ON STABILIZED FINITE ELEMENT METHOD IN PROBLEMS OF AEROELASTICITY *

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Abstract

In this paper we are concerned with the application of the stabilized finite element method to aero-elastic problems. The main attention is paid to the numerical solution of incompressible viscous two dimensional flow around a flexibly supported solid body. Typical velocities in this case are low enough to assume the air flow being incompressible, on the other hand the Reynolds numbers are very high ($10^4 - 10^6$). As the neccessary mesh refinement for standard Galerkin approximation is clearly unfeasible, several possibilities of stabilization procedures (SUPG - streamline upwind/Petrov-Galerkin, GLS - Galerkin Least Squares) is discussed. Moreover the application of the stabilized method to an aeroelastic problem is presented.

1. Introduction

Fluid-structure interaction problems of fluid flow and elastic structures are studied in many technical disciplines - aeroplane industry (e.g., wings deformations), blade machines (turbines, pumps), civil engineering (stability of bridges), etc. In this paper we focus on the problems of fluid flow past a vibrating airfoil or a blade profile. The research in aero-elasticity or hydro-elasticity focuses on the bilateral interaction between moving fluids and structures (see e.g., [6], [15]). Widely used commercial codes, e.g. such as NASTRAN, FLUENT or ANSYS, solve only special problems of aero-elasticity or hydro-elasticity and mainly in the linear domain. The simplified fluid flow description without resolving of fluid flow patterns is usually employed in these packages.

On the contrary, in this paper the description of the fluid flow around a structure is considered and resolved. As the main attention is paid to the numerical solution of the fluid flow on moving meshes, the structural model is simplified and described as a flexibly supported solid body. The further generalization of the structural model can be included in the method.

The mathematical description of the flow field is represented by the system of the Navier-Stokes equations and the continuity equation (see, e.g., [7]). The incompressible flow problems include wide range of complications typical for numerical solution of partial differential equations. The numerical solution is often found with the aid of the finite element method (FEM). As an alternative to the FEM the finite volume

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method (see, e.g., [8], [9], [11]) also could be used. Nevertheless the extension of FVM to higher order schemes is complicated.

The finite element velocity/pressure pair has to be suitably chosen in order to satisfy the Babuška-Brezzi condition (BB; guarantees the stability of the scheme [17]). The other possibility is to employ so-called fully stabilized scheme (e.g., GLS – Galerkin Least Squares). Nevertheless for both approaches the considered high Reynolds numbers ($10^5 - 10^6$) require a correct stabilisation of the convective term, see, e.g., [1], [14], [18]. Both the proper choice of stabilizaton parameters and adaptive grid refinement are required in order to obtain reasonable solutions. As for the mesh construction, we use the approach from [4].

Further, the structural deformation has to be taken into account. The structural deformation/motion causes the deformation of the computational domain. The domain motion can be captured by the Arbitrary Lagrangian-Eulerian (ALE) method, see, e.g., [16]. Finally, the system of the ALE formulation of the Navier-Stokes equations, the continuity equation and the structural model written as a system of ordinary differential equation have to be coupled.

2. Mathematical description of the problem

Mathematical description of an aero-elastic problem consists of coupled mathematical models of fluid flow and structural deformations or body motion. For low velocities the air flow over a profile can be considered as an incompressible viscous fluid flow. The mathematical description of fluid flow in such a case is the system of the Navier-Stokes equations and the continuity equation. This system is satisfied even in the case when the computational domain is deformed. Nevertheless, the discretization of the Navier-Stokes system in this case meets several difficulties (e.g. grid nodes motion, mesh deformations , etc.). In order to overcome this drawback the system of the Navier-Stokes equations is rewritten in the Arbitrary Lagrangian-Eulerian formulation, which allows the time discretization in the moving meshes case.

In this paper two dimensional fluid flow, which interacts with a flexibly supported solid body, is considered. The rotation and vertical displacement are allowed and described by a system of second order ordinary differential equations. Fluid forces acting on the airfoil are involved in the system.

2.1. Fluid model

We start from the system of the incompressible Navier-Stokes equations in a bounded domain $\Omega_t \subset R^2$ and $t \in (0,T)$

