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GOAL ORIENTED A POSTERIORI ERROR ESTIMATES FOR THE DISCONTINUOUS GALERKIN METHOD

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Abstract: This paper is concerned with goal-oriented a posteriori error estimates for discontinous Galerkin discretizations of linear elliptic boundary value problems. Our approach combines the Dual Weighted Residual method (DWR) with local weighted least-squares reconstruction of the discrete solution. This technique is used not only for controlling the discretization error, but also to track the influence of the algebraic errors. We illustrate the performance of the proposed method by numerical experiments.

Keywords: discontinuous Galerkin method, aposteriori error estimates, goal-oriented estimates, algebraic error

MSC: 65M60, 65M70

1. Introduction

Usually a posteriori error estimation techniques measure the error in a norm which is connected with the mathematical formulation of the problem being solved. The concern in practical application may be quite different. The main purpose of the computation may be to calculate a quantity of interest, expressed in the mathematical language as a functional applied to the solution of the solved problem (e.g. drag or lift in the airflow simulations). The dual weighted residual (DWR) method first proposed by Rannacher et al., (for a survey, see e.g. [2]), suggests a way how to connect the error of the target quantity with the solved problem. This is enabled by solving the so-called dual (or adjoint) problem.

Our main goal is to employ the DWR method for designing an efficient adaptive algorithm for solving stationary partial differential equations. We focus on the Poisson problem with Dirichlet boundary conditions in this paper, but most of the work can be extended even to nonlinear problems. Even though the DWR method can be combined with any discretization technique based on the variational formulation, we focus mainly on the discontinuous Galerkin (DG) method, which is based on discontinuous piece-wise polynomial approximation.

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Solving the additional dual problem may lead to an increase in the computational effort. Moreover, the dual solution needs to be in a space $V_h^+ \supsetneq V_h$, where V_h is the original discrete space. On the other hand, the ability of measuring directly the error of the target quantity reduces the computational efforts compared to other estimation techniques.

There are several possibilities for discretization of the dual problem. One can solve the dual problem on a globally refined mesh with higher polynomial degree, see e.g. [8]. This approach gives very precise results, but the computational effort used to solve the dual problem exceeds the cost of the original problem dramatically. Therefore, we present an algorithm based on higher-order reconstruction presented originally in [5], which can be computed locally and therefore much more efficiently.

Further, the presented method naturally allows to integrate estimation of the algebraic errors arising from inexact solution of both the primal and dual problems. We present estimates enabling to keep the discretization and algebraic errors in balance.

Finally, we compare the performance of the presented goal-oriented error estimation method with a classical (not goal-oriented) a posteriori error estimate and we examine the influence of the algebraic errors by a numerical experiment.

2. Problem description

Let $\Omega \in \mathbb{R}^2$ be a bounded polygonal domain. We consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \tag{1a}$$

$$u = u_D \quad \text{on } \partial\Omega \tag{1b}$$

where $u : \Omega \to \mathbb{R}$ is an unknown scalar function defined on Ω.

We use the standard notation for the Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^1(\Omega)$ for the subspace of $H^1(\Omega)$ containing functions with vanishing traces of $\partial\Omega$. Moreover, the space of polynomial functions up to the degree k defined on a domain $M \subset \mathbb{R}^2$ is denoted by $P^k(M)$. We assume that $f \in L^2(\Omega)$ and u_D is trace of some $u^* \in H^1(\Omega) \cap L^\infty(\Omega)$ on $\partial\Omega$.

We say that function $u \in H^1(\Omega)$ is the *weak solution* of problem (1) if it satisfies

$$\begin{aligned} &u - u^* \in H_0^1(\Omega), \\ &a(u, \varphi) = l(\varphi) \quad \forall \varphi \in H_0^1(\Omega), \end{aligned} \tag{2}$$

where $a(u, \varphi) := \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x$, $l(\varphi) := \int_\Omega f \varphi \, \mathrm{d}x$, $u, \varphi \in H^1(\Omega)$.

3. Discretization of the problem

Let $\mathcal{T}_h$ be a partition covering $\overline{\Omega}$ consisting of finite number of closed triangles K with mutually disjoint interiors. The boundary of the element $K \in \mathcal{T}_h$ will be denoted by ∂K and its diameter by $h_K = \operatorname{diam} K$.

