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Persistent URL: http://dml.cz/dmlcz/702931

SOLUTIONS OF HYPERSINGULAR INTEGRAL EQUATIONS OVER CIRCULAR DOMAINS BY A SPECTRAL METHOD

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Abstract

The problem of a solving a class of hypersingular integral equations over the boundary of a nonplanar disc is considered. The solution is obtained by an expansion in basis functions that are orthogonal over the unit disc. A Fourier series in the azimuthal angle, with the Fourier coefficients expanded in terms of Gegenbauer polynomials is employed. These integral equations appear in the study of the interaction of water waves with submerged thin plates.

1. Introduction

The aim of the present study is to consider the semi-analytical solution of a class of two-dimensional hypersingular integral equations. These equations can arise in the study of the interaction of water waves with submerged plates and the method of solution can be classified in the general area of spectral methods.

When the physical problem is two-dimensional and thus, the hypersingular integral equations is onedimensional, an efficient method can be applied for solution based on expansions in terms of Chebyshev polynomials. These problems can be related to scattering by flat [17] and curved [18] submerged plates, and by surface-piercing plates [18], and the trapping of water waves by submerged plates [19]. They used an expansion-collocation method to solve the one-dimensional hypersingular integral equations, in which the unknown is expanded using Chebyshev polynomials of the second kind. This method is very effective, and its convergence has been proved by Golberg [5, 6] and by Ervin & Stephan [1], in various function spaces. Ervin & Stephan [1] obtained the rate of convergence in appropriate Sobolev spaces. See also Frenkel [4] and Kaya & Erdogan [8].

The three-dimensional scattering by a thin disc, in deep water was investigated by Farina and Martin [2] and by Ziebell and Farina [20]. The authors solved the governing twodimensional hypersingular integral equation numerically using a spectral method using as basis functions, Gegenbauer polynomials in the radial variable. Physcally, when the plate is very close to the free surface, resonant frequencies can occur and this phenomenon has been inverstigated by Farina [3].

In this work, we ilustrate the spectral method by choosing the of problem a submerged disc is perturbed out of it original plane, so the disc could be denominated wrinkled or rough. This type of problem has been solved approximately, for circular caps and rough discs by Ziebell and Farina [20].

A similar problem in acoustics has been studied by Jansson [7], where the scattering of an acoustic wave from a thin circular disc was investigated by an integral equation method where the disc is modelled as part of an infinite interface between two half-spaces; this interface is then perturbed. However, this approach causes the behaviour of the solution near the edge of the disc to produce singularities at the edge of the disc.

Before presenting the integral equation that we will focus on, let us present in the next section, the physical and differential problem that originates it.

2. Formulation

A Cartesian coordinate system is chosen, in which z is directed vertically downwards into the fluid. We take the mean free surface lying at $z = 0$. We assume the presence of a submerged body into the fluid with a smooth, closed and bounded surface S. We suppose that the motions of the fluid are of small-amplitude, time-harmonic, that the fluid is incompressible and inviscid, and that the motion is irrotational. We denote ϕ as the potential flow and $[\phi]$ as the discontinuity in ϕ across S. Thus, the time dependent velocity potential is $\mathrm{Re}\{\phi(x, z, t)\}\mathrm{e}^{-\mathrm{i}\omega t}$, where ω is the angular frequency.

The conditions to be satisfied by ϕ are Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \quad \text{in the fluid}$$

along with the free-surface condition

$$K\phi + \frac{\partial\phi}{\partial z} = 0 \ \text{ on } \ z = 0,$$

where $K = \omega^2/g$; g being the acceleration due to gravity.

On the surface of the body, the normal velocity is prescribed by

$$\frac{\partial\phi}{\partial n} = V \ \text{ in } \ S, \tag{1}$$

where V is a given function and $\frac{\partial}{\partial n}$ denotes normal differentiation.

Additionally, ϕ must satisfy a radiation condition:

$$r^{1/2}\left(\frac{\partial\phi}{\partial r} - \mathrm{i}K\phi\right) \to 0 \quad \text{when} \quad r = (x^2 + y^2)^{1/2} \to \infty.$$

The points P, Q denote points in the fluid and the points p, q denote points on the submerged body.

