Hossein Azari; F. Parzlivand; Shu Hua Zhang

A mesh free numerical method for the solution of an inverse heat problem

In: Jan Brandts and J. Chleboun and Sergej Korotov and Karel Segeth and J. Šístek and Tomáš Vejchodský (eds.): Applications of Mathematics 2012, In honor of the 60th birthday of Michal Křížek, Proceedings. Prague, May 2-5, 2012. Institute of Mathematics AS CR, Prague, 2012. pp. 14–30.

Persistent URL: http://dml.cz/dmlcz/702889

Conference Applications of Mathematics 2012
in honor of the 60th birthday of Michal Křížek.
Institute of Mathematics AS CR, Prague 2012

A MESH FREE NUMERICAL METHOD FOR THE SOLUTION OF AN INVERSE HEAT PROBLEM

H. Azari[1], F. Parzlivand[1], S. Zhang[2]

[1] Department of Mathematics Science, Shahid Beheshti University
Tehran, Iran
h_azari@sbu.ac.ir, fparzlivand@yahoo.com
[2] Research Center for Mathematics and Economics,
Tianjin University of Finance and Economics 300222, China
szhang@tjufe.edu.cn (corresponding author)

Abstract

We combine the theory of radial basis functions with the finite difference method to solve the inverse heat problem, and use five standard radial basis functions in the method of the collocation. In addition, using the newly proposed numerical procedure, we also discuss some experimental numerical results.

1. Introduction

In the present work, we study the inverse problem of finding $p(t)$ and $u(x,t)$, which satisfy

$$\begin{aligned} u_t &= u_{xx} + p(t)u_x + f(x,t), \qquad \text{in} \quad Q_T, \\ u(x,0) &= u_0(x), \qquad 0 \le x \le 1, \\ u_x(0,t) &= g_1(t), \qquad 0 \le t \le T, \\ u_x(1,t) &= g_2(t), \qquad 0 \le t \le T, \end{aligned} \tag{1}$$

along with an extra condition

$$u(x^*,t) = h(t), \qquad 0 \le t \le T, \tag{2}$$

where $x^* = 0$ or 1, $Q_T = \{(x,t),\ 0 < x < 1,\ 0 < t < T\}$, $T > 0$, and u_0, $g_1 > 0$, $g_2 < 0$, are known functions.

Recently, considerable efforts have been made in dealing with inverse problems in partial differential equations. These inverse problems not only have the intrinsic mathematical interests, but also have a variety of applications in industry and engineering sciences (cf. [15, 17, 4, 14, 25, 22, 21, 10, 24, 23, 7, 6, 31, 9, 2, 3, 30, 1, 28, 20] for examples). They arise, for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion, and control theory [10, 8, 13, 11, 5, 16].

The existence and the uniqueness of the above inverse problem have been investigated in [12, 8, 13]. Also some other numerical and theoretical discussions about this problem can be found in [26] and [27]. From (1) and (2) we have

$$h'(t) = u_{xx}(0,t) + p(t)u_x(0,t) + f(0,t),$$

and it follows that

$$p(t) = \frac{h'(t) - u_{xx}(0,t) - f(0,t)}{g_1(t)},$$

and thus the inverse problem (1)–(2) is equivalent to the following non–local parabolic problem

$$\begin{aligned} u_t &= u_{xx} + \frac{h'(t) - u_{xx}(0,t) - f(0,t)}{g_1(t)} u_x + f(x,t), && \text{in} \quad Q_T, \\ u(x,0) &= u_0(x), && 0 \le x \le 1, \\ u_x(0,t) &= g_1(t), && 0 \le t \le T, \\ u_x(1,t) &= g_2(t), && 0 \le t \le T, \end{aligned} \tag{3}$$

where $h'(t) > 0$, $u_{xx}(0,t) < 0$, $u_0(x) > 0$, $g_1(t) > 0$, and $g_2(t) < 0$.

2. Radial basis functions

The numerical solution of partial differential equations by Radial Basis Functions (RBFs) methods is based on a scattered data interpolation. Let $x_1, \cdots, x_N \in \Omega \subset \mathbb{R}^d$ be a given set of scattered data. A radial basis function $\phi_i(x) = \phi(\|x - x_i\|_2)$ depends only on the distance between $x \in \mathbb{R}^d$ and a fixed point $x_i \in \mathbb{R}^d$, such that the radial basis function ϕ_i is radially symmetric about the center x_i. Some well–known RBFs are listed in Table 1.

Name of Radial Basis Function	Definition
Multiquadric (MQ)	$\phi(r) = \sqrt{c^2 + r^2}$
Inverse Quadratic (IQ)	$\phi(r) = \frac{1}{c^2+r^2}$
Inverse Multiquadric (IMQ)	$\phi(r) = \frac{1}{\sqrt{c^2+r^2}}$
Gaussian (GA)	$\phi(r) = \exp(-cr^2)$
Thin Plate Splines	$\phi(r) = r^2 \log(r)$

Table 1: Some well–known functions that generate RBFs.

Let r be the Euclidean distance between a fixed point $x_i \in \mathbb{R}^d$ and an arbitrary point $x \in \mathbb{R}^d$, i.e. $r = \|x - x_i\|_2$. A radial function interpolation problem may be described as follows: For given data $f_i = f(x_i)$ $(i = 1, \cdots, N)$ and $x = (x_1, x_2, \cdots, x_d)$, the interpolation RBF approximation is

$$S_f(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) + \Psi(x), \tag{4}$$

where α_i are chosen such that $S_f(x_i) = f_i$, and the above equation can be written without the additional polynomial Ψ. In that case, ϕ must be unconditionally positive definite to guarantee the solvability of the resulting system (e.g., Gaussian or inverse multiquadrics, Sobolev splines or compactly supported functions). However, Ψ is usually required when ϕ is conditionally positive definite, i.e., when ϕ has a polynomial growth towards infinity. Examples are thin plate splines and multiquadrics.

