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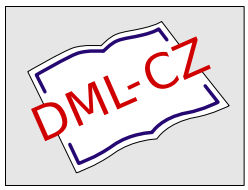

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A FICTITIOUS DOMAIN APPROACH TO THE NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS DEFINED IN STOCHASTIC DOMAINS*

Claudio Canuto, Tomáš Kozubek

1. Introduction

In [2], we present an efficient method for the numerical solution of elliptic PDEs in domains depending on random variables. The key feature is the combination of a fictitious domain approach and a polynomial chaos expansion. The PDE is solved in a larger, fixed domain (the fictitious domain), with the original boundary condition enforced via a Lagrange multiplier acting on a random manifold inside the new domain. A (generalized) Wiener expansion is invoked to convert such a stochastic problem into a deterministic one, depending on an extra set of real variables (the stochastic variables). Discretization is accomplished by standard mixed finite elements in the physical variables and a Galerkin projection method with numerical integration (which coincides with a collocation scheme) in the stochastic variables. A stability and convergence analysis of the method, as well as numerical results, are provided in [2]. The convergence is "spectral" in the polynomial chaos order, in any subdomain which does not contain the random boundaries.

2. Setting of the problem

Let (Ω, F, P) be a complete probability space, where Ω is the set of outcomes, F is the σ-algebra of events and P is the probability measure. For any $\omega \in \Omega$, let $D(\omega) \subset \mathbb{R}^2$ be a bounded domain depending on ω; its boundary $\Gamma(\omega) := \partial D(\omega)$ is assumed to be polygonal or of class $C^{1,1}$, i.e., the boundary is locally represented by functions, whose first derivatives are Lipschitz continuous. We suppose that all domains are contained with their boundaries in a domain $\hat{D} \subset \mathbb{R}^2$, which will serve as the fictitious domain in the fictitious domain formulation (see Figure 1).

For the sake of simplicity, we will be concerned with the following model boundary value problem in $D(\omega)$: Find $u : \overline{D(\omega)} \times \Omega \to \mathbb{R}$ such that almost surely (a.s.) in Ω we have

$$\left\{ \begin{array}{rcl} -\triangle u(\,\cdot\,,\omega) & = & f \ \text{ in } D(\omega), \\ u(\,\cdot\,,\omega) & = & 0 \ \text{ on } \Gamma(\omega), \end{array} \right. \qquad \big(\mathcal{P}(\omega)\big)$$

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where f is a given function in $L^2(\hat{D})$. The case of Neumann or mixed boundary conditions or of random coefficients and data (independent of the random variables describing the domain) could be handled at no extra difficulty.

Solving the discrete problem $\big(\mathcal{P}(\omega)\big)$ for any $\omega \in \Omega$ using, e.g., the finite element method, means that by varying ω we have to: (i) remesh the new domain $D(\omega)$; (ii) assemble the new stiffness matrix and the right hand side vector; (iii) solve the new system of linear equations. Thus the efficiency of solving the discrete problems is crucial. Hereafter, we will explore a fictitious domain method with nonfitted meshes as a possible way to increase efficiency: indeed, this approach avoids completely step (i) and partially step (ii), since the stiffness matrix remains the same for any admissible domain.

3. The fictitious domain (FD) formulation

In this section, we will consider problem $\big(\mathcal{P}(\omega^*)\big)$ for a given event $\omega^* \in \Omega$; we will simplify our notation by setting $D := D(\omega^*)$, $\Gamma := \Gamma(\omega^*)$ and $u = u(\,\cdot\,, \omega^*)$.

Let $\hat{D}$ be the fictitious domain containing $\overline{D}$. The corresponding fictitious domain formulation reads as follows:

$$\left\{\begin{array}{l} \text{Find } (\hat{u}, \lambda) \in V \times M \text{ such that} \\ \int_{\hat{D}} \nabla \hat{u} \cdot \nabla v \, d\mathbf{x} + \langle \lambda, \tau v \rangle_{\Gamma} = \int_{\hat{D}} f v \, d\mathbf{x}, \quad \forall v \in V, \\ \langle \mu, \tau \hat{u} \rangle_{\Gamma} = 0, \quad \forall \mu \in M, \end{array}\right. \tag{$\hat{\mathcal{P}}$}$$

where the symbol $\langle ., . \rangle$ denotes the duality pairing between $M := H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, $\tau : H_0^1(\hat{D}) \to H^{1/2}(\Gamma)$ stands for the trace mapping and V is a closed subspace of $H^1(\hat{D})$. Typical choices for V are: $H^1(\hat{D})$, $H_0^1(\hat{D})$, or $H_P^1(\hat{D}) = \{v \mid \ v \in H^1(\hat{D}),\ v \text{ is periodic on } \partial\hat{D}\}$ if $\hat{D}$ is a cartesian product of intervals.