The free surface Green function for this problem is given by

$$G(P,Q) \equiv G(\xi, \eta, \zeta; x, y, z) = G_0(R, z - \zeta) + G_1(R, z + \zeta), \tag{2}$$

where $R = ((x - \xi)^2 + (y - \eta)^2)^{1/2}$, $G_0(R, z - \zeta) = (R^2 + (z - \zeta)^2)^{-1/2}$ and

$$G_1(R, z + \zeta) = \oint_0^\infty \mathrm{e}^{-k(z+\zeta)} J_0(kR) \frac{k + K}{k - K}\, dk. \tag{3}$$

Here J_0 is the Bessel function of order zero. The path integral defining G_1 above runs below the singularity K. G satisfies the free surface condition, the Laplace equation, and have a weak singularity at $P = Q$.

For any harmonic function ϕ, satisfying $\phi = O(r^{-1})$ as $r \to \infty$, we have from Green's second identity, the following integral representation.

$$\phi(P) = \frac{1}{4\pi} \int_S \left(\phi(q)\frac{\partial}{\partial n_q}G(P, q) - G(P, q)\frac{\partial\phi}{\partial n_q}\right) dS_q, \tag{4}$$

where $\frac{\partial}{\partial n_q}$ denotes normal differentiation at q on S.

Now, for a thin body with surface Ω, denote the two sides of Ω by Ω^+ and Ω_- and define the discontinuity in ϕ across Ω by

$$[\phi] = \lim_{Q \to q^+} \phi(Q) - \lim_{Q \to q^-} \phi(Q),$$

where $q \in \Omega$, $q^- \in \Omega^-$, $q^+ \in \Omega^+$ and Q is a point in the fluid. Thus, equation (4) reduces to

$$\phi(P) = \frac{1}{4\pi} \int_\Omega [\phi(q)]\frac{\partial}{\partial n_q}G(P, q)\, dS, \tag{5}$$

where $n_q = n_q^+$ denotes now the normal unit vector at q on Ω^+. Applying boundary condition (1) to (5) gives

$$\frac{1}{4\pi} \oint_\Omega [\phi(q)]\frac{\partial^2}{\partial n_q \partial n_q}\, G(p, q)\, dS_q = V(p), \quad p \in \Omega, \tag{6}$$

where the integral must be interpreted in the Hadamard finite-part sense. Equation (6) is the governing hypersingular integral equation for $[\phi]$; this is to be solved subject to the edge condition

$$[\phi] = 0 \quad \text{in} \quad \partial\Omega.$$

Now let
$$\Omega : z = F(x, y) + \frac{b}{2}, \ (x, y) \in D,$$

where D is the unit disc in the xy-plane and $\frac{b}{2}$ is the depth to which the body is submerged. Let $p, q \in \Omega$ such that $p = (\xi, \eta, \zeta)$, $q = (x, y, z)$. The normal vector to Ω is then given by
$$\mathbf{N} = \left(-\frac{\partial F}{\partial x}, -\frac{\partial F}{\partial y}, 1 \right)$$

and a unit normal vector is therefore, expressed by $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. Using the notation
$$w(x, y) = [\phi(q)], \tag{7}$$

it can be shown by a direct calculation that formula (5) becomes
$$\phi(\xi, \eta, \zeta) = \frac{1}{4\pi} \int_D w(x, y) \frac{\mathbf{N} \cdot \mathbf{R_F}}{R_F^3} \, dS + \frac{1}{4\pi} \int_D w(x, y)(\nabla G_1 \cdot \mathbf{N}) \, dS, \tag{8}$$

where $\mathbf{R_F} = (\xi - x, y - \eta, \zeta - F(x, y))$, $R_F = |\mathbf{R_F}|$ and $dS = dx\, dy$.

Our goal now is to clarify and understand the governing equation (6). In order to do this, consider the following definitions and notations.
$$F_1 = \frac{\partial F}{\partial x}, \ F_2 = \frac{\partial F}{\partial y} \tag{9}$$

with F_1^0 and F_2^0 being the corresponding functions at (ξ, η). Let also $\Lambda = \frac{F(x,y)-F(\xi,\eta)}{R}$ and $\bar{\Lambda} = \frac{F(x,y)+F(\xi,\eta)}{R}$ and define the angle Θ by $x - \xi = R\cos\Theta$ and $y - \eta = R\sin\Theta$.

