Jan Pospíšil

Numerical approaches to parameter estimates in stochastic differential equations driven by fractional Brownian motion

In: Jan Chleboun and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Prague, May 28-31, 2006. Institute of Mathematics AS CR, Prague, 2006. pp. 208–213.

Persistent URL: http://dml.cz/dmlcz/702839

NUMERICAL APPROACHES TO PARAMETER ESTIMATES IN STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION*

Jan Pospíšil

Abstract

We solve the one-dimensional stochastic heat equation driven by fractional Brownian motion using the modified Euler-Maruyama finite differences method. We use the numerical solution as our observation and we show how to estimate the drift parameter from a one path only.

1. Introduction

In this paper we follow [5] where parameter estimates in stochastic evolution equations driven by fractional Brownian motion were studied. The existence and ergodicity of the strictly stationary solution, which is proved there, is crucial for the parameter (especially the drift) estimates. From an observation of the solution on some time interval $[0, T]$, consistent drift estimates are given for $T \to \infty$. Such a constraint is not necessary for the diffusion estimates that can be calculated for $T < \infty$ using the variation of the solution. A presentation of the diffusion estimates is beyond the scope of this paper and only the drift estimates will be considered. In [5], the results are presented in infinite dimension, however, they apply to finite dimensional case as well.

In this paper we give a brief summary of numerical experiments done to support the obtained results in parameter estimates. To simulate the one-dimensional fractional Brownian motion we use the spectral method proposed by Z. Yin in [6]. The problem of numerical simulations of solutions to SDEs and SPDEs has only recently been addressed. Kloeden and Platen wrote a comprehensive book [2] dedicated to numerical solutions to SDEs. Some of the methods were compared by Higham in [1] and by the author in [4]. We solve the one-dimensional SPDE using the Euler-Maruyama finite differences method that has been modified for the purposes of this paper so that the driving process is considered to be a fractional Brownian motion. We will use the numerical solution as our observation and we will show how to estimate the parameters either from a one path or many paths observation.

*This work was partially supported by the GACR Grant 201/04/0750 and by the MSMT Research Plan MSM 4977751301.

2. Parameter estimates in linear SPDEs

In this section we consider the following initial boundary value problem for linear stochastic heat equation

$$\begin{aligned} dX(t,x) &= \alpha\Delta X(t,x)\,dt + \sigma\, dB^H(t), \quad t \geq 0,\ x \in [0,L],\ L > 0\,, \\ X(0,x) &= x_0(x), \quad x \in [0,L], \\ X(t,0) &= X(t,L) = 0, \quad t \geq 0, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $\sigma > 0$ are real constant parameters, $\Delta = \partial^2/\partial x^2$ is the Laplace operator, $x_0 \in L^2([0,L])$ and $B^H(t), t \geq 0$, is a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

Denote by $e_k(x) = \sqrt{2/L}\sin(k\pi x/L)$ the orthonormal[1] basis for the Laplacian on $[0,L]$ and by $\lambda_k = \alpha k^2\pi^2/L^2$ for $k = 1, 2, \ldots$. Let $S(t)$ be a strongly continuous semigroup generated by the Laplacian. Using this notation, it can be shown that $[S(t)e_k](x) = e_k(x)e^{-\lambda_k t}$. In our estimates below, we can use one function from the basis as our test function $z(x)$. Obviously

$$\begin{aligned} \langle X(t,x), z(x)\rangle_V &= \int_0^L X(t,x)z(x)\,dx \\ |X(t,x)|_V^2 = \langle X(t,x), X(t,x)\rangle_V &= \int_0^L X^2(t,x)\,dx. \end{aligned}$$

We will also need to calculate the covariance operator

$$\begin{aligned} &\langle Q_T e_k(x), e_k(x)\rangle_V \\ &\quad= \sigma^2 \int_0^T \int_0^T \left(\int_0^L \left([S(u)e_k](x)\right)\left([S(v)e_k](x)\right)\,dx \right) \phi(u-v)\,du\,dv \\ &\quad= \sigma^2 \int_0^T \int_0^T e^{-\lambda_k(u+v)}\phi(u-v)\,du\,dv, \end{aligned}$$

where $\phi(u) = H(2H-1)|u|^{2H-2}$ is again the kernel.

Let us now introduce an approach to numerically solve (1). We have to point out that the rest of this section is for illustration purposes only, because there is no result in numerical solution to SPDEs driven by fractional Brownian motion so far. The proposed method below is only a natural modification of a similar method for solving SPDEs driven by the Wiener process, but the method is presented without the knowledge of its convergence.

