Jaroslav Mlýnek

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THE BOX METHOD AND SOME ERROR ESTIMATION*

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Abstract

This article focuses its attention on practical use of the box method for solving certain type of partial differential equations. The heat conduction problem of the oil transformer under stationary load is described by this equation. The knowledge of the transformer operating temperature is important for ensuring correct functionality and lifespan of transformer. We consider an elliptic partial differential equation of second order with the Newton boundary condition on a rectangular domain. The paper contains description of a numerical solution procedure of the heat problem and an estimation of local discretization error. The box method is often called the finite volume method, too. The solution of practical examples are presented as well.

1. Introduction

This paper deals with the stationary heat conduction problem. Our objective is to solve the problem of the relative transformer screening warming with respect to cooling oil of the transformer. The classical formulation of the problem is

$$-\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)+cu=f \tag{1}$$

in a rectangular domain $\Omega \subset R^2$ with the Newton boundary condition

$$\alpha u+\frac{\partial u}{\partial n_A}=g. \tag{2}$$

The derivative with respect to conormal in (2) is defined by the relation

$$\frac{\partial u}{\partial n_A}=a_{11}\frac{\partial u}{\partial x_1}n_1+a_{22}\frac{\partial u}{\partial x_2}n_2 \tag{3}$$

and $n=(n_1,n_2)$ denotes the unit outward normal to $\partial\Omega$. The unknown function u denotes the relative warming of the transformer screening with respect to cooling oil of the transformer, i.e. the difference of temperatures of these two media. We suppose $u \in C^2\left(\bar{\Omega}\right)$, functions a_{11}, a_{22}, c, $f \in C^1\left(\bar{\Omega}\right)$ and $\alpha, g \in\ C\left(\partial\Omega,\right)$, $\alpha\left(s\right)\geq 0$ on $\partial\Omega$. The coefficients a_{11} and a_{22} describe the heat conduction character of the screening in the x_1-axis and x_2-axis directions, respectively.

We will describe a numerical solution procedure of the heat conduction problem model by the box method, local error estimation of the error of this method and practical examples in the following paragraphs.

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2. Use of the box method

We construct the triangulation τ on the closure of rectangle Ω ($X_1 \leq x_1 \leq X_2, Y_1 \leq x_2 \leq Y_2$) in a similar way as if we used the finite element method. We construct a regular rectangular mesh with increments $h_1 = (X_2 - X_1)/p$ and $h_2 = (Y_2 - Y_1)/q$ in the x_1-axis and x_2-axis direction, respectively, where p and q denote the number of segments, to which the region is divided in the x_1-axis and x_2-axis direction, respectively. A general node has coordinates $V_{rs} = [X_1 + rh_1, Y_1 + sh_2]$, where $r \in \{0, 1, \ldots, p\}$, $s \in \{0, 1, \ldots, q\}$. The rectangles with vertices defined at points of mesh create elements of the triangulation τ. We construct a special case of mesh dual to the mesh τ published in [4, p. 215]. Points T_i, $1 \leq i \leq 4$, are midpoints of abscissas defined by the mesh point V_{rs} and adjacent mesh points. Then points S_i, $1 \leq i \leq 4$, are intersection points of axes of the abscissas mentioned. The rectangle corresponds to node V_{rs} and is given by vertices S_1, S_2, S_3 and S_4 thus creating element b_{rs} of the mesh dual to τ (see Fig. 1). If the node V_{rs} lies on the