Projecting onto D, we can rewrite (6) as
$$\frac{1}{4\pi} \oint_D H \, w(q) dA + \frac{1}{4\pi} \int_D W \, w(q) dA = V(p), \quad p \in D, \tag{10}$$

where (see [12])
$$H(\xi, \eta; x, y) = \frac{1}{R^3} \left(\frac{1 + F_1 F_1^0 + F_2 F_2^0}{(1 + \Lambda^2)^{\frac{3}{2}}} \right.$$
$$\left. - 3 \frac{(F_1\cos\Theta + F_2\sin\Theta - 1)(F_1^0\cos\Theta + F_2^0\sin\Theta - 1)}{(1 + \Lambda^2)^{\frac{5}{2}}} \right) \tag{11}$$

and
$$W = \left. \frac{\partial^2 G_1}{\partial n_q \partial n_p} \right|_D = \oint_0^\infty e^{-k\bar{\Lambda}R} e^{-kb} \mathcal{K} \frac{k + K}{k - K} \, dk, \tag{12}$$

55

where

$$
\begin{aligned}
\mathcal{K} \;=\; & F_1 F_1^0 \frac{k}{2R}(2\sin^2\Theta J_1(kR) + kR\cos^2\Theta(J_0(kR) - J_2(kR))) \\
& + F_2 F_2^0 \frac{k}{2R}(2\cos^2\Theta J_1(kR) + kR\sin^2\Theta(J_0(kR) - J_2(kR))) \\
& + (F_2 F_1^0 + F_1 F_2^0)\frac{k}{2R}\cos\Theta\sin\Theta\,(kR(J_0(kR) - J_2(kR)) - 2J_1(kR)) \\
& + (F_1^0 - F_1)k^2\cos\Theta J_1(kR) \\
& + (F_2^0 - F_2)k^2\sin\Theta J_1(kR) \\
& + k^2 J_0(kR).
\end{aligned}
\tag{13}
$$

Equation (10) is the governing equation for the problem of any submerged non planar circular disc Ω in water of infinite depth. Its solution gives the jump in the velocity potential ϕ across Ω. With this information, one can evaluate ϕ at any point P in the fluid by using (8). Equation (10) could be solved numerically, although not by the semi-analytical expansion-collocation method proposed by Farina e Martin [2] for the solution of hypersingular integral equations on a disc. Alternatively, an approximation to the solution could be obtained by a boundary perturbation method. We present such a method next. This method follows the one proposed by Martin [12] for treating the problem of a wrinkled disc in an unbounded fluid.

3. Perturbation method

We now assume that

$$
V(p) = n_3, \quad n_3 = \frac{1}{\sqrt{F_1^2 + F_2^2 + 1}},
\tag{14}
$$

where n_3 is the vertical component of the unit normal vector to the disc. This simplifies the following analysis and corresponds physically a situation where the disc performs heave (vertical) oscillations. Thus the problem stated in Section 2 becomes a radiation problem.

In order to consider a perturbation of the flat disc, we introduce the function f such that

$$
F(x,y) = \epsilon f(x,y),
\tag{15}
$$

where ϵ is a small parameter and f is independent of ϵ. In [12] it is shown that

$$
H = \frac{1}{R^3}\{1 + \epsilon^2 K_2 + O(\epsilon^4)\},
$$

where

$$
K_2 = f_1 f_1^0 + f_2 f_2^0 - \frac{3}{2}\lambda^2 - 3(f_1\cos\Theta + f_2\sin\Theta - \lambda)(f_1^0\cos\Theta + f_2^0\sin\Theta),
$$

$\lambda = (f(x, y) - f(x\,\mathrm{i}, \eta))/R$ and f_j, f_j^0 are defined similarly to F_j, F_j^0; see the comments after (9).

In order to get a similar expression for W, substitute (15) in (12), giving

$$W = W_0 + \epsilon W_1 + \epsilon^2 W_2, \tag{16}$$

where

$$W_0 = \fint_0^\infty \mathrm{e}^{-k(\epsilon(f(x,y)+f(\xi,\eta))+b)} k^2 J_0(kR) \frac{k+K}{k-K}\, dk, \tag{17}$$

$$
\begin{aligned}
W_1 &= [(f_1^0 - f_1)\cos\Theta + (f_2^0 - f_2)\sin\Theta] \\
&\quad \times \fint_0^\infty \frac{k+K}{k-K} \mathrm{e}^{-k(\epsilon(f(x,y)+f(\xi,\eta))+b)} k^2 J_1(kR)\, dk, \tag{18}
\end{aligned}
$$