We assume that there exists $h_0 > 0$ such that $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is a system of triangulations is *shape-regular* and *locally quasi-uniform*. We do not require the conforming properties known from finite element methods. Therefore, the triangulations $\mathcal{T}_h$ could contain so called *hanging nodes*. Over the triangulation $\mathcal{T}_h$ we define the so-called *broken Sobolev space* over the triangulation $\mathcal{T}_h$ as $H^s(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega),\, v\big|_K \in H^s(K),\, \forall K \in \mathcal{T}_h\}$.

The DWR method can be combined with any discretization technique based on the variational formulation of the solved problem such as finite element method, finite volumes method or discontinuous Galerkin.

Here we focus only on the discontinuous Galerkin (DG) method. Since the DG method is very convenient for *hp*-adaptation, we assign to each $K \in \mathcal{T}_h$ its local polynomial degree p_K. Then we define vector $\mathtt{p} := \{p_K;\, K \in \mathcal{T}_h\}$ and as the finite dimensional discrete space we use

$$S_h^{\mathtt{p}} = \{v \in L^2(\Omega);\, v\big|_K \in P^{p_K}(K)\, \forall K \in \mathcal{T}_h\}. \tag{3}$$

We say that the function $u_h \in S_h^{\mathtt{p}}$ is the approximate solution of (2) if

$$a_h(u_h, \varphi_h) = l_h(\varphi_h) \quad \forall \varphi_h \in S_h^{\mathtt{p}}, \tag{4}$$

where $a_h(\cdot,\cdot)$ is a bilinear form resulting from the DG discretization of the problem (2) and l_h is a linear form representing the right-hand side of the equation enriched by some terms resulting from the DG method.

The Dirichlet boundary condition (1b) is not enforced directly, but it is integrated in the forms a_h and l_h by a penalty term. Detailed introduction of this method is not necessary for this paper, so we only stress out the important properties of the DG method when needed. Precise definitions of the forms and description of the properties of the method can be found in the monograph [3].

The crucial requirement on this method needed in this article is to be *consistent,* i.e. the exact solution u of problem (2) also satisfies

$$a_h(u, \varphi) = l_h(\varphi) \quad \forall \varphi \in H^2(\Omega, \mathcal{T}_h). \tag{5}$$

For the purpose of the higher-order reconstruction presented in Section 5 we also define the space $S_h^{\mathtt{p}+1} := \{v \in L^2(\Omega);\, v\big|_K \in P^{p_k+1}(K)\, \forall K \in \mathcal{T}_h\}$. Obviously $S_h^{\mathtt{p}} \subset S_h^{\mathtt{p}+1} \subset H^2(\Omega, \mathcal{T}_h)$.

4. Dual weighted residual method

Our goal is to estimate the error $J(u) - J(u_h)$, where $J : V \cup S_h^{\mathtt{p}} \to \mathbb{R}$ is a linear functional representing the so-called quantity of interest. Typically, this functional represents a regularized value of the solution (or its derivative) in a given point in Ω or an integral over a part of Ω or its boundary $\partial\Omega$.

The trick which enables to plug the functional J into the computation (similarly as the Aubin-Nietche trick used to prove the optimal rate of convergence in L^2-norm)

is the introduction of the so-called *dual* (or adjoint) *problem.* In addition to the original problem we introduce the dual problem:

$$\text{find } z \in V \text{ such that } a_h(\psi, z) = J(\psi) \quad \forall \psi \in H^2(\Omega, \mathcal{T}_h). \tag{6}$$

And also its discrete variant:

$$\text{find } z_h \in S_h^{\mathtt{p}} \text{ such that } a_h(\psi_h, z_h) = J(\psi_h) \quad \forall \psi_h \in S_h^{\mathtt{p}}. \tag{7}$$

Remark 1. *Problem* (6) *may look a bit artificial since it contains the bilinear form* $a_h(\cdot,\cdot)$ *from the discretized problem* (4)*. One could also consider directly the dual problem to the weak formulation* (2)*, i.e.* $a(\psi, z) = l(\psi)$, $\forall \psi \in V$*. If the discretization method is* dually consistent*, i.e. the discrete dual problem is a consistent discretization of the weak dual formulation, then these two definitions coincide. It has been shown in [6] that the dual consistency is essential in order to maintain the optimal convergence order of the method. Dual consistency is maintained only for the symmetric variant of DG method known as SIPG – symmetric interior penalty Galerkin, see [3] for details.*