Define, for $i = 0, 1, \ldots, M$, a space grid by $x_i = ik$, where $k = L/M$. Using the finite difference for Laplacian we obtain the following system of SDEs

$$dX(t,x_i) = \frac{\alpha}{k^2}\left(X(t,x_{i+1}) - 2X(t,x_i) + X(t,x_{i-1})\right)\,dt + \sigma\, d\beta_i^H(t),$$

[1] it means that $\int_0^L e_k^2(x)\,dx = 1$

where $\beta_i^H(t)$ are stochastically independent standard fractional Brownian motions and $i = 1, \ldots, M$. We rewrite the system into the matrix form

$$dX(t) = AX(t)\,dt + \sigma\,dB^H(t),$$

where $X(t)$ is now an $M \times 1$ matrix (vector) with elements $X(t, x_i)$, A is an $M \times M$ matrix and $B^H(t)$ an $M \times 1$ vector of the form

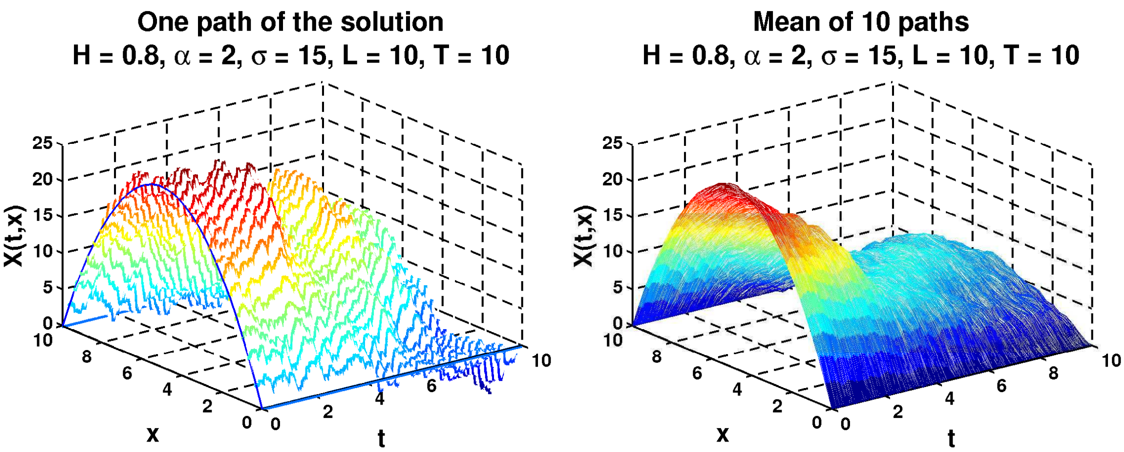
$$A = \frac{\alpha}{k^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}, \quad B^H(t) = \begin{bmatrix} \beta_1^H(t) \\ \beta_2^H(t) \\ \vdots \\ \beta_M^H(t) \end{bmatrix}.$$

Now we can use again the Euler-Maruyama method to generate a sequence (of vectors) (Y_j) approximating the solution $X(t_j)$ by the following *explicit* scheme:

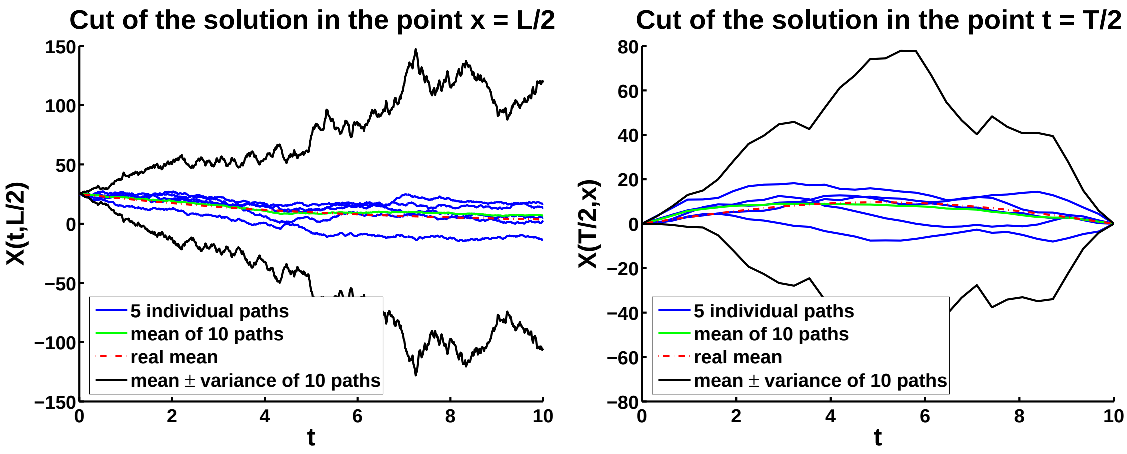
$$\begin{aligned} Y_0 &= x_0 \\ Y_{j+1} &= Y_j + AY_jh + \sigma W_j^H, \quad j = 1, \ldots, N, \end{aligned} \tag{2}$$