and

$$
\begin{aligned}
W_2 &= \left[\frac{\sin^2\Theta}{R} f_1 f_1^0 + \frac{\cos^2\Theta}{R} f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2R}\right] \\
&\quad \times \fint_0^\infty \mathrm{e}^{-k(\epsilon(f(x,y)+f(\xi,\eta))+b)} k J_1(kR) \frac{k+K}{k-K}\, dk \\
&\quad + \left[\cos^2\Theta f_1 f_1^0 + \sin^2\Theta f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2}\right] \\
&\quad \times \frac{1}{2}\fint_0^\infty \mathrm{e}^{-k(\epsilon(f(x,y)+f(\xi,\eta))+b)} k^2 (J_0(kR) - J_2(kR)) \frac{k+K}{k-K}\, dk. \tag{19}
\end{aligned}
$$

Expanding $\mathrm{e}^{-k(\epsilon(f(x,y)+f(\xi,\eta))+b)}$ in Taylor's series, we obtain

$$
\begin{aligned}
W_0 &= W_{00} + \epsilon W_{01} + \epsilon^2 W_{02}, \\
W_1 &= W_{10} + \epsilon W_{11} + \epsilon^2 W_{12}, \\
W_2 &= W_{20} + \epsilon W_{21} + \epsilon^2 W_{22},
\end{aligned}
$$

where

$$W_{00} = \fint_0^\infty \frac{k+K}{k-K} \mathrm{e}^{-kb} k^2 J_0(kR)\, dk \tag{20}$$

and the functions $W_{01}, \ldots, W_{22}$ are given in appendix A.

Substituting (15) in (14) and expanding in Taylor series, we get

$$n_3 = 1 + \frac{1}{2}\left(f_1^2 + f_2^2\right)\epsilon^2 + O(\epsilon^3). \tag{21}$$

Similarly, for w, assume

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \dots. \tag{22}$$

Now, substituting (16) and (22) in (10), with V given by (21), we obtain

$$
\frac{1}{4\pi} \oint_D w_0 \frac{dA}{R^3} + \frac{1}{4\pi} \int_D W_{00}\, w_0\, dA = 1, \tag{23}
$$

$$
\frac{1}{4\pi} \oint_D w_1 \frac{dA}{R^3} + \frac{1}{4\pi} \int_D W_{00}\, w_1\, dA = -\frac{1}{4\pi} \int_D (W_{10} + W_{01})w_0\, dA, \tag{24}
$$

$$
\frac{1}{4\pi} \oint_D w_2 \frac{dA}{R^3} + \frac{1}{4\pi} \int_D W_{00}\, w_2\, dA = -\frac{1}{4\pi} \oint_D K_2 w_0 \frac{dA}{R^3}
$$

$$
-\frac{1}{4\pi} \int_D (W_{02} + W_{11} + W_{20})w_0\, dA
$$

$$
-\frac{1}{4\pi} \int_D (W_{01} + W_{10})w_1\, dA
$$

$$
+\frac{1}{2} \left(f_1^2 + f_2^2 \right). \tag{25}
$$

Note that equation (23) appears in [11, eq. 4.1] and in [2, eq. 17]. Thus, the first order equation of the present perturbation method recovers the governing equation for the *plane* disc: this corresponds to the problem of a horizontal and plane circular disc performing heave oscillations.

By defining the integral operators

$$
H_{ij}w = \int_D W_{ij}w\, dA \quad \forall\, i,j \in \{0,1,2\},
$$

$$
\mathcal{H}w = \oint_D w\, \frac{dA}{R^3},
$$

$$
\mathcal{K}_2 w = \oint_D K_2 w\, \frac{dA}{R^3},
$$

we can write equations (23)-(25) in a more compact form as

$$
(\mathcal{H} + H_{00})\, w_0 = 1, \tag{26}
$$

$$
(\mathcal{H} + H_{00})\, w_1 = -(H_{10} + H_{01})w_0, \tag{27}
$$

$$
(\mathcal{H} + H_{00})\, w_2 = -(\mathcal{K}_2 + H_{02} + H_{11} + H_{20})w_0 - (H_{01} + H_{10})w_1 + \frac{1}{2}(f_1^2 + f_2^2) \tag{28}
$$

Equations (26)–(28) form a sequence of integral equations that approach the solution of the governing equation (10). Note that the simple structure of these equations offers an alternative to the solution of the problem: in order to solve it, one has just to invert the integral operator $H_{00} + \mathcal{H}$. Note further that the function f is only present in the right-hand side of the equations. This means that all the information about the specific geometry of the plate is in these terms of the equations. Thus, it is possible to *pre-solve* the problem for any perturbation of the disc by inverting the operator mentioned above. This can be done efficiently by the numerical method presented in Section 4.