The following manipulation gives us a relation between the error of the quantity of interest and the residual of the solved problem. Thanks to linearity of J, consistency of the DG scheme and the Galerkin orthogonality $a_h(u - u_h, \varphi_h) = a_h(\varphi_h, z - z_h) = 0 \, \forall \varphi_h \in S_h^{\mathtt{p}}$, we get

$$\begin{aligned} J(u) - J(u_h) &= J(u - u_h) = a_h(u - u_h, z) = a_h(u - u_h, z - \varphi_h) \\ &= l_h(z - \varphi_h) - a_h(u_h, z - \varphi_h) =: r_h(u_h)(z - \varphi_h) \quad \forall \varphi_h \in S_h^{\mathtt{p}} \end{aligned} \tag{8}$$

and very similarly

$$\begin{aligned} J(u) - J(u_h) &= a_h(u - u_h, z - z_h) = a_h(u - \psi_h, z - z_h) \\ &= J(u - \psi_h) - a_h(u - \psi_h, z_h) =: r_h^*(z_h)(u - \psi_h) \quad \forall \psi_h \in S_h^{\mathtt{p}}. \end{aligned} \tag{9}$$

Hence the residuals $r_h(u_h)(\cdot)$ and $r_h^*(z_h)(\cdot)$ are equivalent in the following way

$$r_h(u_h)(z - \varphi_h) = r_h^*(z_h)(u - \psi_h) \quad \forall \varphi_h, \psi_h \in S_h^{\mathtt{p}}. \tag{10}$$

Unfortunately, even the "exact" discrete solution satisfying (4) is not available in practical computations due to algebraic errors. Instead, we compute their approximation u_h^a and z_h^a typically resulting from a finite number of iterations of an iterative solver. In this case, Galerkin orthogonality property is violated and hence identities (8) and (9) need to be revised. Similarly to [1], by adding the algebraic error to (8), we get

$$\begin{aligned} J(u) - J(u_h^a) &= a_h(u - u_h^a, z - z_h^a) + a_h(u - u_h^a, z_h^a) \\ &= r_h(u_h^a)(z - z_h^a) + r_h(u_h^a)(z_h^a), \end{aligned} \tag{11}$$

and rewriting (9) gives

$$\begin{aligned} J(u) - J(u_h^a) &= a_h(u - u_h^a, z - z_h^a) + a_h(u - u_h^a, z_h^a) \\ &= a_h(u, z - z_h^a) - a_h(u_h^a, z - z_h^a) + a_h(u - u_h^a, z_h^a) \\ &= l_h(z) - a_h(u, z_h^a) - (J(u_h^a) - a_h(u_h^a, z_h^a)) + l_h(z_h^a) - a_h(u_h^a, z_h^a) \\ &= r_h^*(z_h^a)(u) - r_h^*(z_h^a)(u_h^a) + r_h(u_h^a)(z_h^a). \end{aligned} \tag{12}$$

Here, the expressions $r_h(u_h^a)(z - z_h^a)$ and $r_h^*(z_h^a)(u)$ represent the discretization error, while $r_h(u_h^a)(z_h^a)$ and $r_h^*(z_h^a)(u_h^a)$ represent the algebraic errors of the primal and dual problem, respectively. Unlike $r_h^*(z_h^a)(u_h^a)$ and $r_h(u_h^a)(z_h^a)$, expressions $r_h(u_h^a)(z - z_h^a)$ and $r_h^*(z_h^a)(u)$ are not computable and have to be further approximated.

5. Approximation of the exact solutions u and z

Except for a few very special examples (see e.g. [2, Chapter 3]) exact solution of the dual problem is not computable and has to be approximated. Since the residuals of the (algebraically exact) approximate solutions u_h and z_h equal to zero for all functions from $S_h^{\mathtt{p}}$, functions approximating u and z must be from a richer space than $S_h^{\mathtt{p}}$, otherwise the error estimates (8) and (9) would degenerate.