The reason for introducing the space of the Lagrange multipliers M is to fulfil the requirement that $\hat{u}_{|_D}$ solves $\big(\mathcal{P}(\omega^*)\big)$.

The well-posedness of this problem for any $f \in L^2(\hat{D})$ follows from classical results on abstract saddle-point problems (see [1]). Hence the saddle-point problem $\big(\hat{\mathcal{P}}\big)$ has a unique solution $(\hat{u}, \lambda) \in V \times M$. In addition, $\hat{u}_{|_D} = u$ and $\lambda = \big[\frac{\partial u}{\partial n}\big]$, the jump of the normal derivative of u across Γ.

3.1. Discretization of the FD formulation

Problem $\big(\hat{\mathcal{P}}\big)$ will be approximated by using the mixed finite element method (see [1]). For this purpose the spaces V and M are replaced by suitable finite dimensional subspaces V_h and M_H. More specifically, V_h contains all *continuous piecewise bilinear* functions $\hat{v}_h$ constructed over a uniform rectangulation of $\hat{D}$ and satisfying boundary condition on $\partial\hat{D}$ dependent on the choice of V. Further, M_H contains all *piecewise constant* functions μ_H constructed over a partition of ∂D. For more details we refer to [3].

The resulting algebraic formulation is

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \qquad (\mathbf{P})$$

where $\mathbb{A}$ is *the stiffness matrix*, $\mathbb{B}$ is the matrix coupling the primal variable $\mathbf{u}$ and the Lagrange multiplier $\boldsymbol{\lambda}$, which are the vectors of the nodal values of $\hat{u}_h$ (approximation of $\hat{u}$ from V_h) and of the constant values of λ_H (approximation of λ from M_H), respectively, and $\mathbf{f}$ *is the load vector*.

To solve $(\mathbf{P})$, we use the first equation to eliminate the vector $\mathbf{u} = \mathbb{A}^{-1}(-\mathbb{B}^T\boldsymbol{\lambda}+\mathbf{f})$ from the second one, and we solve the resulting system for $\boldsymbol{\lambda}$, $\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T\boldsymbol{\lambda} = \mathbb{B}\mathbb{A}^{-1}\mathbf{f}$, by a *conjugate gradient method*. The size of $\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T$ is much smaller than the size of $\mathbb{A}$. Generally, we do not need any preconditioning but we are able to construct preconditioners to the Schur complement based on the pseudoinverse and multigrid techniques. The multiplication by $\mathbb{A}^{-1}$ can be realized efficiently, e.g., by Choleski factorization with symmetric approximate minimum degree reordering, multigrid approach, domain decomposition method or by using fast solvers based on *the Fourier Analysis* and *the cyclic reduction*.

4. The stochastic FD formulation

We go back to the stochastic setting. The FD formulation $(\hat{\mathcal{P}})$ suggests the following stochastic FD formulation: Find $\hat{u}(\,\cdot\,,\omega) \in H_0^1(\hat{D})$ and $\lambda(\,\cdot\,,\omega) \in M(\omega) := H^{-1/2}(\Gamma(\omega))$ such that, a.s. in Ω,

$$\begin{cases} \int_{\hat{D}} \nabla\hat{u}(\,\cdot\,,\omega)\cdot\nabla v\,d\mathbf{x} + \langle\lambda(\,\cdot\,,\omega),\tau v\rangle_{\Gamma(\omega)} = \int_{\hat{D}} f v\,d\mathbf{x}, \quad \forall v \in H_0^1(\hat{D}), \\ \langle\mu,\tau\hat{u}(\,\cdot\,,\omega)\rangle_{\Gamma(\omega)} = 0, \quad \forall\mu \in M(\omega). \end{cases} \qquad (\hat{\mathcal{P}}(\omega))$$