4. Alternative expressions and numerical method

In this section we show how to compute a solution of the problem formulated in the section above.

4.1. Alternative expressions for W

The integrands of the integral equations (26)–(28) involve the regular part of the free surface Green function, that is, G_1, and its derivatives. The numerical implementation of these functions are not trivial. Specifically, these integrands present path integrals that involve Bessel functions. Nevertheless, we can express these integrals in terms of Bessel functions and Struve functions which are suitable for more efficient numerical calculation. According to [13] (see also [15, 16]), we have

$$
\begin{aligned}
G_1 &= \fint_0^\infty \frac{k+K}{k-K} e^{-k(z+\zeta)} J_0(kR)\, dk \\
&= K\left[(X^2+Y^2)^{-1/2} - \pi e^{-Y}(H_0(X)+Y_0(X)) - 2\int_0^Y e^{t-Y}(X^2+t^2)^{-1/2} dt \right] \\
&\quad - 2\pi i K e^{-Y} J_0(X),
\end{aligned} \tag{29}
$$

where $X=KR$, $Y=K(z+\zeta)$, H_0 is the Struve function of order 0 and J_0 and Y_0 denote the Bessel functions of the first and second kind, respectively. Expression (29) is suitable for numerical calculation; this has been used is several computer codes for water wave analysis. See for instance [10].

Using (29), it can be shown that the integrands $W_{00}, \ldots, W_{22}$, originally written as (39–45) in appendix A, admit similar representations. For example,

$$
\begin{aligned}
W_{00} &= 2K^2(R^2+b^2)^{-1/2} + (2Kb-1)(R^2+b^2)^{-3/2} + 3b^2(R^2+b^2)^{-5/2} \\
&\quad - \pi K^3 e^{-Kb}(H_0(KR)+Y_0(KR)) - 2K^3 e^{-Kb}\int_0^{Kb} e^t((KR)^2+t^2)^{-1/2}\, dt \\
&\quad + 2\pi i K^3 e^{-Kb} J_0(KR)
\end{aligned} \tag{30}
$$

is an alternative expression for W_{00}, which allows more efficient numerical computation than (39) does. The expression (30) does not involve path integrals whose calculation are computationally expensive. Furthermore, the Struve and Bessel functions present in this alternative term are efficiently computed by approximating orthogonal polynomials; see [14]. Integrals such as the one in (30) can be efficiently computed; see [13] and [15]. Similar alternative expressions for the W_{01}, W_{02}, W_{10} and W_{20} are shown in appendix B.

4.2. Expansion-collocation method
4.2.1. Review of the one-dimensional theory

In two-dimensions, many wave problems involving thin plates can be reduced to an equation of the form

$$
\fint_{-1}^1 \left\{ \frac{1}{(x-t)^2} + H(x,t) \right\} v(t)\, dt = f(x) \quad \text{for } -1 < x < 1, \tag{31}
$$

supplemented by two boundary conditions, which we take to be $v(-1) = v(1) = 0$. Here, v is the unknown function, f is prescribed and the kernel H is known. Assuming that f is sufficiently smooth, the solution v has square-root zeros at the end-points. This suggests that we write

$$v(x) = \sqrt{1 - x^2}\, u(x).$$

Then, we expand u using a set of orthogonal polynomials; a good choice is to use Chebyshev polynomials of the second kind, U_n, defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n = 0, 1, 2, \ldots.$$

This is a good choice because of the formula

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, U_n(t)}{(x-t)^2}\, dt = -(n+1)U_n(x). \tag{32}$$

Thus, we approximate u by

$$\sum_{n=0}^{N} a_n U_n(x),$$

substitute into (31) and evaluate the hypersingular integral analytically, using (32). To find the $(N+1)$ coefficients a_n, we collocate at $(N+1)$ points; good choices are the zeros of T_{N+1} or U_{N+1}, where T_n is a Chebyshev polynomial of the first kind.