The standard approach is to compute the dual problem on a finer mesh and/or with higher polynomial degree. To avoid this costly procedure we exploit a higher order reconstruction of the discrete solutions u_h and z_h, which can be obtained locally and hence much faster. We use the weighted least-square reconstruction, firstly presented in [5].

Let $u_h \in S_h^{\mathtt{p}}$ be the approximate solution of (4). We compute the reconstruction $u_h^+ \in S_h^{\mathtt{p}+1}$ locally for each element $K \in \mathcal{T}_h$ by a weighted least square approximation from the elements sharing at least a vertex with K. We denote this patch of elements $\mathscr{D}_K = \{K' \in \mathcal{T}_h;\ K' \cap K \neq \emptyset\}$.

We compute the function $\boldsymbol{U}_K^+ \in P^{p_K+1}(\mathscr{D}_K)$ by

$$\boldsymbol{U}_K^+ = \underset{U_h \in P^{p_K+1}(\mathscr{D}_K)}{\arg\min} \sum_{K' \in \mathscr{D}_K} \omega_{K'} \|U_h - u_h\|^2_{H^1(K')}. \tag{13}$$

Then we assemble the higher-order reconstruction u_h^+ as an element-wise composition of $\boldsymbol{U}_K^+\big|_K$, i.e. $u_h^+ = \sum_{K \in \mathcal{T}_h} \boldsymbol{U}_K^+\big|_K$.

When choosing the values of the weights $\omega_{K'}$, we distinguish between elements sharing a face and elements having only a common vertex. We set $\omega_{K'} = 1$ if $K' = K$ or if K, K' share a face and $\omega_{K'} = \varepsilon$ if K, K' share only a vertex. The parameter $\varepsilon > 0$ is chosen to be considerably smaller than one.

The computation of z_h^+ is done alike, using function z_h.

6. Error estimates

Exploiting (11), (12) and the reconstructions defined in Section 5, we introduce two kinds of discretization error estimators

$$\eta_{\mathrm{S}} := r_h(u_h^a)(z_h^+ - z_h^a), \quad \eta_{\mathrm{S}}^* := r_h^*(z_h^a)(u_h^+). \tag{14}$$

Furthermore, we define algebraic error estimators

$$\eta_{\mathrm{A}} := r_h(u_h^a)(z_h^a), \qquad \eta_{\mathrm{A}}^* := r_h^*(z_h^a)(u_h^a), \tag{15}$$

which measure the influence of the algebraic errors arising from the inexact solution of the primal and the dual problem, respectively. Since we do not have in hands the true error, but only its approximation, we proceed with iterations of the Krylov solver until these algebraic estimators decrease significantly (10–1000 times) under the level of the discretization error.

Using the definitions of the error estimators (14) and (15) and the relations (11) and (12), we can write the error estimates

$$J(u) - J(u_h^a) \approx \eta_{\mathrm{S}} + \eta_{\mathrm{A}} \tag{16}$$

and

$$J(u) - J(u_h^a) \approx \eta_{\mathrm{S}}^* - \eta_{\mathrm{A}}^* + \eta_{\mathrm{A}}. \tag{17}$$

The functional J has not the properties of a norm and can attain both positive and negative values on different elements. Hence, we have to separate the estimate of the error, where we avoid overestimation, and the local error indicators that have to be positive at each element. Therefore, we define

$$\eta_{\mathrm{S,K}} = |r_h(u_h^a)((z_h^+ - z_h^a)\big|_K)|, \qquad \eta_{\mathrm{S,K}}^* = |r_h^*(z_h^a)(u_h^+\big|_K)|, \qquad K \in \mathcal{T}_h. \tag{18}$$

Either of those can be used as a local error indicator for mesh refinement. Although the primal and dual residuals are theoretically equivalent, see (10), localizations (18) can differ notably and hence may lead to differently refined meshes.

7. Numerical experiments

The problem we solve comes from [7]. We consider Poisson problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega = (0,1) \times (0,1) \\ u\big|_{\partial\Omega} &= 0, \end{aligned} \tag{19}$$

and we set the primal and dual right-hand sides

$$f(v) = -\int_{T_{\mathrm{f}}} \frac{\partial v}{\partial x_1}\,\mathrm{d}x, \qquad J(v) = -\int_{T_{\mathrm{g}}} \frac{\partial v}{\partial x_1}\,\mathrm{d}x, \tag{20}$$

where T_f and T_g are triangles with vertices $T_\mathrm{f} = \{(0,0),(0.5,0),(0,0.5)\}$ and $T_\mathrm{g} = \{(1,1),(0.5,1),(1,0.5)\}$, see Figure 1.

