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REMARKS ON INVERSE OF MATRIX POLYNOMIALS

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Abstract: Analysis of a non-classically damped engineering structure, which is subjected to an external excitation, leads to the solution of a system of second order ordinary differential equations. Although there exists a large variety of powerful numerical methods to accomplish this task, in some cases it is convenient to formulate the explicit inversion of the respective quadratic fundamental system. The presented contribution uses and extends concepts in matrix polynomial theory and proposes an implementation of the inversion problem.

Keywords: matrix polynomial, polynomial eigenvalues, structural vibration

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1. Introduction

Solution of matrix differential equations is closely associated to the theory of matrix polynomials. The very important class of the second order matrix differential equations has a wide variety of applications, among others in vibration analysis in civil or mechanical engineering or in the analysis of oscillation circuits in electrical engineering. The motivation behind this contribution originates from the vibration analysis of non-classically damped engineering structures, which are subjected to a random external excitation. In case of non-stationary excitation, the numerical integration of the differential system gives only a very limited information on the stochastic character of the response. In such cases it is more convenient to formulate the exact or approximate analytical solution, if possible, and to use it for an assessment of the stochastic properties of the system response. Such a procedure is provided by, e.g., the spectral decomposition method [2].

The behaviour of the structure is described by a relation:

$$\mathbf{A}\ddot{\mathbf{U}}(\omega,t)+\mathbf{B}\dot{\mathbf{U}}(\omega,t)+\mathbf{C}\mathbf{U}(\omega,t)=\mathbf{f}(\omega,t) \tag{1}$$

where the coefficient matrices $\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbb{R}^{n\times n}$ are considered to be constant, real and symmetric, $\mathbf{U}(\omega,t)$ is a deterministic function describing transformation of the

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random excitation and $\mathbf{f}$ describes properties of the random excitation. The Laplace transform changes the differential system into an algebraic one

$$(\mathbf{A}p^2 + \mathbf{B}p + \mathbf{C})\mathbf{U}^\star(\omega, p) = \mathbf{Q}(p)\mathbf{U}^\star(\omega, p) = \mathbf{f}^\star(\omega, p)\,, \tag{2}$$

whose solution is given by

$$\mathbf{U}^\star(\omega, p) = \mathbf{Q}(p)^{-1}\mathbf{f}^\star(\omega, p)\,. \tag{3}$$

The inverse of the matrix polynomial $\mathbf{Q}(p)^{-1}$ can be written in a form of a sum [1]

$$\mathbf{Q}(p)^{-1} = \sum_{j=1}^{n}\left(\mathbf{S}_j\frac{1}{p-p_j} + \overline{\mathbf{S}}_j\frac{1}{p-\overline{p}_j}\right) \tag{4}$$

where p_j are the roots of $\det\mathbf{Q}(p)$ (generalized eigenvalues of $\mathbf{Q}$) and matrices $\mathbf{S}_j$ are rank 1 matrices related to the generalized eigenvectors of $\mathbf{Q}$. Solution of (1) is finally given as

$$\mathbf{U}(\omega, t) = \sum_{j=1}^{2n}\mathbf{S}_j\int_0^t \mathsf{e}^{p_j(t-\tau)}\mathbf{f}(\omega,\tau)\mathsf{d}\tau. \tag{5}$$

In the following section, the basics of matrix polynomial theory will be introduced according to the monograph by Gohberg et al. [1]. Sections 3 and 4 will be devoted to the lemma which leads to an advantageous formulation of matrices $\mathbf{S}_j$ in (4) and to a computational algorithm.

2. Basics of the matrix polynomials theory

Definition. *Let* $l > 0$ *and* $\mathbf{A}_j \in \mathbb{R}^{n\times n}$, $j = 0,\ldots,l$, $\mathbf{A}_l \neq 0$ *be square matrices. The* matrix polynomial $\mathbf{L}(\lambda)$ of degree l *is defined as*

$$\mathbf{L}(\lambda) = \sum_{j=0}^{l}\mathbf{A}_j\lambda^j\ . \tag{6}$$

An eigenvalue λ *of the matrix polynomial* $\mathbf{L}(\lambda)$ *is the solution of*

$$\mathbf{L}(\lambda) = 0 \quad or \quad \det\mathbf{L}(\lambda) = 0\,, \tag{7}$$

whilst the corresponding (right) eigenvector $\mathbf{x}$ *and* left eigenvector $\mathbf{y}$ *is any non-zero solution of*

$$\mathbf{L}(\lambda)\mathbf{x} = \sum_{j=0}^{l}\mathbf{A}_j\lambda^j\mathbf{x} = 0 \quad resp. \quad \mathbf{y}^\mathsf{T}\mathbf{L}(\lambda) = \sum_{j=0}^{l}\mathbf{y}^\mathsf{T}\mathbf{A}_j\lambda^j = 0\ . \tag{8}$$

