac{(\boldsymbol{\varepsilon}^{tr} - \varepsilon_1^{tr}\boldsymbol{I})(\boldsymbol{\varepsilon}^{tr} - \varepsilon_2^{tr}\boldsymbol{I})}{(\varepsilon_3^{tr} - \varepsilon_1^{tr})(\varepsilon_3^{tr} - \varepsilon_2^{tr})}, \quad \boldsymbol{E}_{12}^{tr} = \boldsymbol{I} - \boldsymbol{E}_3^{tr},\\
\boldsymbol{F}_{l,\phi}^{tr} &= G(1+\sin\phi)\boldsymbol{E}_{12}^{tr} - 2G(1-\sin\phi)\boldsymbol{E}_3^{tr} + \frac{2}{3}(3K-2G)\sin\phi\boldsymbol{I},\\
\boldsymbol{F}_{l,\psi}^{tr} &= G(1+\sin\psi)\boldsymbol{E}_{12}^{tr} - 2G(1-\sin\psi)\boldsymbol{E}_3^{tr} + \frac{2}{3}(3K-2G)\sin\psi\boldsymbol{I},\\
\mathbb{E}_3^{tr} &= \frac{\mathbb{E}^{tr,2} - (\varepsilon_1^{tr}+\varepsilon_2^{tr})\mathbb{I} - [\boldsymbol{\varepsilon}^{tr}\otimes\boldsymbol{E}_{12}^{tr} + \boldsymbol{E}_{12}^{tr}\otimes\boldsymbol{\varepsilon}^{tr}] + (\varepsilon_1^{tr}+\varepsilon_2^{tr})\boldsymbol{E}_{12}^{tr}\otimes\boldsymbol{E}_{12}^{tr}}{(\varepsilon_3^{tr} - \varepsilon_1^{tr})(\varepsilon_3^{tr} - \varepsilon_2^{tr})}\\
&\quad + \frac{(\varepsilon_1^{tr}+\varepsilon_2^{tr}-2\varepsilon_3^{tr})\boldsymbol{E}_3^{tr}\otimes\boldsymbol{E}_3^{tr} + \varepsilon_3^{tr}[\boldsymbol{E}_{12}^{tr}\otimes\boldsymbol{E}_3^{tr} + \boldsymbol{E}_3^{tr}\otimes\boldsymbol{E}_{12}^{tr}]}{(\varepsilon_3^{tr} - \varepsilon_1^{tr})(\varepsilon_3^{tr} - \varepsilon_2^{tr})}.
\end{aligned}
$$

It is possible to prove that the definitions of $\mathbb{E}_3^{tr}$ introduced here and in Section 4.4 are equivalent under the assumption $\varepsilon_1^{tr} > \varepsilon_2^{tr} > \varepsilon_3^{tr}$. The solution scheme for the return to the left edge reads as:

- $\triangle\lambda \in C_l^{tr}$ and solves $q_l^{tr}(\triangle\lambda) = 0$
- $\sigma_1 = \sigma_2 = \frac{1}{2}(\sigma_1^{tr} + \sigma_2^{tr}) - \triangle\lambda\left[\frac{2}{3}(3K-2G)\sin\psi + G(1+\sin\psi)\right]$
- $\sigma_3 = \sigma_3^{tr} - \triangle\lambda\left[\frac{2}{3}(3K-2G)\sin\psi - 2G(1-\sin\psi)\right]$
- $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi)$
- $\boldsymbol{\sigma} = \sigma_1\boldsymbol{E}_{12}^{tr} + \sigma_3\boldsymbol{E}_3^{tr}$
- $\left\{\begin{aligned} \mathbb{T} &= (\sigma_3 - \sigma_1)\mathbb{E}_3^{tr} + G\boldsymbol{E}_{12}^{tr}\otimes\boldsymbol{E}_{12}^{tr} + 2G\boldsymbol{E}_3^{tr}\otimes\boldsymbol{E}_3^{tr} + \tfrac{1}{3}(3K-2G)\boldsymbol{I}\otimes\boldsymbol{I}\\ &\quad -\frac{1}{L+H_1}\boldsymbol{F}_{l,\psi}^{tr}\otimes\boldsymbol{F}_{l,\phi}^{tr}\end{aligned}\right.$

4.6. Solution scheme for the return to the right edge

For this return, the inequality $\varepsilon_1^{tr} > \varepsilon_2^{tr}$ is guaranteed. We use the following auxilliary and well-defined formulas:

$$
\begin{aligned}
\boldsymbol{E}_1^{tr} &= \frac{(\boldsymbol{\varepsilon}^{tr} - \varepsilon_2^{tr}\boldsymbol{I})(\boldsymbol{\varepsilon}^{tr} - \varepsilon_3^{tr}\boldsymbol{I})}{(\varepsilon_1^{tr} - \varepsilon_2^{tr})(\varepsilon_1^{tr} - \varepsilon_3^{tr})}, \quad \boldsymbol{E}_{23}^{tr} = \boldsymbol{I} - \boldsymbol{E}_1^{tr},\\
\boldsymbol{F}_{r,\phi}^{tr} &= 2G(1+\sin\phi)\boldsymbol{E}_1^{tr} - G(1-\sin\phi)\boldsymbol{E}_{23}^{tr} + \frac{2}{3}(3K-2G)\sin\phi\boldsymbol{I},\\
\boldsymbol{F}_{r,\psi}^{tr} &= 2G(1+\sin\psi)\boldsymbol{E}_1^{tr} - G(1-\sin\psi)\boldsymbol{E}_{23}^{tr} + \frac{2}{3}(3K-2G)\sin\psi\boldsymbol{I},\\
\mathbb{E}_1^{tr} &= \frac{\mathbb{E}^{tr,2} - (\varepsilon_2^{tr}+\varepsilon_3^{tr})\mathbb{I} - [\boldsymbol{\varepsilon}^{tr}\otimes\boldsymbol{E}_{23}^{tr} + \boldsymbol{E}_{23}^{tr}\otimes\boldsymbol{\varepsilon}^{tr}] + (\varepsilon_2^{tr}+\varepsilon_3^{tr})\boldsymbol{E}_{23}^{tr}\otimes\boldsymbol{E}_{23}^{tr}}{(\varepsilon_1^{tr} - \varepsilon_2^{tr})(\varepsilon_1^{tr} - \varepsilon_2^{tr})}\\
&\quad + \frac{(\varepsilon_2^{tr}+\varepsilon_3^{tr}-2\varepsilon_1^{tr})\boldsymbol{E}_1^{tr}\otimes\boldsymbol{E}_1 + \varepsilon_1^{tr}[\boldsymbol{E}_{23}^{tr}\otimes\boldsymbol{E}_1^{tr} + \boldsymbol{E}_1^{tr}\otimes\boldsymbol{E}_{23}^{tr}]}{(\varepsilon_1^{tr} - \varepsilon_2^{tr})(\varepsilon_1^{tr} - \varepsilon_3^{tr})}.
\end{aligned}
$$

It is possible to prove that the definitions of $\mathbb{E}_1^{tr}$ introduced here and in Section 4.4 are equivalent under the assumption $\varepsilon_1^{tr} > \varepsilon_2^{tr} > \varepsilon_3^{tr}$. The solution scheme for the return to the right edge reads as:

- $\triangle\lambda \in C_r^{tr}$ and solves $q_r^{tr}(\triangle\lambda) = 0$
- $\sigma_1 = \sigma_1^{tr} - \triangle\lambda \left[\frac{2}{3}(3K-2G)\sin\psi + 2G(1+\sin\psi)\right]$
- $\sigma_2 = \sigma_3 = \frac{1}{2}(\sigma_2^{tr} + \sigma_3^{tr}) - \triangle\lambda \left[\frac{2}{3}(3K-2G)\sin\psi - G(1-\sin\psi)\right]$
- $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi)$
- $\boldsymbol{\sigma} = \sigma_1 \boldsymbol{E}_1^{tr} + \sigma_3 \boldsymbol{E}_{23}^{tr}$
- $\left\{ \begin{aligned} \mathbb{T} = {} & (\sigma_1 - \sigma_3)\mathbb{E}_1^{tr} + 2G\boldsymbol{E}_1^{tr} \otimes \boldsymbol{E}_1^{tr} + G\boldsymbol{E}_{23}^{tr} \otimes \boldsymbol{E}_{23}^{tr} + \tfrac{1}{3}(3K-2G)\boldsymbol{I} \otimes \boldsymbol{I} \\ & -\frac{1}{R+H_1}\boldsymbol{F}_{r,\psi}^{tr} \otimes \boldsymbol{F}_{r,\phi}^{tr} \end{aligned} \right.$

4.7. Solution scheme for the return to the apex

- $\triangle\lambda \in C_a^{tr}$ and solves $q_a^{tr}(\triangle\lambda) = 0$
- $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3}(\sigma_1^{tr} + \sigma_2^{tr} + \sigma_3^{tr}) - \triangle\lambda[2K\sin\psi]$
- $\bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle\lambda(2\cos\phi)$
- $\boldsymbol{\sigma} = \sigma_1 \boldsymbol{I}$
- $\mathbb{T} = K\left(1 - \dfrac{A}{A+H_1}\right)\boldsymbol{I} \otimes \boldsymbol{I}$

5. Conclusion

The subdifferential-based constitutive solution scheme for the Mohr-Coulomb model was introduced. This technique has several advantages in comparison to the current ones. First, it enabled a deeper analysis of the constitutive problem. Second, a priori decision criteria were derived for each position of the unknown stress tensor on the yield surface. Finally, for each return type, we specified the necessary conditions on multiplicity of $\varepsilon_1^{tr}, \varepsilon_2^{tr}, \varepsilon_3^{tr}$. Such conditions are crucial for the correct definition of the consistent tangent operator $\mathbb{T}$. Without this knowledge, an additional branching in the definition of $\mathbb{T}$ must be introduced as in [2, Appendix A].

The presented solution schemes were implemented in Matlab codes for the analysis of slope stability in 2D and 3D. The codes are publicly available in [6] and the used numerical techniques are described in [4].

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