Both f and J are in $H^{-1}(\Omega)$ and the solutions have singularities along the lines connecting points $[0,0.5],[0.5,0]$ and $[0.5,1],[1,0.5]$ for the primal and dual problems, respectively. We discretized the problem by SIPG DG method with piece-wise quadratic polynomials. We compare the numerical results obtained by goal-oriented estimates with u_h^+, z_h^+ computed with the least-squares reconstruction (DWR) and with globally increased polynomial degree (DWR_P). The third approximation was computed with a classical (not goal-oriented) error estimation technique (RES). This method, first proposed in [4], estimates dual norm of the residual of the discrete solution.

Meshes after 25 steps of mesh adaptation are showed in Figure 1. In each adaptation step we refined 10% of elements with the largest local error indicators. In Figure 2, we compare the decrease of the error $J(u) - J(u_h^a)$ for all three algorithms on adaptively refined meshes. The adaptive RES technique does not take into account the singularity of the dual problem. For this reason it does not refine the mesh in the upper-right corner and it cannot decrease the error of the target quantity bellow the level 10^{-6}. The goal-oriented algorithm reduces the error more steadily. The computation with the least-squares reconstruction behaves comparably to the more expensive algorithm DWR_P. Our goal-oriented algorithm almost achieves the optimal theoretical rate of convergence $O((\#\mathcal{T}_h)^{-2})$ proved in [7].

In the second experiment, Figure 3, we compare the decrease of the true error of the quantity of interest $J(u) - J(u_h^a)$ with estimates $\eta_\mathrm{S}, \eta_\mathrm{S}^*$ of the discretization error and estimates $\eta_\mathrm{A}, \eta_\mathrm{A}^*$ of the algebraic errors given by (14)–(15). These results were obtained by the SIPG method with quadratic polynomials on fixed uniform mesh with 256 elements. At each step (outer iterations, which are marked on the horizontal axis in Figure 3) we simultaneously performed 8 iterations of the algebraic solver (GMRES with ILU preconditioning) for primal and 50 iterations for the dual problem, respectively. In other words, at the outer step i, the situation after $i \times 8$ and $i \times 50$ steps of the algebraic solver for the primal and the dual problem, respectively, is plotted.

Figure 3 nicely illustrates the relations (12) and (17). Since we perform more iterations of the algebraic solver in each (outer) step, thus both $|z_h - z_h^a|$ and the

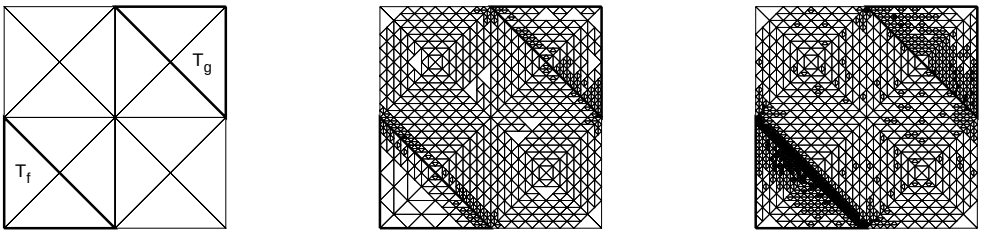

Figure 1: The initial mesh (left) and final meshes produced by the DWR method (center) and DWR_P method (right), respectively.