We assume that, a.s., $\Gamma(\omega)$ is obtained from a reference $C^{1,1}$ or polygonal boundary Γ_0 as the image of a piecewise smooth invertible mapping $\gamma_0(\omega)$. More precisely, we assume that $\Gamma(\omega) = \gamma_0(\omega)(\Gamma_0)$, where $\gamma_0(\omega)$ belongs to $C^{1,p}(\Gamma_0)$ (the space of all continuous and piecewise continuously differentiable mappings $\gamma : \Gamma_0 \to \mathbb{R}^2$) and its inverse $\gamma_0(\omega)^{-1}$ exists and belongs to $C^{1,p}(\Gamma(\omega))$. The function $\gamma_0 : \Omega \to C^{1,p}(\Gamma_0)$ is assumed to be a random variable belonging to $L^\infty(\Omega, dP; C^{1,p}(\Gamma_0))$, i.e., γ_0 is a jointly measurable function on the Borel sets of $\Gamma_0 \times \Omega$ for which there exists a constant $g_0 > 0$ such that $\|\gamma_0(\omega)\|_{C^{1,p}(\Gamma_0)} \le g_0$ a.s. in Ω; the same occurs for the inverse mapping, i.e., $\|\gamma_0(\omega)^{-1}\|_{C^{1,p}(\Gamma(\omega))} \le g_0$ a.s. in Ω.

Let $\mathbb{E}[X] = \int_\Omega X(\omega)\,dP(\omega)$ be the expected value of a real-valued random variable X. Let $L^2(\Omega, dP) = \{X : \Omega \to \mathbb{R} \,|\, X \text{ is a random variable such that } \mathbb{E}[X^2] < +\infty\}$ be the space of second order random variables over the probability space (Ω, F, P). We denote by $L^2(\Omega, dP;\ H_0^1(\hat{D}))$ the space of the random variables $v : \Omega \to H_0^1(\hat{D})$ (i.e., $v : \hat{D}\times\Omega \to \mathbb{R}$ is jointly measurable and $v(\,\cdot\,,\omega) \in H_0^1(\hat{D})$ a.s. in Ω) with finite second order moment $\mathbb{E}\left[\|v\|^2_{H_0^1(\hat{D})}\right] = \int_{\hat{D}} \mathbb{E}[|\nabla v|^2]\,d\mathbf{x} < +\infty$. The definition of the space $L^2(\Omega, dP; H^{-1/2}(\Gamma_0))$ is similar. Finally, the space $L^2(\Omega, dP; H^{-1/2}(\Gamma))$ is defined as follows: $\mu \in L^2(\Omega, dP; H^{-1/2}(\Gamma))$ means that $\mu_0 \in L^2(\Omega, dP; H^{-1/2}(\Gamma_0))$, where $\mu_0(\omega) \in H^{-1/2}(\Gamma_0)$ is defined a.s. in Ω by the conditions $\langle\mu_0, v_0\rangle_{\Gamma_0} = \langle\mu, v_0\circ\gamma_0^{-1}\rangle_{\Gamma(\omega)}$ for all $v_0 \in H^{1/2}(\Gamma_0)$.

With such notation at hand, the stochastic FD formulation given at the beginning of the section can be made precise as follows: Find $\hat{u} \in L^2(\Omega, dP; H_0^1(\hat{D}))$ and $\lambda \in L^2(\Omega, dP; H^{-1/2}(\Gamma))$ such that

$$\begin{cases} \mathbb{E}\left[\int_{\hat{D}} \nabla\hat{u} \cdot \nabla v \, d\mathbf{x}\right] + \mathbb{E}\left[\langle \lambda, \tau v\rangle_\Gamma\right] = \mathbb{E}\left[\int_{\hat{D}} f v \, d\mathbf{x}\right], \quad \forall v \in L^2(\Omega, dP; H_0^1(\hat{D})), \\ \mathbb{E}\left[\langle \mu, \tau\hat{u}\rangle_\Gamma\right] = 0, \quad \forall \mu \in L^2(\Omega, dP; H^{-1/2}(\Gamma)). \end{cases} \tag{$\hat{\mathcal{P}}^S$}$$

Our next step will be to transform this stochastic problem into a purely deterministic one. This will be accomplished by expanding the random variables into polynomial chaos.