If $\mathcal{P}_q^d$ denotes the space of d–variate polynomials of order not exceeding q, and letting the polynomials $P_1, \cdots, P_m$ be the basis of $\mathcal{P}_q^d$ in $\mathbb{R}^d$, then the polynomial Ψ is usually written in the following form:

$$\Psi(x) = \sum_{i=1}^{m} \zeta_i P_i(x), \tag{5}$$

where $m = (q - 1 + d)!/(d!(q - 1)!)$.

To determine the coefficients $(\alpha_1, \cdots, \alpha_N)$ and $(\zeta_1, \cdots, \zeta_m)$, the collocation method is used. However, in addition to the N equations resulting from collocating equation (4) at the N points, an extra m equations are required. This is insured by the m conditions for (4),

$$\sum_{j=1}^{N} \alpha_j P_i(x_j) = 0, \qquad i = 1, \cdots, m. \tag{6}$$

In a similar representation as (4), for any linear partial differential operator $\mathcal{L}$, $\mathcal{L}u$ can be approximated by

$$\mathcal{L}u(x) \simeq \sum_{i=1}^{N} \alpha_i \mathcal{L}\phi(x, x_i) + \mathcal{L}\Psi(x).$$

3. Implementation of the meshless method

In this section, we combine the theory of radial basis functions with the finite difference method to solve the non–local parabolic problem (3).

Since our problem depends on time, the idea of the proposed numerical scheme is to interpolate the unknown function u by the following RBFs ϕ_j $(j = 1, 2, \ldots, N)$:

$$u(x, t) \simeq \sum_{j=1}^{N} \alpha_j(t) \phi_j(x), \tag{7}$$

where $\alpha_j(t)$ are the unknown coefficients depending on time. Since each radial basis function does not depend on time, the time derivative of u is simply given in terms of the time derivatives of the coefficients:

$$\frac{\partial u(x,t)}{\partial t} \simeq \sum_{j=1}^{N} \frac{d\alpha_j(t)}{dt}\phi_j(x),$$

and the first and second partial derivatives of u with respect to x are respectively given as follows:

$$\frac{\partial u(x,t)}{\partial x} \simeq \sum_{j=1}^{N} \alpha_j(t)\frac{d\phi_j(x)}{dx},$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq \sum_{j=1}^{N} \alpha_j(t)\frac{d^2\phi_j(x)}{dx^2}.$$

First, let us discretize (3) according to the following θ–method

$$\begin{aligned}
\frac{u(x,t+\delta t)-u(x,t)}{\delta t} =&\theta\left[\nabla^2 u(x,t+\delta t)+\frac{h'(t+\delta t)}{g_1(t+\delta t)}\nabla u(x,t+\delta t)\right.\\
&-\frac{1}{g_1(t+\delta t)}\nabla^2 u(0,t+\delta t)\nabla u(x,t+\delta t)\\
&\left.-\frac{1}{g_1(t+\delta t)}f(0,t+\delta t)\nabla u(x,t+\delta t)+f(x,t+\delta t)\right]\\
&+(1-\theta)\left[\nabla^2 u(x,t)+\frac{h'(t)}{g_1(t)}\nabla u(x,t)\right.\\
&-\frac{1}{g_1(t)}\nabla^2 u(0,t)\nabla u(x,t)\\
&\left.-\frac{1}{g_1(t)}f(0,t)\nabla u(x,t)+f(x,t)\right],
\end{aligned} \tag{8}$$

where $u(x,t)$ is the temperature at the position x and at time t, ∇ the gradient differential operator, $0 \le \theta \le 1$, and δt is the time step size. Rearranging equation (8), using the notation $u(x,t^n) = u^n$ where $t^n = t^{n-1} + \delta t$, we obtain

$$\begin{aligned}
\frac{u^{n+1}-u^n}{\delta t} =&\theta\left[\nabla^2 u^{n+1}+\frac{h^{n+1}}{g_1^{n+1}}\nabla u^{n+1}-\frac{1}{g_1^{n+1}}\nabla^2 u_0^{n+1}\nabla u^{n+1}\right.\\
&\left.-\frac{1}{g_1^{n+1}}f_0^{n+1}\nabla u^{n+1}+f^{n+1}\right]\\
&+(1-\theta)\left[\nabla^2 u^n+\frac{h^n}{g_1^n}\nabla u^n-\frac{1}{g_1^n}\nabla^2 u_0^n\nabla u^n-\frac{1}{g_1^n}f_0^n\nabla u^n+f^n\right],
\end{aligned} \tag{9}$$

where $g_1^n = g_1(t^n)$, $h^n = h'(t^n)$, $f^n = f(x,t^n)$, $f_0^n = f(0,t^n)$, and $u_0^n = u(0,t^n)$. The nonlinear term in the above equation is linearized by using the following term [29]:

$$(\nabla u \cdot \nabla^2 u)^{n+1} = (\nabla u)^{n+1}(\nabla^2 u)^n + (\nabla u)^n(\nabla^2 u)^{n+1} - (\nabla u)^n(\nabla^2 u)^n.$$

Rearranging equation (9), we have

$$\begin{aligned} u^{n+1} &- \xi\left[\nabla^2 u^{n+1} + h^{n+1}\gamma^{n+1}\nabla u^{n+1} - \gamma^{n+1}\nabla^2 u_0^{n+1}\nabla u^n\right. \\ &\left.- \gamma^{n+1}\nabla^2 u_0^n \nabla u^{n+1} - \gamma^{n+1} f_0^{n+1}\nabla u^{n+1}\right] \\ &= u^n - \beta\left[\nabla^2 u^n + h^n\gamma^n\nabla u^n - \gamma^n\nabla^2 u_0^n\nabla u^n - \gamma^n f_0^n\nabla u^n\right] \\ &+ \xi\gamma^{n+1}\nabla^2 u_0^n\nabla u^n + \xi f^{n+1} - \beta f^n, \end{aligned} \tag{10}$$

where $\xi = \theta\delta t$, $\beta = -(1-\theta)\delta t$, and $\gamma^n = \frac{1}{g_1^n}$.