4.2.2. The two-dimensional theory

The governing equations (26)–(28), obtained by the perturbation method in section 3, can be written in the same form, which is

$$(\mathcal{H} + H_{00})u = g, \tag{33}$$

where g is a known function, which can involve solutions of lower order problems. As a particular case, the plane disc equation (26) has an axisymmetric solution and can be solved by reducing it to a non singular one dimensional Fredholm integral equation of the second kind [11, eq. 7.6]. A simple numerical method can be used for this equation; for instance a Nyström method combined with the Gauss-Legendre quadrature rule, as employed by Martin and Farina [11]. However, as the solutions of equations (27) and (28) are not axisymmetric, we need a more general method of solution. We employ the expansion-collocation method used by Farina and Martin [2] for solving an equation of the form of (33). In fact, this method does not require that $V = 1$. This forcing could be any function of two variables; for instance, this could represent an incident wave and in this way, the problem would be a scattering one. In order to describe the expansion-collocation method, introduce cylindrical polar coordinates (r, θ, z), so that $x = r\cos\theta$ and $y = r\sin\theta$. Then, the disc is given by

$$D = \{(r, \theta, z) : 0 \le r < 1, -\pi \le \theta < \pi, z = b/2\}. \tag{34}$$

If we write $\xi = s\cos\alpha$, $\eta = s\sin\alpha$, we have

$$R^3 = [r^2 + s^2 - 2rs\cos(\theta - \alpha)]^{3/2}.$$

Hence we can write (33) as

$$\frac{1}{4\pi} \oint_D u(s,\alpha)\left\{\frac{1}{R^3} + W_{00}(r,\theta;s,\alpha;b,K)\right\} s\,ds\,d\alpha = g(r,\theta), \quad (r,\theta) \in D, \qquad (35)$$

We shall expand u using the basis functions B_k^m, defined by

$$B_k^m(r,\theta) = P_{m+2k+1}^m(\sqrt{1-r^2})\,\mathrm{e}^{\mathrm{i}m\theta}, \qquad k, m = 0, 1, \ldots,$$

where P_n^m is an associated Legendre function. The radial part of these basis functions can also be expressed in terms of Gegenbauer polynomials.

The functions $\{B_k^m\}$ are orthogonal over the unit disc with respect to the weight $(1-r^2)^{-1/2}$.

The next formula, due to Krenk [9] is essential in the construction of the method:

$$\frac{1}{4\pi} \oint_S \frac{1}{R^3} B_k^m(s,\alpha)\,s\,ds\,d\alpha = C_k^m \frac{B_k^m(r,\theta)}{\sqrt{1-r^2}}, \qquad (36)$$

where

$$C_k^m = -\frac{\pi}{4}\frac{(2k+1)!}{(2m+2k+1)!}[P_{m+2k+1}^{m+1}(0)]^2$$

Equation (36) allows us to evaluate the hypersingular integrals analytically[1]. To exploit (36), we expand $[\phi]$ in terms of the functions B_k^m. For brevity, we write

$$[\phi] = w \approx \sum_{k,m}^{N} a_k^m B_k^m := \sum_{k=0}^{N_1}\sum_{m=0}^{N_2} a_k^m B_k^m. \qquad (37)$$

Substituting (37) in the integral equation (35) and then evaluating the hypersingular integrals analytically using (36), we obtain

$$\sum_{k,m}^{N} a_k^m \left\{C_k^m \frac{B_k^m(r,\theta)}{\sqrt{1-r^2}} + \frac{1}{4\pi}\int_S B_k^m(s,\alpha)W_{00}(r,\theta;s,\alpha;d,K)\,s\,ds\,d\alpha\right\} = g(r,\theta),$$

$$(r,\theta) \in D. \qquad (38)$$

---

[1]Another consequence of formula (36) is that the functions $B_k^m(r,\theta)/\sqrt{1-r^2}$ can bee seen as eigenfunctions of the integral operator $\bar{\mathcal{H}}$ defined by

$$\bar{\mathcal{H}}v(r,\theta) = \oint_D \frac{1}{R^3}v(s,\alpha)\,\sqrt{1-s^2}\,s\,ds\,d\alpha.$$

It remains to determine the unknown coefficients a_k^m. We use a collocation method, in which evaluation of (38) at $(N_1+1)(N_2+1)$ points on the disc gives a linear system for the coefficients a_k^m. For a discussion on the choice of the collocation points on a disc and other numerical issues on the collocation-expansion method, including its analogue for two-dimensional water wave problems, see [2]. Numerical results showing the effectiveness of the method were presented by Ziebell and Farina [20] for spherical caps and rough discs.