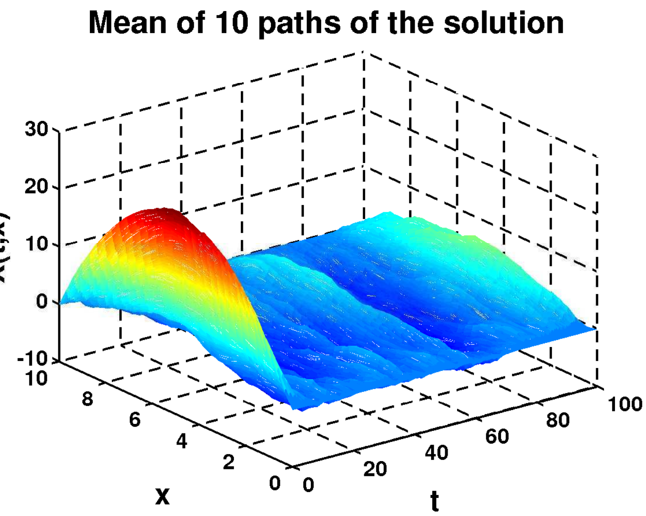
where $W_j^H = B^H(t_{j+1}) - B^H(t_j)$ are the increments of fractional Brownian motion. Like in the previous section, it must be pointed out that it was not the purpose of this paper to study convergence of this numerical scheme.

In the following figure on the left we can see one sample path of the solution $X(t, x)$ to (1) with initial condition $x_0(x) = x(L - x)$, $x \in [0, L]$ for particular values of H, α, σ, L and T. Picture on the right shows the mean of $P = 10$ paths of the solution.



In the next figure we can see the cuts of the solution in the points $x = L/2$ and $t = T/2$ respectively. Several individual paths are drawn together with their mean and variance. Note that some of the path and even the mean could be also negative.





In figure on the right we can see the mean of $P = 10$ paths of the solution to (1) over a larger time interval ($T = 100$). We can see that the influence of the initial condition vanishes rather quickly and the solution converges to the strictly stationary solution.

Remark 2.1. To ensure the convergence in the explicit scheme, it is necessary to control some relation between the time and space steps. For a deterministic PDE, i.e. when $\sigma = 0$, it is known [3] that the relation is the following

$$\alpha \frac{h}{k^2} \leq 1/2. \tag{3}$$

To overcome this difficulty, we can modify (2) to get the *implicit* scheme:

$$\begin{aligned} Y_0 &= x_0 \\ Y_{j+1} &= Y_j + AY_{j+1}h + \sigma W_j^H, \quad j = 1, \dots, N \end{aligned} \tag{4}$$

and calculate Y_{j+1} by solving the following systems of equations

$$(I - Ah)Y_{j+1} = Y_j + \sigma W_j^H, \quad j = 1, \dots, N,$$

where I denotes the identity matrix. Instead of calculating each of the unknown vector Y_j by a separate trivial formula, we must now solve this system of equations to give the values simultaneously. This task is however not very difficult, because the matrix $(I - Ah)$ has a special form, it is a three-diagonal symmetric positive definite matrix. From the theory of PDEs, it is known that the implicit scheme has one big advantage, namely there is no such constraint as (3). In [5] it was believed that something similar holds also for this implicit scheme for SPDEs with additive noise. However, additional numerical experiments showed rather unstable behaviour also for the implicit scheme. Therefore, a relation similar to (3) (depending probably also on H) will have to be taken into account.

In both schemes (2) and (4) there has been a slight modification to take into account the boundary conditions.

Let us now suppose that we have one path observation $X^{x_0}(t,x), t \in [0,T], T \gg 1$, of the solution to (1). For the test purposes we use again the already calculated numerical solution as our observation. From this path we want to estimate the value of the parameter α. We may either consider that we know the parameter σ or we can use its estimate from the previous section. To estimate the parameter α we will again use [5], Theorem 3.2.1.

Let $z(x) = e_1(x) = \sqrt{2/L}\sin(\pi x/L)$, $x \in [0,L]$.