Assuming that there are $(N-2)$ interpolation points, then $u(x, t^n)$ can be approximated by

$$u^n(x) \simeq \sum_{j=1}^{N-2} \alpha_j^n \phi_j(x) + \alpha_{N-1}^n x + \alpha_N^n, \tag{11}$$

where $\alpha_j(t^n) = \alpha_j^n$. To determine the interpolation coefficients $(\alpha_1, \cdots, \alpha_N)$, we employ the collocation method by applying (11) at every point x_i $(i = 1, \cdots, N-2)$. Thus, we have

$$u^n(x_i) \simeq \sum_{j=1}^{N-2} \alpha_j^n \phi_j(x_i) + \alpha_{N-1}^n x_i + \alpha_N^n. \tag{12}$$

The additional conditions due to (6) are written as:

$$\sum_{j=1}^{N-2} \alpha_j^n = \sum_{j=1}^{N-2} \alpha_j^n x_j = 0. \tag{13}$$

Writing (12) together with (13) in a matrix form, we have

$$[u]^n = \mathbf{A}[\alpha]^n, \tag{14}$$

where $[u]^n = [u_1^n \ \cdots \ u_{N-2}^n \ 0 \ 0]^T$, $[\alpha]^n = [\alpha_1^n \cdots \alpha_N^n]^T$, and $\mathbf{A} = [a_{ij}, 1 \le i, j \le N]$ is given by

$$\mathbf{A} = \begin{bmatrix} \phi_{1,1} & \cdots & \phi_{1,N-2} & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \phi_{N-2,1} & \cdots & \phi_{N-2,N-2} & x_{N-2} & 1 \\ x_1 & \cdots & x_{N-2} & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

There are $p = (N-4)$ internal (domain) points and two boundary points. Therefore, the $(N \times N)$ matrix $\mathbf{A}$ can be split into

$$\mathbf{A} = A_d + A_{b1} + A_{b2} + A_e,$$

where

$$\begin{aligned}
A_d &= [a_{ij} \quad \text{for} \quad (2 \le i \le N-3,\ 1 \le j \le N) \quad \text{and} \quad 0 \quad \text{elsewhere}],\\
A_{b1} &= [a_{ij} \quad \text{for} \quad (i = 1,\ 1 \le j \le N) \quad \text{and} \quad 0 \quad \text{elsewhere}],\\
A_{b2} &= [a_{ij} \quad \text{for} \quad (i = N-2,\ 1 \le j \le N) \quad \text{and} \quad 0 \quad \text{elsewhere}],\\
A_e &= [a_{ij} \quad \text{for} \quad (N-1 \le i \le N,\ 1 \le j \le N) \quad \text{and} \quad 0 \quad \text{elsewhere}].
\end{aligned}$$

Using the notation $\mathcal{L}\mathbf{A}$ to designate the matrix of the same dimension as $\mathbf{A}$ and containing the elements $\tilde{a}_{ij} = \mathcal{L}a_{ij},\ 1 \le i,\ j \le N$, then (10) together with boundary conditions can be written, in the matrix form, as follows:

$$\begin{aligned}
[C_\xi + B + D + A_e][\alpha]^{n+1} =& [C_\beta][\alpha]^n + (a+c)(\nabla H[\alpha]^n)\cdot(\nabla^2 G[\alpha]^n)\\
&+ b(\nabla H[\alpha]^n) + \xi f^{n+1} - \beta f^n + F^{n+1},
\end{aligned} \tag{15}$$

where $a = \beta\gamma^n$, $b = \beta\gamma^n f_0^n$, $c = \xi\gamma^{n+1}$,

$$H = \nabla A_{b1} + \nabla A_{b2} + \nabla A_d, \quad B = \nabla A_{b1} + \nabla A_{b2}, \quad C_\xi = A_d - \xi\nabla^2 A_d - \xi h^{n+1}\gamma^{n+1}\nabla H$$

$$D = c[\nabla H[\alpha]^n \nabla^2 G + \nabla^2 G[\alpha]^n \nabla H + f_0^{n+1}\nabla H], \quad C_\beta = A_d - \beta\nabla^2 A_d - \beta h^n \gamma^n \nabla H,$$

and

$$G = [\phi_1''(0)\ \cdots\ \phi_{N-2}''(0)\ 0\ 0], \qquad F^n = [(g_1)_1^n\ 0\ \cdots\ 0\ (g_2)_{N-2}^n\ 0\ 0]^T.$$

Assuming $M = [C_\xi + B + D + A_e]$, in general the well–posedness of (15) and the solvability of such a system are open, and see the paper of Fasshauer [18] for details. However, recently Franke and Schaback [19] gave the first the convergence proof and the error bound for the solution of the partial differential equation with collocation and radial basis functions. They have showed that the radial basis functions have to be much smoother than the smoothness required for a weak solution of the differential operator. As far as the Laplace operator and the thin plate splines or the multiquadrics are concerned, the requirements are met to guarantee the positive definiteness of the resulting matrix and therefore insure the solvability of the system, and see Reference [19] for details.