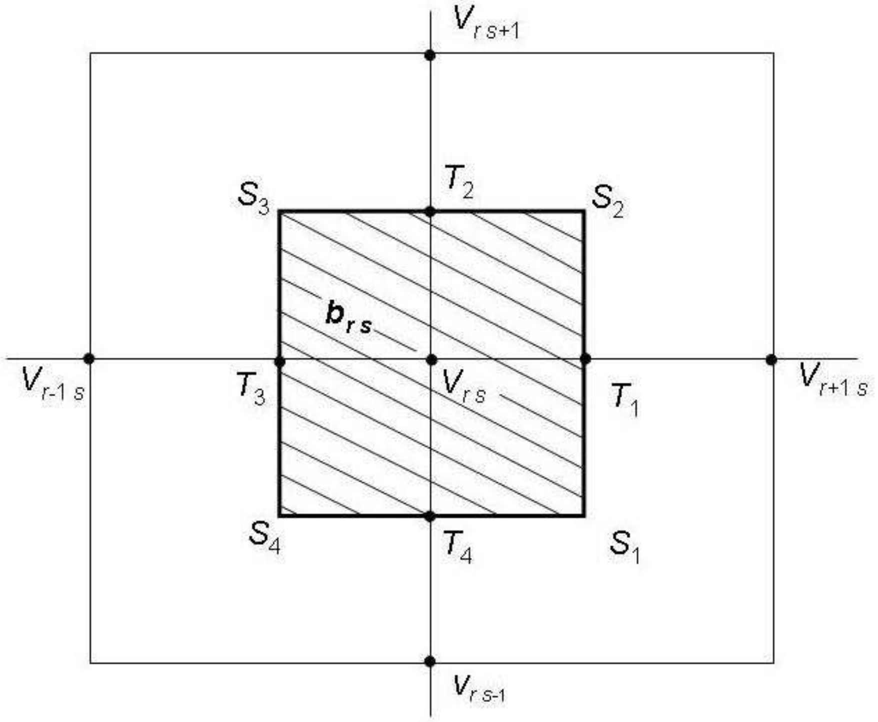

Fig. 1: *Element b_{rs} of the dual mesh corresponding to node V_{rs}.*

boundary of Ω, the element b_{rs} is modified in the corresponding way. In particular, the case when the node V_{rs} lies at "corner" of Ω is in Fig. 2. The elements b_{rs} are characterized by two conditions: $\bar{\Omega} = \bigcup b_{rs}$, where $0 \leq r \leq p$, $0 \leq s \leq q$, and int $b_{rs} \cap$ int $b_{kl} = \emptyset$ for $V_{rs} \neq V_{kl}$.

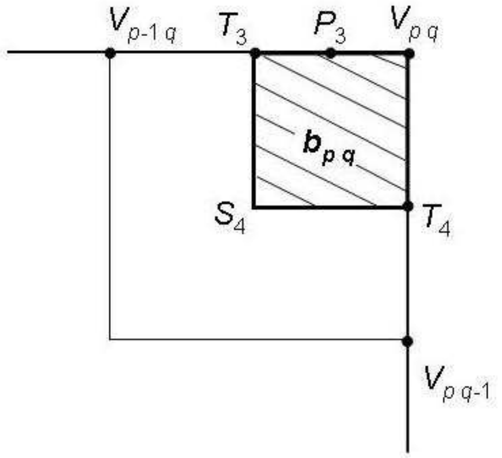

Fig. 2: *Element b_{pq} of the dual mesh corresponding to "corner" node V_{pq}.*

We can transfer the term cu to the right hand side of the equation (1) and integrate the left and right hand sides over the element b_{rs}. Then we get

$$\int_{b_{rs}} \left[-\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)\right] \mathrm{d}x = \int_{b_{rs}} (f - cu)\,\mathrm{d}x. \tag{4}$$

Using now Green's formula on the left hand side of the relation (4), we find that

$$\int_{b_{rs}} \left[-\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)\right] \mathrm{d}x = -\int_{b_{rs}} \frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) \mathrm{d}x -$$

$$-\int_{b_{rs}} \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right) \mathrm{d}x = -\int_{\partial b_{rs}} a_{11}\frac{\partial u}{\partial x_1} n_1\,\mathrm{d}s - \int_{\partial b_{rs}} a_{22}\frac{\partial u}{\partial x_2} n_2\,\mathrm{d}s.$$

Then the relation (4) can be modified to read

$$-\int_{\partial b_{rs}} a_{11}\frac{\partial u}{\partial x_1} n_1\,\mathrm{d}s - \int_{\partial b_{rs}} a_{22}\frac{\partial u}{\partial x_2} n_2\,\mathrm{d}s = \int_{b_{rs}} (f - cu)\,\mathrm{d}x. \tag{5}$$