5. (Wiener) polynomial chaos

This section is devoted to recalling some basic facts about polynomial chaos (see, e.g., [4]), as well as to setting the notation.

Let $Y_1(\omega), \ldots, Y_k(\omega), \ldots$ be a sequence of independent standard Gaussian random variables with zero mean and unit variance, i.e., $\mathbb{E}[Y_k] = 0$, $\mathbb{E}[Y_k Y_\ell] = \delta_{k\ell}$ for all $k, \ell \geq 1$. On the other hand, given a real variable y, let $\{H_n(y)\}_{n\geq 0}$ be the sequence of Hermite polynomials on the real line, satisfying

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n(y) H_m(y) \, \mathrm{e}^{-y^2/2} dy = \delta_{nm}, \qquad n, m \geq 0,$$

where δ_{nm} is the Kronecker symbol. Next, denote by $\mathbf{y} = (y_k)_{k\geq 1} \in \mathbb{R}^{\mathbb{N}_0}$ any infinite sequence of real variables, and by $\boldsymbol{\nu} = (\nu_k)_{k\geq 1} \in \mathbb{N}^{\mathbb{N}_0}$ any infinite sequence of integers which is "finite", i.e., such that $\nu_k > 0$ only for a finite number of indices; let $|\boldsymbol{\nu}| = \sum_{k\geq 1} \nu_k$. Define the multidimensional Hermite polynomials of order $|\boldsymbol{\nu}|$ as $H_{\boldsymbol{\nu}}(\mathbf{y}) = \prod_{k=1}^{\infty} H_{\nu_k}(y_k)$; note that the definition is meaningful since $H_0(y) \equiv 1$, hence, $H_{\boldsymbol{\nu}}(\mathbf{y})$ actually depends only on a finite number of components of $\mathbf{y}$. These polynomials are mutually orthonormal, in the following sense:

$$(H_{\boldsymbol{\nu}}, H_{\boldsymbol{\mu}}) := \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_{\nu_k}(y_k) H_{\mu_k}(y_k) \, \mathrm{e}^{-y_k^2/2} dy_k = \delta_{\boldsymbol{\nu}\boldsymbol{\mu}}, \quad \forall \boldsymbol{\nu}, \boldsymbol{\mu}.$$

Setting $\mathbf{Y}(\omega) := (Y_k(\omega))_{k\geq 1}$ for all $\omega \in \Omega$, the random variables $\mathcal{H}_{\boldsymbol{\nu}} : \omega \mapsto H_{\boldsymbol{\nu}}(\mathbf{Y}(\omega))$ are independent and with unit variance, since $\mathbb{E}[\mathcal{H}_{\boldsymbol{\nu}}\mathcal{H}_{\boldsymbol{\mu}}] = (H_{\boldsymbol{\nu}}, H_{\boldsymbol{\mu}}) = \delta_{\boldsymbol{\nu}\boldsymbol{\mu}}$, $\forall \boldsymbol{\nu}, \boldsymbol{\mu}$. They form the so-called *Wiener chaos* (sometimes termed *homogeneous chaos* or *Hermite chaos*). The Cameron-Martin theorem states that the family $\{\mathcal{H}_{\boldsymbol{\nu}}\}$ so defined forms an orthonormal basis of the space $L^2(\Omega, dP)$ of the second order random variables over a Gaussian space. The precise result is as follows.

Theorem 5.1 *Let $\Phi \in L^2(\Omega, dP)$ and let $\Phi_{\boldsymbol{\nu}} = \mathbb{E}[\Phi\mathcal{H}_{\boldsymbol{\nu}}]$ for any finite $\boldsymbol{\nu}$. Then,* $\Phi = \sum_{\boldsymbol{\nu} \text{ finite}} \Phi_{\boldsymbol{\nu}}\mathcal{H}_{\boldsymbol{\nu}}$ in $L^2(\Omega, dP)$.

This means, for instance, that we have $\mathbb{E}\left[\left(\Phi - \sum_{|\boldsymbol{\nu}|\leq N} \Phi_{\boldsymbol{\nu}}\mathcal{H}_{\boldsymbol{\nu}}\right)^2\right] \to 0$ as $N \to \infty$.