Two matrix polynomials $\mathbf{M}(\lambda)$ *and* $\mathbf{N}(\lambda)$ *are* equivalent, $\mathbf{M}(\lambda) \simeq \mathbf{N}(\lambda)$, *if there exist two matrix polynomials* $\mathbf{E}(\lambda)$ *and* $\mathbf{F}(\lambda)$ *with constant determinants such that*

$$\mathbf{M}(\lambda) = \mathbf{E}(\lambda)\mathbf{N}(\lambda)\mathbf{F}(\lambda). \tag{9}$$

A linearization *of a matrix polynomial* $\mathbf{L}(\lambda)$ *of dimension* n *and degree* l *is a linear matrix polynomial* $\mathbf{E}\lambda - \mathbf{H}$ *of dimension* nl *where*

$$\mathbf{L}(\lambda) \simeq (\mathbf{E}\lambda - \mathbf{H}) . \tag{10}$$

The linearization matrix $\mathcal{A} \in \mathbb{R}^{nl \times nl}$ *is such a matrix that* $\mathbf{L}(\lambda) \simeq \mathbf{I}\lambda - \mathcal{A}$.

The concept of linearization is traditionally used for computation of eigenvalues of a matrix polynomial using standard methods for the linear eigenvalue problem [4]. The linearization is not uniquely defined. However, all linearizations share the same set of eigenvalues. The commonly used linearization assumes $\mathbf{E} = \mathbf{I}$ and uses a block--matrix $\mathbf{H}$ consisting of terms $-\mathbf{A}_l^{-1}\mathbf{A}_i, i = 0, \ldots, l-1$ in the last row and identity matrices in positions of the first superdiagonal. However, there exist also other forms, suitable for particular purposes. One of the most interesting examples is the symmetric linearization, which assures symmetry of the matrices $\mathbf{E}$ and $\mathbf{H}$ due to symmetry in individual matrices $\mathbf{A}_i$, see [3].

Definition. *A* standard pair *of a matrix polynomial is a pair of matrices* $(\mathbf{X}, \mathbf{T})$, $\mathbf{X} \in \mathbb{C}^{n \times nl}, \mathbf{T} \in \mathbb{C}^{nl \times nl}$ *such that the matrix* $\mathbf{Z}$ *of dimension* $(nl \times nl)$, *where*

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{XT} \\ \vdots \\ \mathbf{A}_l \mathbf{X}\mathbf{T}^{l-1} \end{pmatrix}$$

is regular and

$$\sum_0^l \mathbf{A}_j \mathbf{X}\mathbf{T}^j = 0 \ .$$

The standard pairs are not unique. However, if $\mathbf{T}$ is diagonal (or in a Jordan form in the case where some eigenvalues have higher multiplicity), the matrix $\mathbf{X}$ will be uniquely defined. Its columns will be formed by eigenvectors corresponding to the respective eigenvalues. Such a standard pair $(\mathbf{X}, \mathbf{T})$ is called a *Jordan pair*.

Definition A Jordan triple *is called a triple of matrices* $(\mathbf{X}, \mathbf{T}, \mathbf{Y})$, *where* $(\mathbf{X}, \mathbf{T})$ *is a Jordan pair and* $\mathbf{Y} \in \mathbb{C}^{nl \times n}$ *satisfies:*

$$\begin{aligned} \mathbf{X}\mathbf{T}^i\mathbf{Y} = \mathbf{0} \quad i = 0, \ldots, l-1 \ , \\ \mathbf{A}_l \mathbf{X}\mathbf{T}^{l-1}\mathbf{Y} = \mathbf{I} \ . \end{aligned} \tag{11}$$

3. Inverse of matrix polynomial

Lemma 1. *Let all eigenvalues of the matrix polynomial* $\mathbf{L}(\lambda)$ *be non-zero and the leading coefficient matrix be regular. Then the rows of the matrix* $\mathbf{Y}$ *of the Jordan triple* $(\mathbf{X}, \mathbf{T}, \mathbf{Y})$ *form the left eigenvectors of* $\mathbf{L}(\lambda)$, *i.e.*

$$\sum_{j=0}^k \mathbf{T}^j \mathbf{Y}\mathbf{A}_j = 0 \ . \tag{12}$$

Proof: Let $l = 2$. The proof assumes the linearization

$$\left[-\begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}\lambda + \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{A}_0 & -\mathbf{A}_1 \end{pmatrix}\right].$$