It follows from rewriting (15) in the following form

$$\begin{aligned}
[\alpha]^{n+1} =& M^{-1}[C_\beta][\alpha]^n + (a+c)M^{-1}(\nabla H[\alpha]^n)\cdot(\nabla^2 G[\alpha]^n)\\
&+ bM^{-1}(\nabla H[\alpha]^n) + M^{-1}(\xi f^{n+1} - \beta f^n + F^{n+1}),
\end{aligned} \tag{16}$$

and making use of (14) that the vector temperature $[u]^{n+1}$ is computed from $[\mathbf{u}]^n$ by using

$$\begin{aligned}
[u]^{n+1} =& \mathbf{A}M^{-1}[C_\beta]\mathbf{A}^{-1}[u]^n + (a+c)M^{-1}(\nabla H\mathbf{A}^{-1}[u]^n)\cdot(\nabla^2 G\mathbf{A}^{-1}[u]^n)\\
&+ bM^{-1}(\nabla H\mathbf{A}^{-1}[u]^n) + M^{-1}(\xi f^{n+1} - \beta f^n + F^{n+1}),
\end{aligned}$$

where $[u]^0 = \mathbf{A}[\alpha]^0$, and $[\alpha]^0$ can be computed by the initial condition. Finally, the approximate value of $p(t)$ is given by

$$p(t^n) = \frac{h'(t^n) - u_{xx}(0,t^n) - f(0,t^n)}{g_1(t^n)},$$

where

$$u_{xx}(0,t^n) = \sum_{j=1}^{N} \alpha_j^n \frac{d^2\phi_j(0)}{dx^2}.$$

4. Numerical experiments

To show the efficiency of the new method on the inverse parabolic partial differential equation, three examples are given. These tests are chosen such that their analytical solutions are known. However, the method developed in this paper can be applied to more complicated problems. Since the equation (16) is valid for any value of $0 \le \theta \le 1$, we will use $\theta = 0$ (thus the scheme is explicit, and the stability limitation is $\Delta t \le \frac{1}{2}(\Delta x)^2$), $\theta = \frac{1}{2}$ (the scheme is the famous Crank–Nicholson), and $\theta = 1$ (the scheme is implicit).

We use the $\mathbf{L}_2$ and the $\mathbf{L}_\infty$ error norms to measure the difference between the numerical and analytical solutions. Let $\tilde{u}$ denote the approximated solution. The $\mathbf{L}_2$ error norm is defined by

$$\mathbf{L}_2 = \|u - \tilde{u}\|_2 = \sqrt{\frac{1}{N}\sum_{j=1}^{N} |u_j - \tilde{u}_j|^2},$$

and the $\mathbf{L}_\infty$ error norm is defined by

$$\mathbf{L}_\infty = \|u - \tilde{u}\|_\infty = \max_{1\le j\le N} |u_j - \tilde{u}_j|.$$

Example 4.1. *We wish to solve the inverse problem (1)–(2) with the following conditions:*

$$\begin{aligned} u_0(x) &= (0.5 - x)x, \\ g_1(t) &= 0.5, \\ g_2(t) &= -1.5, \\ h(t) &= 2\sin(t) - 2t, \\ f(x,t) &= 2\cos(t)(0.5 + 2x), \end{aligned}$$

$x^* = 0$, *and* $T = 1$, *for which the exact solution is*

$$\begin{aligned} u(x,t) &= -(x - 0.5)x + 2\sin(t) - 2t, \\ p(t) &= 2\cos(t). \end{aligned}$$

t	L_∞–error (u)	L_2–error (u)
0	1×10^{-10}	1×10^{-18}
0.1	4×10^{-6}	3×10^{-6}
0.2	7×10^{-5}	7×10^{-5}
0.4	4×10^{-5}	4×10^{-5}
0.6	1×10^{-4}	1×10^{-4}
0.8	6×10^{-5}	3×10^{-5}
1	1×10^{-4}	1×10^{-4}

Table 2: The L_∞ and the L_2 errors for u with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$ for Example 4.1 and using the GA–RBF and the Crank–Nicholson scheme.

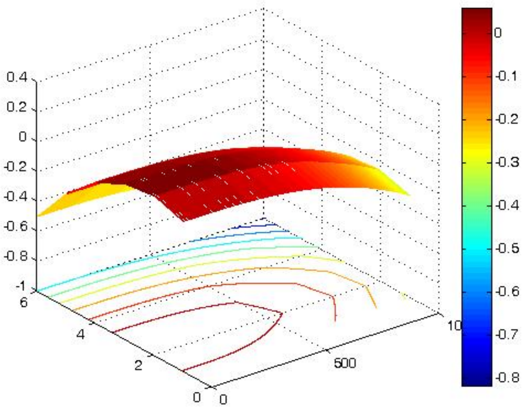
t	L_∞–error (p)
0	1×10^{-7}
0.1	1×10^{-6}
0.2	2×10^{-6}
0.4	1×10^{-5}
0.6	4×10^{-5}
0.8	9×10^{-5}
1	1×10^{-4}

Table 3: The errors between the analytical solution and the estimated solution p, with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$, for Example 4.1, and using the GA–RBF and the Crank–Nicholson scheme.

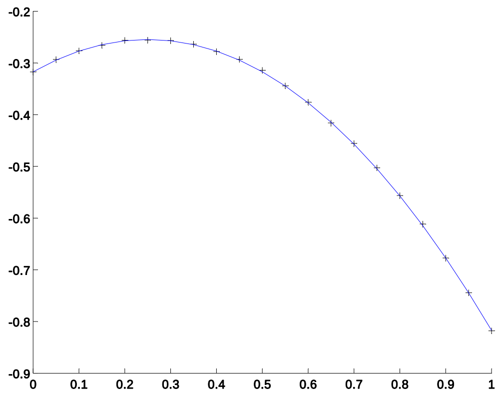
The L_∞ and the L_2 errors are displayed for u in Table 2 for $t = 0,\ 0.1,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1, by using the GA–RBF and the Crank–Nicholson scheme. Also, the corresponding errors between the analytical and the estimated function p are listed in Table 3. The graph of the analytical and the estimated functions for u in $t = 1$ is given in Figure 1(b). In addition, the maximum error variations of the algorithm with different radial basis functions are depicted in Figure 1 and Figure 2.

Example 4.2. *In this example, we consider the inverse problem (1)–(2) with the following conditions:*

$$\begin{aligned} u_0(x) &= 2 + (0.5 - x)x, \\ g_1(t) &= 0.5, \\ g_2(t) &= -1.5, \\ h(t) &= 2t^4 - 2t, \\ f(x,t) &= 8t^3(0.5 + 2x), \end{aligned}$$



(a) Numerical solution of $u(x, t)$



(b) The analytical and the numerical solutions of $u(x, 1)$.