The Cameron-Martin theorem states that $\Phi(\omega) = \varphi(\mathbf{Y}(\omega))$, where $\varphi : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}$ is formally defined as $\varphi(\mathbf{y}) = \sum_{\boldsymbol{\nu} \text{ finite}} \Phi_{\boldsymbol{\nu}} H_{\boldsymbol{\nu}}(\mathbf{y})$. In many situations of interest, Φ will be possible to express using a finite number of random variables $Y_k(\omega)$, say using $\mathbf{Y}_K(\omega) := (Y_1(\omega), \dots, Y_K(\omega))$; then, $\Phi(\omega) = \varphi(\mathbf{Y}_K(\omega))$ with $\varphi : \mathbb{R}^K \to \mathbb{R}$ defined as $\varphi(\mathbf{y}) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^K} \Phi_{\boldsymbol{\nu}} H_{\boldsymbol{\nu}}(\mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^K$ and satisfying

$$\frac{1}{(\sqrt{2\pi})^K} \int_{\mathbb{R}^K} \varphi^2(\mathbf{y})\, \mathrm{e}^{-\mathbf{y}^T\mathbf{y}/2} d\mathbf{y} < +\infty.$$

Thus, for our variable Φ, the condition $\Phi \in L^2(\Omega, dP)$ is equivalent to $\varphi \in L^2_\varrho(\mathbb{R}^K)$, where the weight function ϱ is defined as $\varrho(\mathbf{y}) = \frac{1}{(\sqrt{2\pi})^K} \mathrm{e}^{-\mathbf{y}^T\mathbf{y}/2}$. The variable $\mathbf{y}$ will be termed the *stochastic* variable, whereas the spatial variables $\mathbf{x}$ and s will be referred to as the *deterministic* variables.

So far, we have focussed on Gaussian random variables. Similar representations can be given for second order random variables over other probabilistic spaces admitting a density function. The system of orthonormal polynomials which gives rise to a *generalized polynomial chaos*, similar to the Wiener chaos, is determined by the density function; for instance, the uniform density obviously leads to the Legendre polynomials. We refer to [4] for more details.

In general terms, a second order random variable Φ depending on a finite number K of mutually independent real random variables $Y_1(\omega), \dots, Y_K(\omega)$ with zero mean and unit variance with respect to a density function ρ, can be represented as

$$\Phi(\omega) = \varphi(\mathbf{Y}_K(\omega)), \qquad \mathbf{Y}_K(\omega) := (Y_1(\omega), \dots, Y_K(\omega)), \tag{1}$$

where $\varphi = \varphi(\mathbf{y})$ satisfies $\varphi \in L^2_\varrho(\mathbf{I})$: here, $\mathbf{I} = I^K$, where I is the interval of the real line on which ρ is defined, and $\varrho(\mathbf{y}) = \prod_{k=1}^K \rho(y_k)$. Since $L^2_\varrho(\mathbf{I}) = \bigotimes_{k=1}^K L^2_\rho(I)$, a natural orthonormal basis $\{\psi_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu} \in \mathbb{N}^K}$ in this space is provided by the tensor product of a one-dimensional family of orthonormal functions $\{\psi_n\}_{n \in \mathbb{N}}$ in $L^2_\rho(I)$; we assume that these functions are algebraic polynomials, as it occurs in the most relevant situations.

6. The deterministic formulation of $\left(\hat{\mathcal{P}}^S\right)$

We go back to the stochastic formulation $\left(\hat{\mathcal{P}}^S\right)$. We assume that the boundary $\Gamma(\omega)$ of $D(\omega)$ depends on ω via K mutually independent real random variables $Y_1(\omega), \dots, Y_K(\omega)$ with zero mean and unit variance with respect to a density function ρ defined on some interval $I \subseteq \mathbb{R}$. Let $\mathbf{Y}_K(\omega)$ and ϱ be defined as above. Since we assumed in Section 4 that $\Gamma(\omega) = \gamma_0(\omega)(\Gamma_0)$, equation (1) easily yields $\gamma_0(\omega) = \gamma_0^*(\mathbf{Y}_K(\omega))$, where $\gamma_0^* = \gamma_0^*(\mathbf{y})$ is a family of $C^{1,p}(\Gamma_0)$-mappings defined in $\mathbf{I} = I^K$, with inverses $\gamma_0^*(\mathbf{y})^{-1}$ in $C^{1,p}(\Gamma^*(\mathbf{y}))$. Thus, $\Gamma^*(\mathbf{y}) = \gamma_0^*(\mathbf{y})(\Gamma_0)$ is a parametrization of the set of the admissible boundaries of the stochastic domains $D(\omega)$.