Let $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{XT} \end{pmatrix}$. Because $(\mathbf{X}, \mathbf{T})$ is a Jordan pair it holds that

$$\begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{A}_0 & -\mathbf{A}_1 \end{pmatrix}\mathbf{Z} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}\mathbf{ZT}. \tag{13}$$

Right multiplication by $\mathbf{Z}^{-1}$ and further transformation leads to

$$\mathbf{Z}^{-1}\begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{A}_2^{-1} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{A}_0 & -\mathbf{A}_1 \end{pmatrix} = \mathbf{TZ}^{-1}. \tag{14}$$

Let $\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix}$ and $\mathbf{Z}^{-1} = \begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix}$, where $\mathbf{Z}_i$ $i = 1, \ldots, 4$ are square blocks of dimension n. Expansion of the last expression (14) gives

$$\begin{pmatrix} -\mathbf{Z}_2\mathbf{A}_2^{-1}\mathbf{A}_0 & \mathbf{Z}_1 - \mathbf{Z}_2\mathbf{A}_2^{-1}\mathbf{A}_1 \\ -\mathbf{Z}_4\mathbf{A}_2^{-1}\mathbf{A}_0 & \mathbf{Z}_3 - \mathbf{Z}_4\mathbf{A}_2^{-1}\mathbf{A}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1\mathbf{Z}_1 & \mathbf{T}_1\mathbf{Z}_2 \\ \mathbf{T}_2\mathbf{Z}_3 & \mathbf{T}_2\mathbf{Z}_4 \end{pmatrix}. \tag{15}$$

Now, comparing first columns

$$\begin{aligned} \mathbf{Z}_1 &= -\mathbf{T}_1^{-1}\mathbf{Z}_2\mathbf{A}_2^{-1}\mathbf{A}_0 , \\ \mathbf{Z}_3 &= -\mathbf{T}_2^{-1}\mathbf{Z}_4\mathbf{A}_2^{-1}\mathbf{A}_0 , \end{aligned}$$

and substituting into the second columns (15) then writing in the matrix form leads to

$$-\mathbf{T}^{-1}\begin{pmatrix} \mathbf{Z}_2\mathbf{A}_2^{-1} \\ \mathbf{Z}_4\mathbf{A}_2^{-1} \end{pmatrix}\mathbf{A}_0 - \begin{pmatrix} \mathbf{Z}_2\mathbf{A}_2^{-1} \\ \mathbf{Z}_4\mathbf{A}_2^{-1} \end{pmatrix}\mathbf{A}_1 = \mathbf{T}\begin{pmatrix} \mathbf{Z}_2 \\ \mathbf{Z}_4 \end{pmatrix}.$$

Denoting $\mathbf{Y} = \mathbf{Z}^{-1}\begin{pmatrix} 0 \\ \mathbf{A}_2^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_2\mathbf{A}_2^{-1} \\ \mathbf{Z}_4\mathbf{A}_2^{-1} \end{pmatrix}$, and multiplying by the matrix $\mathbf{T}$ from the left hand side leads to

$$-\mathbf{YA}_0 - \mathbf{TYA}_1 = \mathbf{T}^2\mathbf{YA}_2. \quad \square$$

The proof for general l can be performed in a similar manner: the key step is the expansion of the $\mathbf{Z}^{-1} = \big(\mathbf{Z}_1 \ldots \mathbf{Z}_l\big)$, where $\mathbf{Z}_i$ $i = 1, \ldots, l$ are the column blocks.

In the next section, it will be supposed that $\mathbf{A}_l$ is regular. The inverse matrix polynomial can be written using its Jordan triple in the following form [1]:

$$(\mathbf{L}(\lambda))^{-1} = \mathbf{X}(\lambda\mathbf{I} - \mathbf{T})^{-1}\mathbf{Y} \tag{16}$$

If $\mathbf{T}$ is diagonal, e.g., if all eigenvalues λ_i are distinct, it holds

$$(\lambda\mathbf{I} - \mathbf{T})^{-1} = \operatorname{diag}\left(\frac{1}{\lambda - \lambda_i}\right)$$

and equation (16) can be rewritten as

$$(\mathbf{L}(\lambda))^{-1} = \sum_{j=1}^{ln} \frac{1}{\lambda - \lambda_j} \mathbf{x}_j \mathbf{y}_j^{\mathsf{T}} \tag{17}$$

where $\mathbf{x}_j$ are columns of $\mathbf{X}$ and $\mathbf{y}_j^{\mathsf{T}}$ are rows of $\mathbf{Y}$.