Figure 1: The GA–RBF and the Crank–Nicholson scheme with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$.

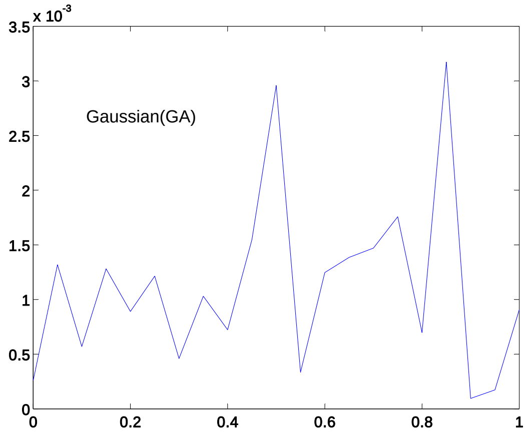
t	L_∞–error (u)	L_2–error (u)
0	8×10^{-10}	21×10^{-21}
0.1	26×10^{-6}	21×10^{-6}
0.2	29×10^{-5}	26×10^{-5}
0.4	24×10^{-4}	22×10^{-4}
0.6	82×10^{-4}	76×10^{-4}
0.8	18×10^{-3}	16×10^{-3}
1	25×10^{-3}	20×10^{-3}

Table 4: the L_∞ and the L_2 errors for u, with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$, by using the MQ–RBF and the explicit scheme.

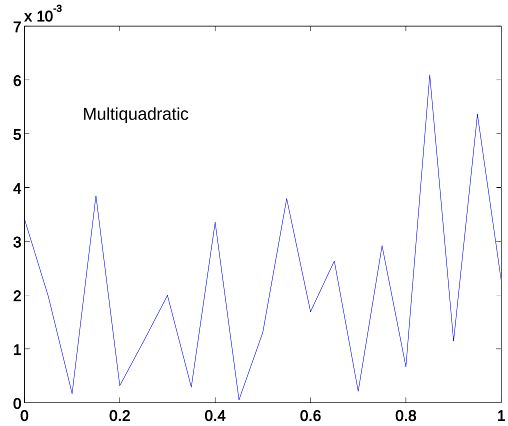
$x^* = 0$, *and* $T = 1$, *for which the exact solution is*

$$u(x, t) = -(x - 0.5)x + 2t^4 - 2t,$$
$$p(t) = 4t^3.$$

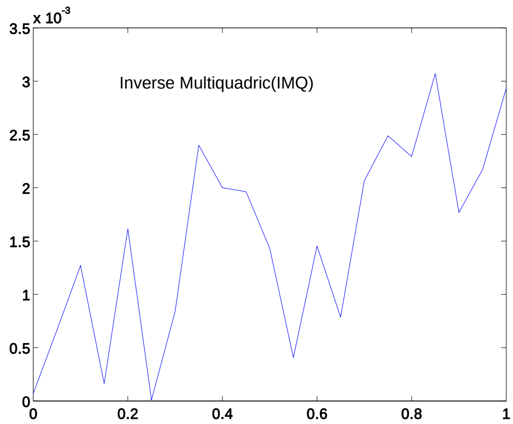
The L_∞ and the L_2 errors are obtained for u in Table 4 for $t = 0, 0.1, 0.2, 0.4, 0.6, 0.8$ and 1, by using the MQ–RBF and the explicit scheme. Also, the corresponding errors between the analytical and the estimated functions p are listed in Table 5. The graph of the analytical and the estimated functions for u in $t = 0.4$ is given in Figure 3(b). Moreover, the maximum error variations of the algorithm with different radial basis functions are presented in Figure 3 and Figure 4.



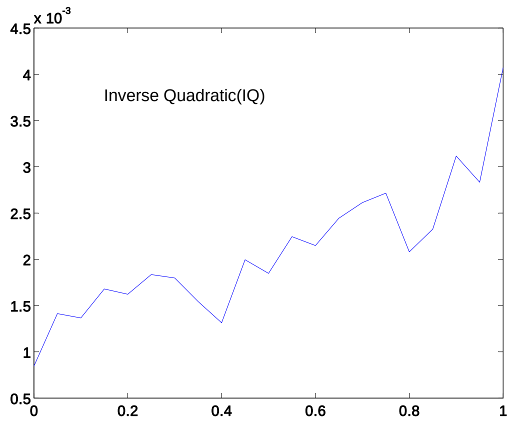
(a) The absolute errors for u by using the GA–RBF



(b) The absolute errors for u by using the MQ–RBF



(c) The absolute errors for u by using the IMQ–RBF



(d) The absolute errors for u by using the IQ–RBF

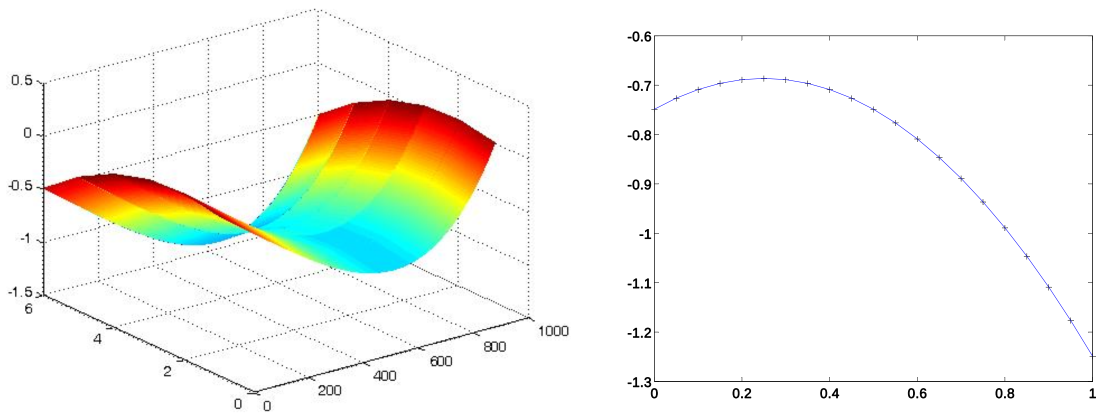
Figure 2: The absolute errors for $u(x,t)$ with $T = 1$, $c = 0.0005$, $dt = 0.001$, $dx = 0.2$ and the Crank–Nicholson scheme.