Since $\hat{u}$ and λ depend on ω only through $\Gamma(\omega)$, the Doob-Dynkin lemma assures that this dependence takes place via $\mathbf{Y}_K(\omega)$, i.e., we have $\hat{u}(\,\cdot\,, \omega) = \hat{u}^*(\,\cdot\,, \mathbf{Y}_K(\omega))$

and $\lambda(\,\cdot\,,\omega) = \lambda^*(\,\cdot\,,\mathbf{Y}_K(\omega))$, where $\hat{u}^*(\,\cdot\,,\mathbf{y}) \in H_0^1(\hat{D})$ and $\lambda^*(\,\cdot\,,\mathbf{y}) \in H^{-1/2}(\Gamma^*(\mathbf{y}))$, a.e. in $\mathbf{I}$. Condition $\hat{u} \in L^2(\Omega, dP; H_0^1(\hat{D}))$ is then equivalent to $\hat{u}^* \in L^2_\varrho(\mathbf{I}; H_0^1(\hat{D}))$; similarly, $\lambda \in L^2(\Omega, dP; H^{-1/2}(\Gamma))$ is equivalent to $\lambda^* \in L^2_\varrho(\mathbf{I}; H^{-1/2}(\Gamma^*))$ (with obvious meaning of the notation).

We now recall the formula $\mathbb{E}[\Phi] = \int_{\mathbf{I}} \varphi(\mathbf{y})\varrho(\mathbf{y})\,d\mathbf{y}$ which holds for all random variables $\Phi(\omega) = \varphi(\mathbf{Y}_K(\omega))$ with $\varphi \in L^1_\varrho(\mathbf{I})$. By applying this formula several times, we transform the stochastic problem $\left(\hat{\mathcal{P}}^S\right)$ into the following deterministic problem: Find $\hat{u}^* \in L^2_\varrho(\mathbf{I}; H_0^1(\hat{D}))$ and $\lambda^* \in L^2_\varrho(\mathbf{I}; H^{-1/2}(\Gamma^*))$ such that

$$\begin{cases} \int_{\mathbf{I}} \int_{\hat{D}} \nabla \hat{u}^* \cdot \nabla v^*\, d\mathbf{x}\, \varrho(\mathbf{y})\, d\mathbf{y} + \int_{\mathbf{I}} \langle \lambda^*, \tau v^* \rangle_{\Gamma^*(\mathbf{y})} \varrho(\mathbf{y})\, d\mathbf{y} = \int_{\mathbf{I}} \int_{\hat{D}} f v^*\, d\mathbf{x}\, \varrho(\mathbf{y})\, d\mathbf{y}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v^* \in L^2_\varrho(\mathbf{I}; H_0^1(\hat{D})), \quad \left(\hat{\mathcal{P}}^D\right) \\ \int_{\mathbf{I}} \langle \mu^*, \tau \hat{u}^* \rangle_{\Gamma^*(\mathbf{y})} \varrho(\mathbf{y})\, d\mathbf{y} = 0, \quad \forall \mu^* \in L^2_\varrho(\mathbf{I}; H^{-1/2}(\Gamma^*)). \end{cases}$$

7. Discretization of the deterministic formulation

Discretization is accomplished by standard mixed finite elements in the physical variables as in Section 3 and a Galerkin projection method with numerical integration (which coincides with a collocation scheme) in the stochastic variables. Thus instead of solving very large algebraic saddle-point system resulting from the discretization of $\left(\hat{\mathcal{P}}^D\right)$, we will solve n deterministic problems $\left(\hat{\mathcal{P}}\right)$ for n different configurations of the stochastic domain $D(\mathbf{y})$, where n is the number of Gauss (collocation) points $\mathbf{y}_\mathbf{q}$. We can simply parallelize all computations. For more details see [2], where a stability and convergence analysis of the method have been presented. We showed that, in any subdomain that does not contain the random boundaries, the convergence is "spectral" in the polynomial chaos order.