5. Discussion

We have presented a spectral method for solving a class of hypersingular equations over a nonplanar circular disc. The motivation of the problem comes from a interaction of water waves with a submerged thin non-planar surface. By using a boundary perturbation method, we formulate the problem in terms of sequence of hypersingular integral equations, $(\mathcal{H} + H_{00})w_n = g_n$, over a flat disc. This approach allows the application of a efficient semi-analytical method where the solution is expanded in terms of Gegenbauer polynomials. This is the analogue of a spectral method used for the solutions of one-dimensional hypersingular integral equations in terms of Chebyshev polynomials.

Acknowledgements

Appendices

A. Expansion terms for W

$$W_{00} = \fint_0^\infty \frac{k+K}{k-K}\mathrm{e}^{-kb}k^2 J_0(kR)\,dk, \tag{39}$$

$$W_{01} = -(f(x,y) + f(\xi,\eta))\fint_0^\infty \frac{k+K}{k-K}\mathrm{e}^{-kb}k^3 J_0(kR)\,dk, \tag{40}$$

$$W_{02} = \frac{1}{2}(f(x,y) + f(\xi,\eta))^2\fint_0^\infty \frac{k+K}{k-K}\mathrm{e}^{-kb}k^4 J_0(kR)\,dk, \tag{41}$$

$$W_{10} = -[(f_1 - f_1^0)\cos\Theta + (f_2 - f_2^0)\sin\Theta]\fint_0^\infty \frac{k+K}{k-K}k^2\mathrm{e}^{-kb} J_1(kR)\,dk, \tag{42}$$

$$W_{11} = [(f_1 - f_1^0)\cos\Theta + (f_2 - f_2^0)\sin\Theta](f(x,y) + f(\xi,\eta))\fint_0^\infty \frac{k+K}{k-K}k^3\mathrm{e}^{-kb} J_1(kR)\,dk, \tag{43}$$

$$W_{12} = -\frac{1}{2}[(f_1 - f_1^0)\cos\Theta + (f_2 - f_2^0)\sin\Theta](f(x,y) + f(\xi,\eta))^2 \fint_0^\infty \frac{k+K}{k-K} k^4 \mathrm{e}^{-kb} J_1(kR)\, dk,$$

$$(44)$$

$$\begin{aligned}
W_{20} =& \left[\frac{\sin^2\Theta}{R} f_1 f_1^0 + \frac{\cos^2\Theta}{R} f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2R}\right] \fint_0^\infty \mathrm{e}^{-kb} k J_1(kR) \frac{k+K}{k-K}\, dk \\
&+ \left[\cos^2\Theta f_1 f_1^0 \sin^2\Theta f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2}\right] \\
&\times \frac{1}{2}\fint_0^\infty \mathrm{e}^{-kb} k^2 (J_0(kR) - J_2(kR)) \frac{k+K}{k-K}\, dk,
\end{aligned}$$

$$(45)$$

$$\begin{aligned}
W_{21} =& (f(x,y) + f(\xi,\eta)) \left\{ \left[-\frac{\sin^2\Theta}{R} f_1 f_1^0 - \frac{\cos^2\Theta}{R} f_2 f_2^0 + (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2R}\right] \right. \\
&\times \fint_0^\infty \mathrm{e}^{-kb} k^2 J_1(kR) \frac{k+K}{k-K}\, dk \\
&+ \left[-\cos^2\Theta f_1 f_1^0 - \sin^2\Theta f_2 f_2^0 + (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2}\right] \\
&\left. \times \frac{1}{2}\fint_0^\infty \mathrm{e}^{-kb} k^3 (J_0(kR) - J_2(kR))\frac{k+K}{k-K}\, dk \right\},
\end{aligned}$$