Example 4.3. *We consider the inverse problem (1)–(2) with the following conditions:*

$$
\begin{aligned}
u_0(x) &= (0.5 - x)x, \\
g_1(t) &= 0.5, \\
g_2(t) &= -1.5, \\
h(t) &= 2t^2 - 2t, \\
f(x,t) &= 4t(0.5 + 2x), \\
x^* &= 0,
\end{aligned}
$$

t		L_∞–error (p)
0		1×10^{-6}
0.1		2×10^{-5}
0.2		4×10^{-5}
0.4		8×10^{-5}
0.6		1×10^{-4}
0.8		2×10^{-4}
1		1×10^{-3}

Table 5: The errors between the analytical solution and the estimated solution p, with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$, by using the MQ–RBF and the explicit scheme.



(a) The numerical solution of $u(x,t)$

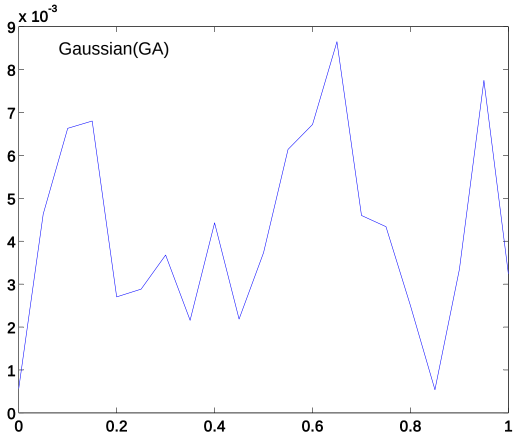
(b) The analytical and the numerical solutions of $u(x, 0.4)$.

Figure 3: The MQ–RBF and the explicit scheme with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$.

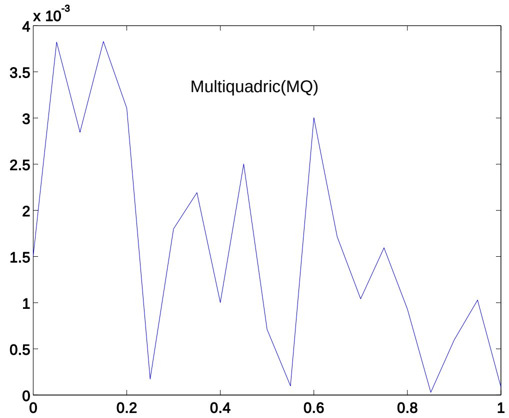
and $T = 1$, for which the exact solution is

$$u(x,t) = -(x - 0.5)x + 2t^2 - 2t,$$
$$p(t) = 4t.$$

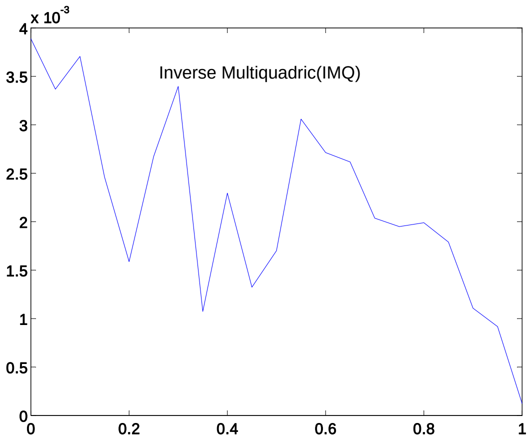
The L_∞ and the L_2 errors are obtained for u in Table 6 for $t = 0, 0.1, 0.2, 0.4, 0.6, 0.8$ and 1, by using the IMQ–RBF and the implicit scheme. Also, the corresponding errors of the analytical and the estimated functions p are listed in Table 7. The graph of the analytical and the estimated functions for u in $t = 0.1$ is given in Figure 5(b). In addition, the maximum error variations of the algorithm with different radial basis functions are given in Figure 5 and Figure 6.



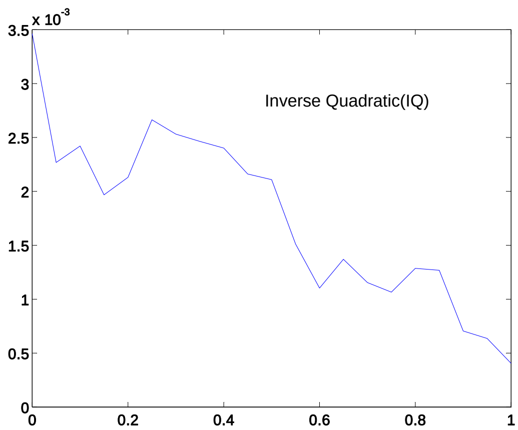
(a) The absolute errors for u by using the GA–RBF



(b) The absolute errors for u by using the MQ–RBF



(c) The absolute errors for u by using the IMQ–RBF



(d) The absolute errors for u by using the IQ–RBF

Figure 4: The absolute errors for $u(x,t)$ with $T = 1$, $c = 0.0005$, $dt = 0.001$, $dx = 0.2$ and the Crank–Nicholson scheme.