8. Numerical examples

In this section, we illustrate the efficiency of our approach on a model example with nonhomogeneous Dirichlet boundary condition for which we do not know an analytic solution. Therefore basic Monte Carlo (MC) simulation without using any special optimization technique is used to validate the result.

Example 1. Let $\hat{D} := (0,1) \times (0,1)$ be the fictitious domain. Let $\mathbf{y} = (y_1, y_2)$ be a stochastic vector variable, associated with two independent normal distributions $Y_k \sim N[\overline{y}, \sigma]$, $k = 1, 2$, with $\overline{y} = (a+b)/2$, $\sigma = (b-a)/8$; the density function $\rho(y_k)$ is truncated from $\mathbb{R}$ to the interval $I = [a, b]$, $a = 0.25$ and $b = 0.35$. In a polar coordinate system centered at $\mathbf{x}_0 = (0.5, 0.5)$, consider the control points C_k, $k = 0, \ldots, 15$, whose angles are $\varphi_k = k\pi/8$ and whose radii are constant, $r_k = 0.3$, except for $k = 5$ and $k = 6$: for these control points, the radii are given by the variables y_1 and y_2, respectively (see Figure 1). The boundary $\Gamma(\mathbf{y})$ is obtained by connecting the control points via a piecewise Bèzier curve of the second order, identified by the Bèzier triples (M_k, C_{k+1}, M_{k+1}), with $M_k = (C_k + C_{k+1})/2$ for $k = 0, \ldots, 15$ and

$C_{16} = C_0,\ M_{16} = M_0$. All possible configurations of the stochastic domain $D(\mathbf{y})$ are obtained by moving the control nodes C_5 and C_6 along the depicted lines.

We consider the problem

$$\left\{ \ -\triangle u(\mathbf{x},\mathbf{y}) = 60 \ \text{ in } D(\mathbf{y}), \quad u(\mathbf{x},\mathbf{y}) = g \ \text{ on } \Gamma(\mathbf{y}), \right. \qquad \big(\mathcal{P}(\mathbf{y})\big)$$

where $g(\varphi) = 0$, $\varphi \in [-\pi, 0]$ and $g(\varphi) = 1 - \cos(2\varphi)$, $\varphi \in (0, \pi)$.

Figures 2 and 3 provide comparisons between the results produced by basic Monte Carlo (MC) simulation, for different numbers of trials N, and second order Polynomial Chaos (PC) results, obtained by solving 9 independent deterministic problems. The results are depicted along the line $L = \{(x_1, \frac{1}{2})|\ x_1 \in [0,1]\}$. The two vertical dot and dash lines bound the domain $D(\mathbf{y})$ which is fixed in this cross-section for all $\mathbf{y} \in I^2$. While the Monte Carlo approximation of the mean value is good already for moderate numbers of trials, an acceptable approximation of the variance is obtained only with a number of trials in the order of several hundreds.

For more examples and deeper understanding we refer to [2].

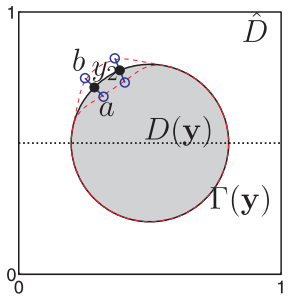

Fig. 1: *Geometry of* $D(\mathbf{y})$.

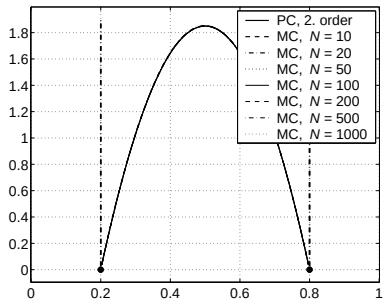

Fig. 2: *PC vs MC, h fixed: mean.*

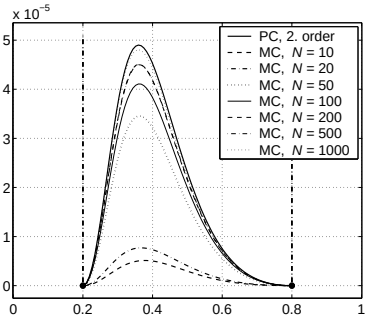

Fig. 3: *PC vs MC, h fixed: variance.*

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