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_i) + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\rho u_i u_j) &= \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j}, \qquad i = 1,2, \qquad \text{in } \Omega_t \\ \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} &= 0, \end{aligned} \tag{1}$$

where ρ is the constant air density, $\mathbf{u}$ is the air velocity with components u_1 and u_2, τ is the stress tensor defined as

$$\tau_{ij} = \rho \left[p\delta_{ij} + \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right], \tag{2}$$

where p denotes the kinematic pressure (i.e. static pressure divided by the air density ρ) and ν denotes the kinematic air viscosity. The system of equations (1) is equipped with boundary and initial conditions (for details see, e.g., [18]). Furthermore, the deformations of the computational domain arising from the structural deformation can be treated with the aid of Arbitrary Lagrangian-Eulerian method. The ALE method is based on the ALE mapping A_t of a reference configuration Ω_{ref} onto the current configuration Ω_t, with the domain velocity $\mathbf{w}_g = \partial A_t/\partial t \circ A_t^{-1}$. In the domain Ω_t the Navier-Stokes system (1) is rewritten in the ALE form [16]

$$\begin{aligned} \frac{D^A}{Dt}(\rho u_i) + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\rho u_j u_i) - \sum_{j=1}^{2} w_{g,j} \frac{\partial}{\partial x_j}(\rho u_i) &= \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j}, \qquad i = 1, 2, \qquad \text{in } \Omega_t \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{3}$$

where $w_{g,j}$ are the components of the domain velocity $\mathbf{w}_g$. The symbol $\frac{D^A}{Dt}$ denotes the ALE derivative, i.e., the derivative with respect to original configuration. The mesh motion as an outcome of ALE mapping is shown in Figure 1.

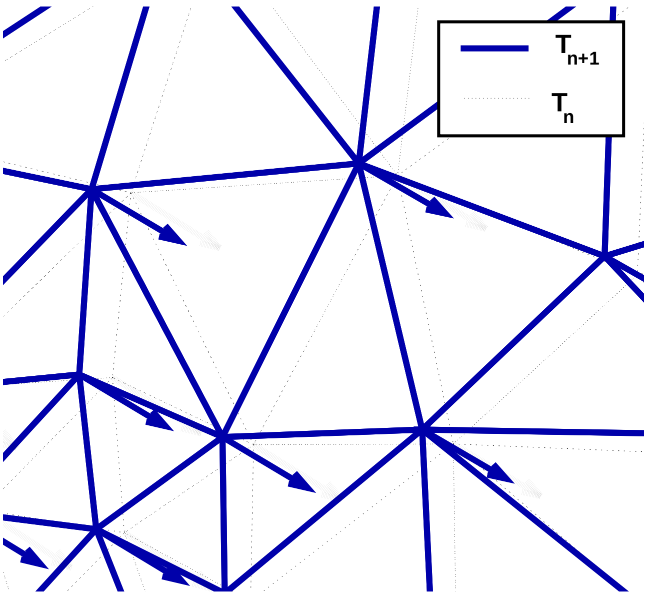

Fig. 1: *Mesh motion and fluid velocity vectors at time moments T_n and T_{n+1}.*

The ALE derivative can be discretized by the following second order difference

$$\frac{D^A u_i}{Dt} \approx \frac{3u_i^{n+1} - 4\hat{u}_i{}^n + \hat{u}_i{}^{n-1}}{2\tau}, \qquad i = 1, 2, \tag{4}$$

where by $\hat{u}_i^k$ we denote the velocity component transformed from the domain Ω_{t_k} onto the domain $\Omega_{t_{n+1}}$, i.e.

$$\hat{u}_i^k = u_i^k \circ A_{t_k} \circ A_{t_{n+1}}^{-1}.$$

Next, the problem (3) is reformulated in a weak sense, which is suitable for the solution with the aid of the finite element method. Defining the velocity spaces $\left(H^1(\Omega)_0\right)^2 \subset X_D \subset X = \left(H^1(\Omega)\right)^2$ (i.e., velocities from X_D are zero on the Dirichlet part Γ_D of boundary $\partial\Omega$) and the pressure space $M \subset L^2(\Omega)$, it is easy to see that the solution $U = (\mathbf{u}, p) \in (X, M)$ of problem (3) satisfies

$$a(U, V) = f(V) \qquad \forall\; V = (\mathbf{v}, q) \in (X_D, M), \tag{5}$$

where

$$\begin{aligned} a(U, V) &= \frac{3}{2\tau}(\mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \Big((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}\Big) - (p, \nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{u}, q), \\ f(V) &= \frac{1}{2\tau}\left(4\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}\right) \end{aligned}$$

and by $(\cdot,\cdot)$ we denote the scalar product in the spaces $L^2(\Omega)$ and $\left[L^2(\Omega)\right]^2$. Moreover, we require that $\mathbf{u}$ satisfies the Dirichlet boundary conditions. The solution $U = (\mathbf{u}, p)$ represents a solution on time level $n+1$, i.e. $\mathbf{u}^{n+1} := \mathbf{u}$ and $p^{n+1} := p$.

Further, the use of the Galerkin FEM restricts the weak formulation from the couple of spaces (X, M) and (X_D, M) to approximate spaces (X_h, M_h) and $(X_{D,h}, M)$: find $U_h \in (X_h, M_h)$ such that

$$a(U_h, V_h) = f(V_h) \qquad \forall\; V_h = (\mathbf{v}, q) \in (X_{D,h}, M_h). \tag{6}$$

The couple (X_h, M_h) of finite element spaces should satisfy the BB condition, which guarantees the stability of the scheme. Nevertheless, this assumption can be lately omitted. Although the Galerkin discretization (6) leads to the second order accuracy, the approximate solution may suffer from spurious oscillations for high Reynolds numbers. In order to avoid this drawback, it is well known that further stabilization is required.