By respecting the character of the underlying physical problem, it is possible to assume that all matrices $\mathbf{A}_i$ are symmetrical and that $\mathbf{T}$ is diagonal and regular with distinct elements. The matrix $\mathbf{Y}$ is defined by the conditions (11). It remains to show that there exists a matrix $\mathbf{D}$ such that $\mathbf{Y}^{\mathsf{T}}\mathbf{D} = \mathbf{X}$.

$$\begin{aligned}
\mathbf{0}^{\mathsf{T}} &= \left(\sum \mathbf{A}_i \mathbf{Y}^{\mathsf{T}} \mathbf{D} \mathbf{T}^i\right)^{\mathrm{T}} && (18)\\
&= \sum \mathbf{T}^i \mathbf{D} \mathbf{Y} \mathbf{A}_i && (19)\\
&= \sum \mathbf{D}^{\mathsf{T}^i} \mathbf{T}^i \mathbf{D} \mathbf{Y} \mathbf{A}_i = \sum \mathbf{T}^i \mathbf{Y} \mathbf{A}_i && (20)
\end{aligned}$$

where the symbol $\mathbf{D}^{\mathsf{T}^i}$ means i-multiple transpositions.

The last equation (20) implies symmetry of $\mathbf{D}$, i.e. $\mathbf{DT} = \mathbf{TD}$ and thus for elements d_{ij} of $\mathbf{D}$ it holds: $d_{i,j} = 0 \Leftrightarrow \frac{t_i}{t_j} \neq 1$.

This means that if the diagonal elements of $\mathbf{T}$ are distinct, the matrix $\mathbf{D}$ is diagonal and regular. The same result can be reached using a different reasoning: due to Lemma 1 the third term of the Jordan triple is formed by the left eigenvectors. For symmetric matrices $\mathbf{A}_j$ the right and left generalized eigenvectors coincide. This means that the corresponding columns of $\mathbf{Y}^{\mathsf{T}}$ and $\mathbf{X}$ differ by multiplicative constants and so the matrix $\mathbf{D}$ has to be diagonal.

Under the assumptions introduced above, it is possible to find such eigenvectors $\mathbf{X}$ that $(\mathbf{X}, \mathbf{T}, \mathbf{X}^{\mathsf{T}})$ forms the Jordan triple. The conditions (11) attain the form:

$$\begin{aligned}
\mathbf{X}\mathbf{T}^i\mathbf{X}\mathsf{T} &= 0 \quad i = 0, \ldots, l-1\ , \\
\mathbf{A}_l \mathbf{X} \mathbf{T}^{l-1} \mathbf{X}^{\mathsf{T}} &= \mathbf{I}\ .
\end{aligned} \tag{21}$$

The only unknown step in the procedure is selection of the proper scaling constants of the eigenvectors $\mathbf{X}$.

4. Formulation of the algorithm

The inverse of a matrix polynomial $\mathbf{L}(\lambda)$ can be formulated using the following procedure

1. Solve the linear eigenvalue problem with some linearization matrix to obtain a pair of matrices $(\widetilde{\mathbf{X}}, \mathbf{T})$.

2. Find the diagonal matrix $\mathbf{D}$ such that $\mathbf{X} = \widetilde{\mathbf{X}}\mathbf{D}$ and $(\mathbf{X}, \mathbf{T})$ satisfy (21). Its existence was proven before.
   According to (11) we now have
$$\begin{pmatrix} \widetilde{\mathbf{X}}\mathbf{D}\widetilde{\mathbf{X}}^{\mathsf{T}} \\ \mathbf{A}_2\widetilde{\mathbf{X}}\mathbf{T}\mathbf{D}\widetilde{\mathbf{X}}^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix} . \tag{22}$$
   Substituting $\Delta = \widetilde{\mathbf{X}}\mathbf{D}$ into (22) the equation transforms into
$$\begin{pmatrix} \widetilde{\mathbf{X}} \\ \mathbf{A}_2\widetilde{\mathbf{X}}\mathbf{T} \end{pmatrix} \Delta^{\mathsf{T}} = \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix} . \tag{23}$$
   Because $(\Delta)_{ij} = x_{ij}d_j$ and x_{ij} are known, it is sufficient to solve the system (23) for only one column of Δ and the corresponding column of the right hand side. Selection of such a column depends on the distribution of non-zero elements of rows of the matrix $\widetilde{\mathbf{X}}$.
3. The diagonal elements of $\mathbf{D}$ are computed as ratios
$$d_{ii} = \Delta_{ji}/x_{ij} \ , \tag{24}$$
   supposing that the j-th column has been used. Finally, set $\mathbf{X} = \mathbf{Y}^{\mathsf{T}} = \widetilde{\mathbf{X}}\sqrt{\mathbf{D}}$.
4. The inverse of the matrix polynomial can be computed using relation (17) where both $\mathbf{x}_j$ and $\mathbf{y}_j^{\mathsf{T}}$ are columns of $\mathbf{X}$.

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