5. Conclusions

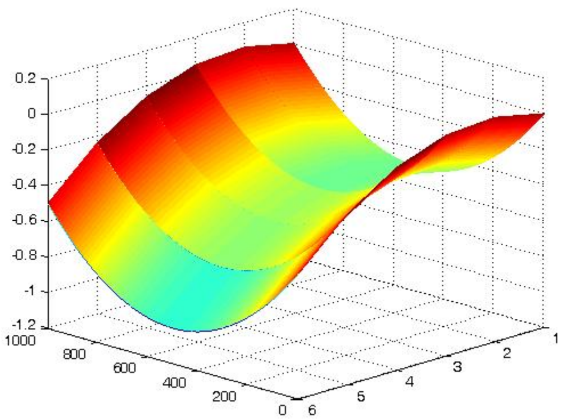
Radial basis functions are used to solve an inverse parabolic equation. The meshless property of the RBFs method is the most important advantage of this scheme over the traditional mesh dependent techniques such as finite difference methods, finite element methods, and boundary element methods. The mesh free nature of the new technique allows us to solve the problems with non–regular geometry. A comparison with some well known finite difference methods for numerical solution of the inverse parabolic problem shows that the present method is more accurate. In conclusion we mention that the RBFs technique can be extended to similar two and three dimensional inverse parabolic problems subject to temperature overspecification.

t	L_∞–error (u)	L_2–error (u)
0	1×10^{-10}	57×10^{-23}
0.1	38×10^{-5}	37×10^{-5}
0.2	39×10^{-4}	39×10^{-4}
0.4	78×10^{-4}	74×10^{-4}
0.6	11×10^{-4}	9×10^{-4}
0.8	12×10^{-3}	78×10^{-4}
1	4×10^{-3}	25×10^{-3}

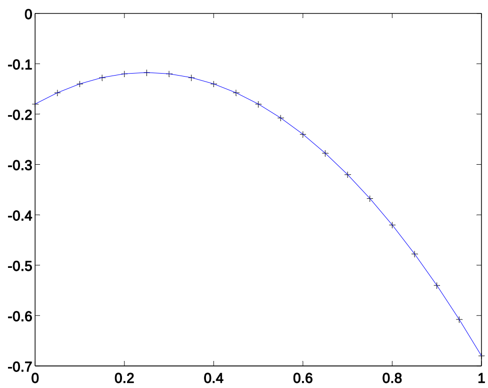
Table 6: The L_∞ and the L_2 errors for u, with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$, by using the IMQ–RBF and the implicit scheme.

t	L_∞–error (p)
0	3×10^{-6}
0.1	4×10^{-5}
0.2	9×10^{-5}
0.4	1×10^{-4}
0.6	2×10^{-4}
0.8	4×10^{-4}
1	4×10^{-3}

Table 7: The errors between the analytical solution and the estimated solution p, with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$, by using the IMQ–RBF and the implicit scheme.

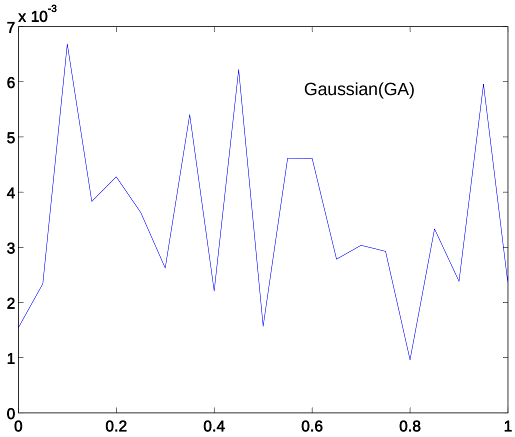


(a) The numerical solution of $u(x, t)$

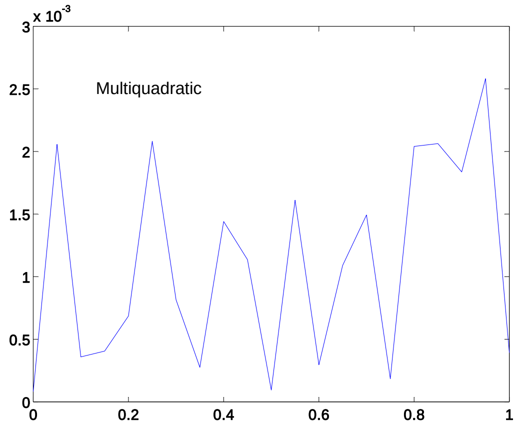


(b) The analytical and the numerical solutions of $u(x, 0.1)$.

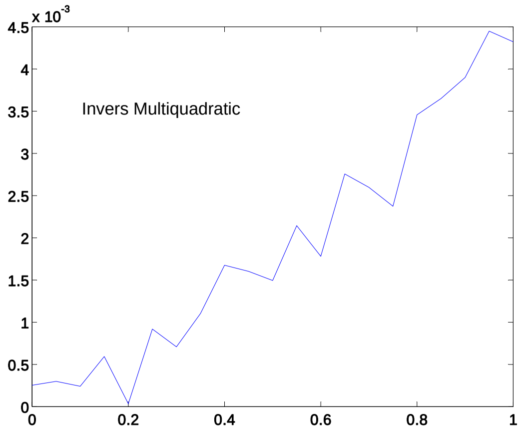
Figure 5: The IMQ–RBF and the implicit scheme with $c = 0.0005$, $dt = 0.001$, $dx = 0.2$.



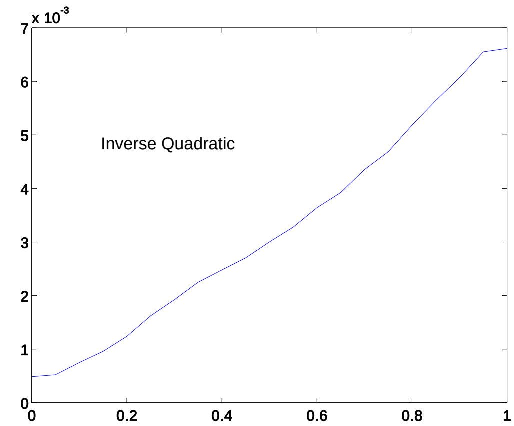
(a) The absolute errors for u by using the GA–RBF



(b) The absolute errors for u by using the MQ–RBF



(c) The absolute errors for u by using the IMQ–RBF



(d) The absolute errors for u by using the IQ–RBF

Figure 6: The absolute errors for $u(x,t)$ with $T = 1$, $c = 0.0005$, $dt = 0.001$, $dx = 0.2$ and the Crank–Nicholson scheme.