The stabilization introduces the additional stabilizing terms defined by

$$\begin{aligned} L_{h,n}(U, V) &= \sum_K \delta_K \left(\frac{3}{2\tau}\mathbf{u} - \nu\triangle\mathbf{u} + (\tilde{\mathbf{w}}\cdot\nabla)\,\mathbf{u} + \nabla p, \psi_{\tilde{\mathbf{w}}}(V)\right)_K, \\ F_{h,n}(V) &= \sum_K \delta_K \left(\frac{1}{2\tau}\left(4\mathbf{u}^n - \mathbf{u}^{n-1}\right), \psi_{\tilde{\mathbf{w}}}(V)\right)_K, \\ P_{h,n}(U, V) &= \sum_{K\in\tau_h} \tau_k \Big(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}\Big) \end{aligned} \tag{7}$$

where $\tilde{\mathbf{w}}$ stands for transport velocity $\tilde{\mathbf{w}} := \mathbf{u} - \mathbf{w}_g^{(n+1)}$ and n indicates that discretization of the problem is treated on the time level t_n. The linear term $P_{h,n}$ depends on time level only by the choice of constants τ_K. Several suitable choices of the test function $\psi_{\tilde{\mathbf{w}}}(V)$ are allowed.

- Streamline Upwind/Petrov-Galerkin (SUPG) method (see [1], [14])

$$\psi_{\tilde{\mathbf{w}}}(V) = (\tilde{\mathbf{w}} \cdot \nabla)\,\mathbf{v},$$

- Galerkin Least-Squares (GLS) method (see [13], [10]),

$$\psi_{\tilde{\mathbf{w}}}(V) = -\nu\Delta\mathbf{v} + (\tilde{\mathbf{w}} \cdot \nabla)\,\mathbf{v} + \nabla q,$$

- Douglas-Wang (see, e.g., [5]) or Orthogonal Subgrid Scale (OSS) (see, e.g., [3]) method,

$$\psi_{\tilde{\mathbf{w}}}(V) = \nu\Delta\mathbf{v} + (\tilde{\mathbf{w}} \cdot \nabla)\,\mathbf{v} + \nabla q.$$

The parameter δ_K is a function of local (element) Reynolds number Re^{loc} based on the transport velocity $\tilde{\mathbf{w}}$. The proper setting of the stabilization parameters usually strongly depends on the employed couple of finite elements, local mesh quality, transport velocity, local viscosity, etc. The resulting stabilized system also includes so called grad-div stabilization P_h

$$a(U_h, V_h) + L_{h,n}(U_h, V_h) + P_h(U_h, V_h) = f(V_h) + F_{h,n}(V_h). \tag{8}$$

Although the BB-stable pair of the finite element spaces was claimed, in the case of GLS- or OSS-stabilizations, this assumption can be overcome. For details, numerical analysis and selection of parameters δ_K and τ_K see, e.g., [18], [10].

2.2. Structural model

The airfoil can oscillate in the vertical direction and in the angular direction around the so-called elastic axis. Such a motion is usually described by a linearized system of ordinary differential equations, see, e.g., [6]. In order to simulate the airfoil oscillations with large displacements we have to introduce geometrical nonlinearities into the equations of motion (see [12], [6])

$$\begin{aligned} m\ddot{h} + k_{hh}h + S_\alpha\,\ddot{\alpha}\,\cos\alpha - S_\alpha\dot{\alpha}^2\sin\alpha &= -L(t), \\ S_\alpha\ddot{h}\cos\alpha + I_\alpha\ddot{\alpha} + k_{\alpha\alpha}\alpha &= M(t), \end{aligned} \tag{9}$$

where $L(t)$ denotes the aerodynamic lift force (upwards positive), $M(t)$ denotes the aerodynamic torsional moment (clockwise positive), m is the mass of the airfoil, S_α, I_α are the inertia, static moments around the elastic axis, k_{hh} and $k_{\alpha\alpha}$ are the bending/torsional stiffness, α is the rotational displacement around the elastic axis