$$(46)$$

$$\begin{aligned}
W_{22} =& \frac{1}{2}(f(x,y) + f(\xi,\eta))^2 \left\{ \left[\frac{\sin^2\Theta}{R} f_1 f_1^0 + \frac{\cos^2\Theta}{R} f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2R}\right] \right. \\
&\times \fint_0^\infty \mathrm{e}^{-kb} k^3 J_1(kR) \frac{k+K}{k-K}\, dk \\
&+ \left[\cos^2\Theta f_1 f_1^0 \sin^2\Theta f_2 f_2^0 - (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2}\right] \\
&\left. \times \frac{1}{2}\fint_0^\infty \mathrm{e}^{-kb} k^4 (J_0(kR) - J_2(kR))\frac{k+K}{k-K}\, dk \right\}.
\end{aligned}$$

$$(47)$$

B. Alternatives expressions

$$\begin{aligned}
W_{01} =& (f(x,y) + f(\xi,\eta))\Big[-2bK^3(R^2 + b^2)^{-1/2} \\
&+ (2K - 2K^2 b)(R^2 + b^2)^{-3/2} + (9b - 6Kb^2)(R^2 + b^2)^{-5/2} \\
&- 15b^3(R^2 + b^2)^{-7/2} + \pi K^4 \mathrm{e}^{-Kb}(H_0(KR) + Y_0(KR)) \\
&+ 2\, K^3 \mathrm{e}^{-Kb}\int_0^{Kb} \mathrm{e}^t((KR)^2 + t^2)^{-1/2}\, dt + 2\pi\mathrm{i}K^4\mathrm{e}^{-Kb} J_0(KR)\Big],
\end{aligned}$$

$$(48)$$

$$W_{02} = \frac{1}{2}(f(x,y) + f(\xi,\eta))^2 \Big[-2K^2(R^2+b^2)^{-3/2} + (6K^2b^2 + 3Kb^2 - 18b + 5)$$
$$\times (R^2+b^2)^{-5/2} + (-75K^2b^4 + 30Kb^3 - 15b^2)(R^2+b^2)^{-7/2}$$
$$+ 105K^2b^6(R^2+b^2)^{-9/2} - \pi K^5 e^{-Kb}(H_0(KR) + Y_0(KR))$$
$$- 2K^3 e^{-Kb} \int_0^{Kb} e^t((KR)^2 + t^2)^{-1/2}\,dt - 2\pi i K^5 e^{-Kb} J_0(X) \Big], \qquad (49)$$

$$W_{10} = [(f_1 - f_1^0)\cos\Theta + (f_2 - f_2^0)\sin\Theta](f(x, y + f(\xi,\eta))\Big[-2KR(R^2+b^2)^{-3/2}$$
$$- 3R(R^2+b^2)^{-5/2} + \pi K^3 e^{-Kb}\left(H_1(KR) + Y_1(KR) - \frac{2}{\pi} \right)$$
$$+ 2K^4 R\, e^{-Kb} \int_0^{Kb} e^{-Kb}((KR)^2 + t^2)^{-3/2}\,dt$$
$$- 2\pi i K^4 e^{-Kb} J_1(KR) \Big], \qquad (50)$$

and

$$W_{20} = \left[\frac{-\sin^2\Theta}{R} f_1 f_1^0 - \frac{\cos^2\Theta}{R} f_2 f_2^0 + (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2R} \right] \Big[-R(R^2+b^2)^{-3/2}$$
$$- \pi K^2 e^{-Kb}(H_0'(X) + Y_0'(X)) + 2K^3 e^{-Kb} R \int_0^{Kb} e^t((KR)^2 + t^2)^{-3/2}\,dt$$
$$+ 2\pi i K^2 e^{-Kb} J_1(KR) \Big] + \Big[-\cos^2\Theta f_1 f_1^0 - \sin^2\Theta f_2 f_2^0 + (f_2 f_1^0 + f_1 f_2^0)\frac{\sin(2\Theta)}{2} \Big]$$
$$\Big[3R(R^2+b^2)^{-5/2} - \frac{1}{2}\pi K^3 e^{-Kb}\Big(H_2(KR) + Y_2(KR) - H_0(KR) - Y_0(KR)$$
$$- \frac{KR}{2\sqrt{\pi}\,\Gamma(5/2)} \Big) + 2K^3 e^{-Kb} \int_0^{Kb} e^t((KR)^2 + t^2)^{-3/2}\,dt$$
$$- 6K^3 e^{-Kb} \int_0^{Kb} e^t((KR)^2 + t^2)^{-5/2}\,dt - \pi i K^3 e^{-Kb}(J_2(KR) - J_0(KR)) \Big], \qquad (51)$$

where Γ denotes the Gamma function.

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