The left hand side of the equation (5) describes the quantity of heat supplied from or delivered to the boundary of the element b_{rs}, the right hand side expresses the waste heat arising in the element b_{rs}. In case of the boundary mesh point V_{rs}, the equation of type (5) is modified. Using equations of type (5) at all mesh points V_{rs} and applying suitable approximations of derivatives and integrals, we obtain a system of linear algebraic equations with a band matrix. The solution of this system gives us the approximation of warming at nodes V_{rs} of the mesh. Now we will concentrate on the approximation of equations of type (5) and the local approximation error at node V_{rs}.

Because the element b_{rs} is a rectangle, in case of internal element we can use the approximation (see Fig. 1)

$$a_{11}\left(T_1\right)\frac{\partial u\left(T_1\right)}{\partial x_1}n_1 \approx a_{11}\left(T_1\right)\frac{u(V_{r+1s})-u(V_{rs})}{h_1}. \tag{6}$$

With respect to the supposed smoothness of function u, we reach the $O(h_1^2)$-order error in the approximation (6). Similar approximations can be carried out for points T_2, T_3 and T_4 in Fig. 1.

We focus now on the boundary element, for example element b_{rq}, where $1 \le r \le p-1$. Using relations (2) and (3), we obtain the expression

$$a_{22}(V_{rq})\frac{\partial u}{\partial x_2}(V_{rq})n_2 = g(V_{rq}) - \alpha(V_{rq})u(V_{rq}). \tag{7}$$

In case of the boundary "corner" element (see Fig. 2), we can form an approximation of the value $u(P_3)$ (where the auxiliary point P_3 is midpoint of abscissa T_3V_{pq}) from the values $u(V_{p-1q})$ and $u(V_{pq})$ using Lagrange's interpolation polynomial of the first degree. As we suppose $u \in C^2(\bar{\Omega})$, the error order of approximation is $O(h_1^2)$ (see [3, p. 64]) and the value

$$a_{22}(P_3)\frac{\partial u}{\partial x_2}(P_3)n_2$$

can be approximated through the use of the relation (2).

Now we target at the approximation of integrals in equations of type (5). We apply the midpoint rule to the approximation. If function $v \in C^2_{[a,\,b]}$ then the midpoint rule gives

$$\left|\int\limits_a^b v(x)\,\mathrm{d}x - v\left(\frac{a+b}{2}\right)(b-a)\right| \le M\,(b-a)^3\,,$$

where $M \in \mathbf{R}$ (see, for example, [1, p. 178]). In case of the internal element b_{rs}, we use points T_i, $1 \le i \le 4$, as midpoints for the integration over the boundary of the element b_{rs} on the left hand side of the equation (5) . The right hand side of the equation (5) is approximated in the form

$$\int\limits_{b_{rs}} (f - cu)\,\mathrm{d}x \approx (f(V_{rs}) - c\,(V_{rs})u(V_{rs}))h_1h_2.$$

In case of the boundary element we use the expressions of type (7) for the approximation of integrals, too. At the boundary "corner" element it is possible to use auxiliary points of type P_3.

Let us set h =max(h_1, h_2). Applying the above mentioned procedure, we find that the local approximation error of the equation of type (5) for every element b_{rs} is $O(h^2)$, where $r \in \{0, 1, \ldots, p\}$, $s \in \{0, 1, \ldots, q\}$. General questions of the box method error estimation are solved in [2].