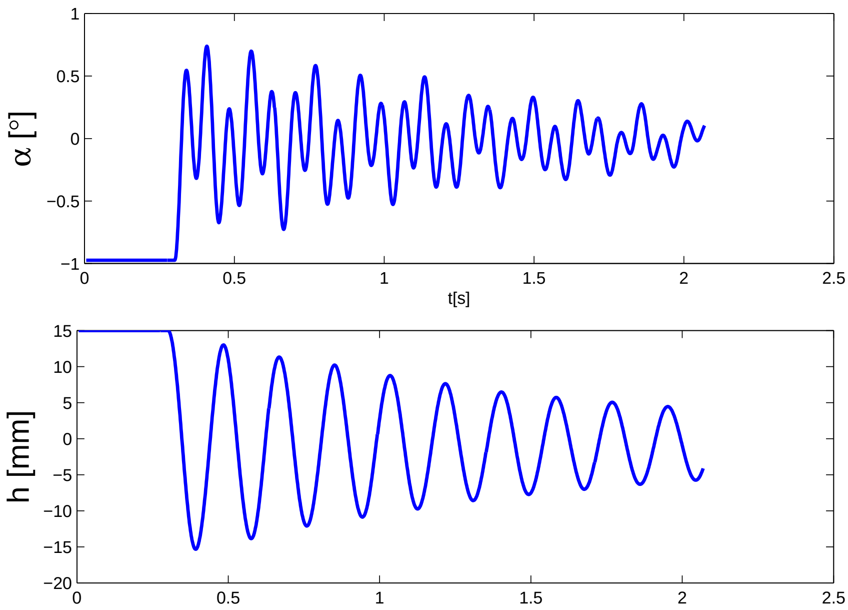

Fig. 2: *Airfoil response for the far field velocity* $U_\infty = 5\ m\,s^{-1}$*, the vibrations arising from the initial deflection are damped at time by the aerodynamical forces.*

(clockwise positive), h is the vertical displacement of the elastic axis (downwards positive) and c is the airfoil chord.

The system of equations (9) is tranformed to a first-order ODE system and then solved by the fourth-order Runge-Kutta method. The aerodynamic lift force L acting in the vertical direction and the torsional moment M are defined by

$$L(t) = -\int_{\Gamma_{Wt}} \sum_{j=1}^{2} \tau_{2j} n_j dS, \qquad M(t) = -\int_{\Gamma_{Wt}} \sum_{i,j=1}^{2} \tau_{ij} n_j r_i^{\mathrm{ort}} dS, \tag{10}$$

where

$$\tau_{ij} = \rho \left[-p\delta_{ij} + \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right], r_1^{\mathrm{ort}} = -(x_2 - x_{T2}),\ r_2^{\mathrm{ort}} = x_1 - x_{T1}, \tag{11}$$

$\mathbf{n} = (n_1, n_2)$ is the unit outer normal to $\partial\Omega_t$ on Γ_{Wt} (pointing into the airfoil) and $x_T = (x_{T1}, x_{T2})$ is the position of the elastic axis (lying in the interior of the airfoil) and ρ is the fluid density.

3. Numerical results

The numerical simulation was performed for structural constants taken from [2]. The considered far field velocity range was $5\ \mathrm{m\,s^{-1}}$ – $50\ \mathrm{m\,s^{-1}}$, the kinematic air viscosity $\nu = 1.5 \cdot 10^{-5}\ \mathrm{m\,s^{-2}}$ and the considered airfoil length was $c = 0.3$ m. The resulting Reynolds number Re then was in range $10^4 - 10^5$. The computation was

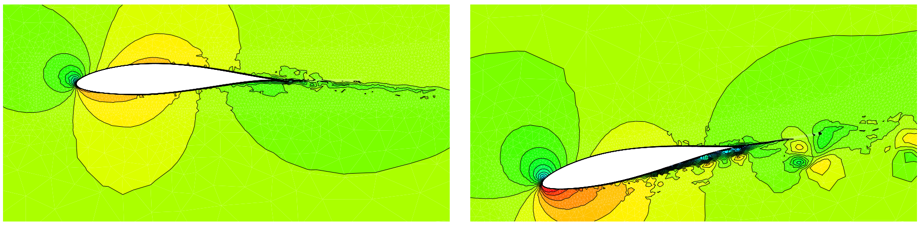

Fig. 3: *Isolines of the velocity magnitude at two different time instants around the vibrating airfoil. The far field velocity in this case was* $U_\infty = 40\ m\,s^{-1}$.

performed for the Taylor-Hood pair of finite elements (piecewise quadratic velocity components/piecewise linear pressure). The SUPG method on a triangular highly anisotropic grid generated by ANGENER (see [4]) was employed. The numerical results show a satisfactory agreement with conclusions of [2]. Figure 2 shows the coupled problem airfoil response for the far field velocity $U_\infty = 5\ \mathrm{m\,s^{-1}}$. The airfoil vibrations starting from the initial deflection are damped to the stable state. Besides the computation of deflection parameters h and α also the time-space approximation of the velocity components u_1, u_2 and the pressure p has to be resolved. Moreover, in every time step the solution of the nonlinear problem has to be found. Figure 3 shows the isolines of the velocity magnitude for two different time instants.

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