First of all we consider a *reference* equation (1) with the parameter $\alpha = 1$. For this equation we calculate numerically

$$\langle Q_T z, z\rangle = \langle Q_T e_1, e_1\rangle_V = \sigma^2 \int_0^T \int_0^T e^{-\lambda_1(u+v)}\phi(u-v)\,du\,dv.$$

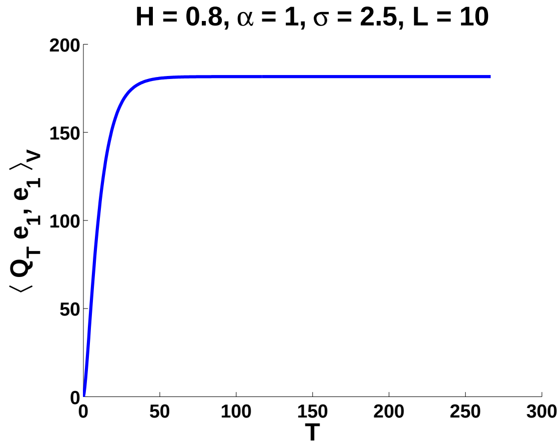
We now turn back to the equation (1) with unknown parameter α (i.e. not necessarily equal to one). From observed path of the solution we have to calculate the average

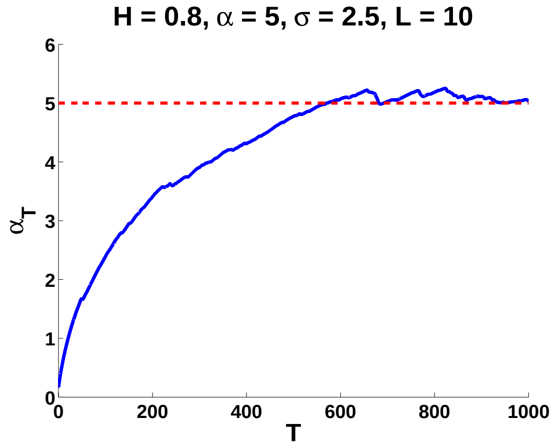
$$\frac{1}{T}\int_0^T \left|\langle X^{x_0}(t,x), z(x)\rangle_V\right|^2 dt = \frac{1}{T}\int_0^T \left|\int_0^L X^{x_0}(t,x)z(x)\,dx\right|^2 dt$$

for sufficiently large T. Using Theorem 3.2.1 from [5], we are now able to calculate the estimate

$$\hat{\alpha}_T := \left(\frac{\langle Q_\infty z, z\rangle_V}{\frac{1}{T}\int_0^T \left|\langle X^{x_0}(t,x), z\rangle_V\right|^2 dt}\right)^{\frac{1}{2H}} = \left(\frac{\langle Q_\infty e_1, e_1\rangle_V}{\frac{1}{T}\int_0^T \left|\int_0^L X^{x_0}(t,x)e_1(x)\,dx\right|^2 dt}\right)^{\frac{1}{2H}}.$$

In the following figure on the left, we can see how $\langle Q_T e_1, e_1\rangle_V$ converges to $\langle Q_\infty e_1, e_1\rangle_V$ for particular values of α, σ, H and L. In picture on the right, we can see how $\hat{\alpha}_T$ converges to the true value of parameter α for large values of T and for particular value of H (σ and L appears in the solution). We can see that a large time has to be considered to obtain a reasonable estimate and due to the fluctuations also some average.





3. Concluding remarks

It has to be pointed out that there are still no appropriate convergence results for the numerical methods used, therefore we used the modified Euler-Maruyama method only for demonstration purposes. Moreover, for some combinations of parameter constants, especially α and σ, the results of these numerical experiments are not so convincing. Hence, further research in the area of numerical solution to stochastic evolution equations driven by fractional Brownian motion is needed.

References

[1] D.J. Higham: *An algorithmic introduction to numerical simulation of stochastic differential equations.* SIAM Rev. **43**, 3, 2001, 525–546 (electronic).

[2] P.E. Kloeden and E. Platen: *Numerical solution of stochastic differential equations.* Applications of Mathematics (New York), Springer-Verlag, Berlin, **23**, 1992.

[3] K.W. Morton and D.F. Mayers: *Numerical solution of partial differential equations. An introduction.* Cambridge University Press, Cambridge, 1994.

[4] J. Pospíšil: *An introduction to numerical methods for solving stochastic differential equations.* In: J. Chleboun, P. Přiklryl, K. Segeth (eds), *Proceedings to Programms and Algorithms of Numerical Mathematics* **11**, Mathematical Institute AS CR, Prague, 2002, pp. 190–201.

[5] J. Pospíšil: *On parameter estimates in stochastic evolution equations driven by fractional Brownian motion.* Ph.D. Thesis, University of West Bohemia in Plzeň, 2005, iv+88.

[6] Z.-M. Yin: *New methods for simulation of fractional Brownian motion.* J. Comput. Phys. **127**, 1, 1996, 66–72.