Acknowledgements

This work was supported in part by the National Basic Research Program (2012CB95 5804), the Shahid Beheshti University and the Scientific Computation Research Group, and the National Natural Science Foundation of China (10771158 and 11171251).

References

[1] Azari, H.: Numerical procedures for the determination of an unknown coefficient in parabolic differential equations. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **9** (2002), 555–573.

[2] Azari, H., Allegretto, W., Lin, Y., and Zhang, S.: Numerical procedures for recovering a time dependent coefficient in a parabolic differential equation. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **11** (2004), 181–199. First Industrial Mathematics Session.

[3] Azari, H. and Zhang, S.: Identifying a time dependent unknown coefficient in a parabolic inverse problem. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **12** (2005), 32–43.

[4] Cannon, J. R. and Ewing, R. E.: Determination of a source term in a linear parabolic partial differential equation. Z. Angew. Math. Phys. **27** (1976), 393–401.

[5] Cannon, J. R., Lin, Y. P., and Wang, S.: Determination of a control parameter in a parabolic partial differential equation. J. Austral. Math. Soc. Ser. B **33** (1991), 149–163.

[6] Cannon, J. R. and Lin, Y.: Determination of a parameter $p(t)$ in some quasilinear parabolic differential equations. Inverse Problems **4** (1988), 35–45.

[7] Cannon, J. R. and Lin, Y.: Determination of parameter $p(t)$ in Hölder classes for some semilinear parabolic equations. Inverse Problems **4** (1988), 595–606.

[8] Cannon, J. R. and Lin, Y.: An inverse problem of finding a parameter in a semilinear heat equation. J. Math. Anal. Appl. **145** (1990), 470–484.

[9] Cannon, J. R., Lin, Y., and van der Hoek, J.: Semi-linear heat equation subject to the specification of energy. In: *Modeling and analysis of diffusive and advective processes in geosciences*, pp. 40–51. SIAM, Philadelphia, PA, 1992.

[10] Cannon, J. R., Lin, Y., and Xu, S.: Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations. Inverse Problems **10** (1994), 227–243.

[11] Cannon, J. R. and Yin, H. M.: A class of nonlinear nonclassical parabolic equations. J. Differential Equations **79** (1989), 266–288.

[12] Cannon, J. R. and Pérez Esteva, S.: Determination of the coefficient of u_x in a linear parabolic equation. Inverse Problems **10** (1994), 521–531.

[13] Cannon, J. R. and Yin, H. M.: On a class of nonlinear parabolic equations with nonlinear trace type functionals. Inverse Problems **7** (1991), 149–161.

[14] Chadam, J. M. and Yin, H. M.: Determination of an unknown function in a parabolic equation with an overspecified condition. Math. Methods Appl. Sci. **13** (1990), 421–430.

[15] Colton, D., Ewing, R., and Rundell, W. (Eds.): *Inverse problems in partial differential equations.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[16] Day, W. A.: Extensions of a property of the heat equation to linear thermoelasticity and other theories. Quart. Appl. Math. **40** (1982/83), 319–330.

[17] Engl, H. W. and Rundell, W. (Eds.): *Inverse problems in diffusion processes.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[18] Fasshauer, G. E.: Solving partial differential equations by collocation with radial basis functions. Surface Fitting and Multiresolution Methods (1997).

[19] Franke, C. and Schaback, R.: Solving partial differential equations by collocation using radial basis functions. Appl. Math. Comput. **93** (1998), 73–82.

[20] Hettlich, F. and Rundell, W.: Identification of a discontinuous source in the heat equation. Inverse Problems **17** (2001), 1465–1482.

[21] Lin, Y.: An inverse problem for a class of quasilinear parabolic equations. SIAM J. Math. Anal. **22** (1991), 146–156.

[22] Lin, Y.: Analytical and numerical solutions for a class of nonlocal nonlinear parabolic differential equations. SIAM J. Math. Anal. **25** (1994), 1577–1594.

[23] Lin, Y. and Liu, J. H.: Stability of nonlocal diffusion equations. Dynam. Contin. Discrete Impuls. Systems **5** (1999), 53–66. Differential equations and dynamical systems (Waterloo, ON, 1997).

[24] Lin, Y. and Tait, R. J.: Finite-difference approximations for a class of nonlocal parabolic boundary value problems. J. Comput. Appl. Math. **47** (1993), 335–350.

[25] Lin, Y., Xu, S., and Yin, H. M.: Finite difference approximations for a class of non-local parabolic equations. Internat. J. Math. Math. Sci. **20** (1997), 147–163.

[26] Pani, A. K. and Anderssen, R. S.: Finite element methods for identification of parameters in parabolic problems. In: *Mini-Conference on Inverse Problems in Partial Differential Equations (Canberra, 1990)*, *Proc. Centre Math. Appl. Austral. Nat. Univ.*, vol. 31, pp. 208–221. Austral. Nat. Univ., Canberra, 1992.

[27] Pani, A. K. and Anderssen, R. S. (Eds.): *Mini-Conference on Inverse Problems in Partial Differential Equations*, *Proceedings of the Centre for Mathematics and its Applications, Australian National University*, vol. 31. Australian National University Centre for Mathematics and its Applications, Canberra, 1992.

[28] Ramm, A. G.: An inverse problem for the heat equation. Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 973–976.

[29] Rubin, S. G. and Graves, Jr., R. A.: A cubic spline approximation for problems in fluid mechanics. Technical Report NASA-TR-R-436 (1975), 93p.

[30] Shidfar, A. and Azari, H.: An inverse problem for a nonlinear diffusion equation. Nonlinear Anal. **28** (1997), 589–593.

[31] Wang, S. and Lin, Y.: A finite-difference solution to an inverse problem for determining a control function in a parabolic partial differential equation. Inverse Problems **5** (1989), 631–640.