3. Practical numerical examples

Now we will solve a real-life technical problem of finding the screening warming with respect to cooling oil of the screening (cooling oil flows around the screening) by using the above mentioned box method. Transformer screening is warmed in consequence of existing eddy currents and it is considered in the form of a thin-walled cylinder. The temperature field is supposed to be rotationally symmetric (see Fig. 3). Hence, the warming problem can be solved in the screening cross section on two dimensional untypical closed rectangular domain Ω. Then the equation (1) with the boundary condition (2) can be written in the form

$$\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)=-q(x_1,x_2) \tag{8}$$

with the Newton boundary condition

$$a_{11}\frac{\partial u}{\partial x_1}n_1+a_{22}\frac{\partial u}{\partial x_2}n_2+\alpha u=\alpha k(x_2-Y_1) \tag{9}$$

on the rectangle Ω. As mentioned above, the solution u represents the warming of the screening with respect to cooling oil, a_{11} and a_{22} are real values in this case; $q(x_1,x_2)$ is the volume density of losses. In the boundary condition (9), the function α means the heat transfer coefficient on the boundary of the domain, a real constant k allows to express the variable temperature of oil in the vicinity of the screening in the x_2-axis direction. The equation (8) with the boundary condition (9) is suitable to solve as a 2D problem. The solution u depends on the functions $q(x_1,x_2)$ and $\alpha(x_1,x_2)$, too. If q depends only on the variable x_2 and α is constant function on $\partial\Omega$, then this problem can be solved as a 1D problem.

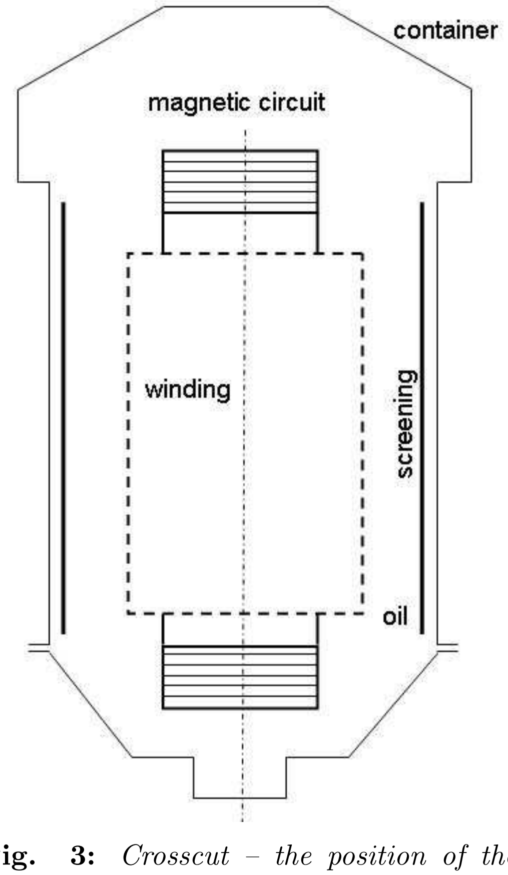

Fig. 3: *Crosscut – the position of the screening in the transformer container.*

Example 1

The function q is given by the relation $q(x_1,x_2)=\rho\,\delta^2(x_1,x_2)$, where ρ is the specific resistance of the screening material, δ denotes the density of eddy currents. Input parameters: $X_1=1.273$m, $X_2=1.280$m, $Y_1=0.000$m, $Y_2=1.200$m, $a_{11}=3$W/mK, $a_{22}=20$W/mK, $\rho=0.143\times10^{-6}\Omega$m, $\alpha(x_1,x_2)=50$W/m^{2}K for $x_2\neq Y_1=0$m and

$\alpha(x_1, x_2) = 0\text{W/m}^2\text{K}$ for $x_2 = Y_1 = 0\text{m}$, $k = 0\text{K/m}$, the current density $\delta(x_1,\ x_2)$ is given by means of 45 values between $0.1158 \times 10^6\text{Am}^{-2}$ and $0.1993 \times 10^7\text{Am}^{-2}$, the current density at the nodes is computed by means of linear interpolation.