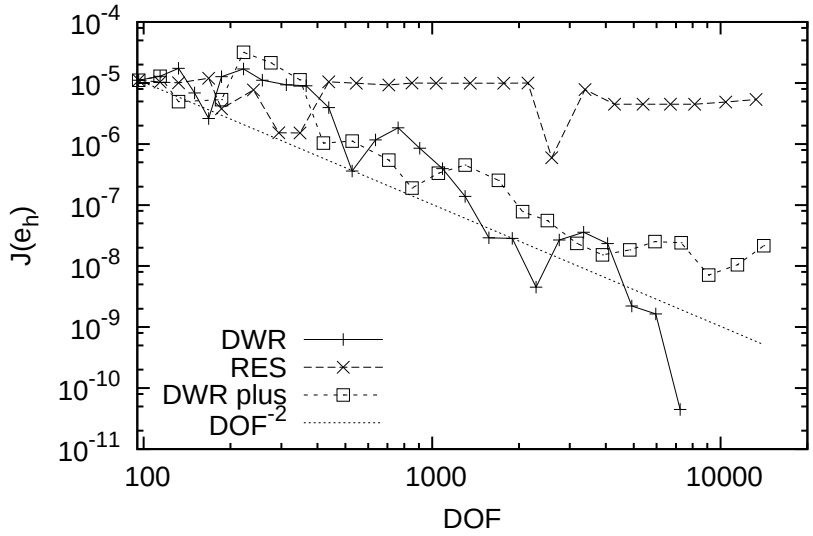

Figure 2: Error convergence of the RES, DWR and DWR_P methods compared to the theoretical rate.

algebraic error of the dual problem ($\approx \eta_{\mathrm{A}}^*$) is negligible already after a few steps. On the other hand, the algebraic error of the primal problem ($\approx \eta_{\mathrm{A}}$) decreases slower. Therefore, in the first 10 steps the error of the target quantity $J(u) - J(u_h^a)$ is mainly caused by the algebraic errors in the solution of the primal problem, hence its decrease corresponds to the decrease of η_{A}. Only when the estimate η_{A} decreases under the level of the discretization error, $J(u) - J(u_h^a)$ stops at the level of $J(u) - J(u_h) \approx \eta_{\mathrm{S}}^*$.

For example in the step 6 (i.e, after 48 and 300 iterations of the algebraic solver for the primal and dual problem, respectively) both the error $J(u) - J(u_h^a)$ and the estimate η_{S} are still strongly influenced by the algebraic errors of u_h^a. On the the contrary, the dual estimate η_{S}^* is already at the level of the exact discretization error $J(u) - J(u_h)$. In other words, the algebraical inexactness in the discrete solution u_h^a influences the primal estimate η_{S} more seriously than the dual estimate η_{S}^*.

Similar (but reversed) behavior was observed in the opposite case, when $u_h^a \approx u_h$ but z_h^a is far from z_h. This indicates that even quite inexact approximation z_h^a of z_h could be sufficient for the primal estimate η_{S}. On the other hand, if we knew that for some reason the dual algebraic problem was easier to solve, we should use the dual estimate η_{S}^* which can give better results for rough approximations of u_h. Finally, we note that this is possible only thanks to the equivalence (10) between the primal and the dual residual, which also implies that $J(u_h) = a_h(u_h, z_h) = l_h(z_h)$, hence we are able to obtain an approximation of $J(u)$ even without computing u_h at all.

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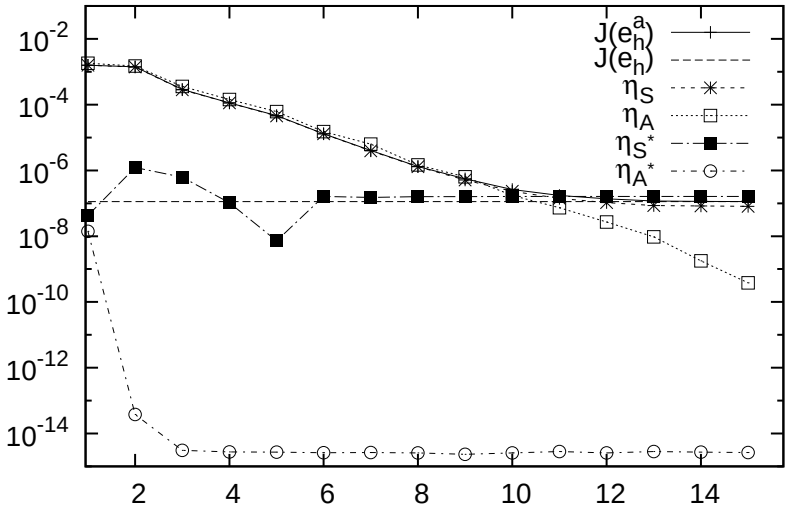

Figure 3: Decrease of error estimates during iterations of the algebraic solver.

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