Table 1 lists approximate values of warming at chosen nodes computed by using the box method. The values of warming u first of all depend on the input values of function δ. The given values $\delta(x_1,\ x_2)$ are decreasing in the x_1-direction for the constant value of variable x_2. Hence, the computed values of warming u are decreasing in the x_1-axis direction (computed warming u at the nodes with $x_1 = 1.27416\text{m}$ is slightly higher than at the nodes with $x_1 = X_1 = 1.273\text{m}$ in consequence of cooling oil).

x_2[m]	$X_1 = 1.273$m	$x_1 = 1.27416$m	$x_1 = 1.27533$m	$X_2 = 1.280$m
$Y_2 = 1.200$	3.993	4.021	3.965	3.731
1.104	6.333	6.399	6.267	5.807
1.008	7.940	8.014	7.866	7.299
0.912	10.100	10.197	10.003	9.278
0.816	12.311	12.429	12.193	11.311
0.720	12.394	12.483	12.305	11.455
0.624	13.413	13.516	13.310	12.187
0.528	13.445	13.545	13.345	12.418
0.432	13.466	13.570	13.362	12.430
0.336	13.211	13.313	13.109	12.195
0.240	12.957	13.054	12.860	11.967
0.144	12.640	12.725	12.555	11.692
0.048	12.491	12.581	12.401	11.541
$Y_1 = 0.000$	12.289	12.356	12.213	11.378

Tab. 1: *The values of the screening warming in K for selected nodes, $h_1 = 0.0011667m$ and $h_2 = 0.04800m$.*

Example 2

The volume density of losses q depends on x_2 only. Input parameters: $X_1 = 0.860\text{m}$, $X_2 = 0.868\text{m}$, $Y_1 = 0.033\text{m}$, $Y_2 = 1.900\text{m}$, $a_{11} = 3\text{W/mK}$, $a_{22} = 20\text{W/mK}$, the volume density of losses $q(x_2)$ is given by means of 36 values between $0.4904 \times 10^2\text{W/m}^3$ and $0.9348 \times 10^6\text{W/m}^3$, values of the function q at the nodes are computed by means of linear interpolation, $\alpha(x_1, x_2) = 50\text{W/m}^2\text{K}$ for $x_1 \neq X_2 = 0.868\text{m}$ and $\alpha(x_1, x_2) = 15\text{W/m}^2\text{K}$ for $x_1 = X_2 = 0.868\text{m}$, $k = 0\text{K/m}$.

The volume density of losses q depends only on x_2-axis in this example. The computed values of warming u first of all depend on the impute values of function q. The value of the heat transfer coefficient $\alpha(x_1, x_2)$ is lower for $x_1 = X_2$ than for the other parts of $\partial\Omega$. Hence, the screening is more cooled on the part of $\partial\Omega$ with $x_1 = X_1$ than on the part of $\partial\Omega$ with $x_1 = X_2$. Therefore, computed values of warming u are a little lower near the part of $\partial\Omega$ with $x_1 = X_1$ than near the part of $\partial\Omega$ with $x_1 = X_2$ in the x_1-axis direction. Table 2 lists approximate values of the warming at chosen nodes computed by using the box method.

x_2[m]	$X_1 = 0.860$m	$x_1 = 0.864$m	$x_1 = 0.866$m	$X_2 = 0.868$m
$Y_2 = 1.900$	71.369	74.530	74.640	74.585
1.783	26.997	28.214	28.270	28.242
1.666	6.753	7.056	7.068	7.062
1.550	2.247	2.348	2.352	2.350
1.433	1.736	1.814	1.818	1.816
1.316	3.105	3.244	3.250	3.247
1.200	3.601	3.762	3.768	3.765
1.083	3.144	3.285	3.289	3.287
0.966	2.855	2.983	2.989	2.986
0.850	2.962	3.095	3.101	3.098
0.733	3.457	3.612	3.618	3.615
0.616	3.449	3.602	3.608	3.605
0.500	1.503	1.571	1.575	1.573
0.383	2.319	2.422	2.426	2.424
0.266	2.449	3.604	3.612	3.608
0.150	17.614	18.413	18.451	18.432
$Y_1 = 0.033$	60.035	62.690	62.782	62.736

Tab. 2: *The values of the screening warming in K for selected nodes,* $h_1 = 0.002m$ *and* $h_2 = 0.029167m$.

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