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MAXWELL-KELVIN MODEL FOR HIGHLOFT MATERIALS *

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Abstract

Compression behaviour and elastic recovery of highloft materials are described by the Maxwell-Kelvin rheological model. We present an algorithm how to determine input parameters for this rheological model using experimental data.

1. Introduction

It was shown [1] that compressional resistance and elastic recovery (Fig.1) of highloft nonwovens (low density fibrous network structures characterised by a high ratio of thickness to weight per unit area) can be described by a rheological model composed of Maxwell and Kelvin models arranged in series (M-K model), Fig.2.

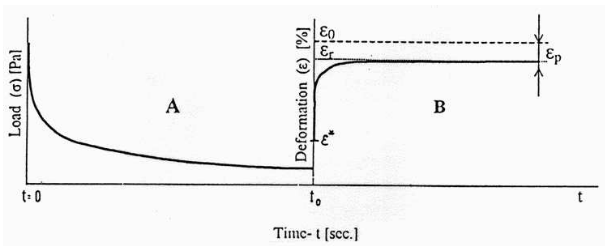

Fig. 1: *Behaviour of a highloft material in loading-recovery test.*

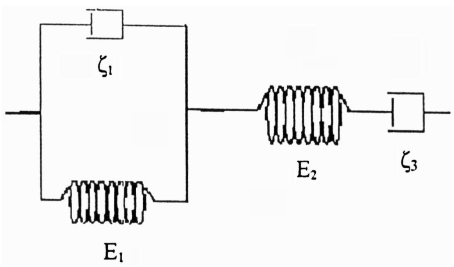

Fig. 2: *Maxwell-Kelvin model.*

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2. Model description

Resulting deformation ε of this model is the sum of deformations ε_1 of the Kelvin model and $(\varepsilon_2+\varepsilon_3)$ of the Maxwell one, where ε_2 describes deformation of its elastic part. Both parts, Maxwell and Kelvin, are under the same stress σ [2]

$$\sigma = E_2\varepsilon_2 = \zeta_3\frac{d\varepsilon_3}{dt} = E_1\varepsilon_1 + \zeta_1\frac{d\varepsilon_1}{dt}, \tag{1}$$

where E_1, E_2 are Young moduli of springs (elastic elements) and ζ_1, ζ_2 are viscosities of viscosity elements. The stress-deformation relation is determined by the differential equation [2]

$$\frac{d^2\sigma}{dt^2} + \Big[E_2(\frac{1}{\zeta_1}+\frac{1}{\zeta_3}) + \frac{E_1}{\zeta_1}\Big]\frac{d\sigma}{dt} + \frac{E_1}{\zeta_1}\frac{E_2}{\zeta_3}\sigma = E_2\frac{d^2\varepsilon}{dt^2} + \frac{E_1}{\zeta_1}E_2\frac{d\varepsilon}{dt}. \tag{2}$$

(A) **Loading modus**

The material is compressed at time $t = 0$ and kept for some time t_0.

As $\varepsilon(t) = \varepsilon_0$ for $t < t_0$, the right side of the equation (2) is equal to zero and we get

$$\frac{d^2\sigma}{dt^2} + \Big[E_2(\frac{1}{\zeta_1}+\frac{1}{\zeta_3}) + \frac{E_1}{\zeta_1}\Big]\frac{d\sigma}{dt} + \frac{E_1}{\zeta_1}\frac{E_2}{\zeta_3}\sigma = 0. \tag{3}$$

The solution of the equation (3) under the initial conditions $\sigma(t=0) = \sigma_0$, $\frac{\sigma}{dt}(t=0) = v_0$ is

$$\sigma(t) = \frac{\sigma_0 k_2 - v_0}{k_2 - k_1}e^{k_1 t} - \frac{\sigma_0 k_1 - v_0}{k_2 - k_1}e^{k_2 t}, \tag{4}$$

where

$$k_{1,2} = -\frac{1}{2}\Big[\frac{1}{\zeta_1}(E_1+E_2) + \frac{E_2}{\zeta_3} \pm \sqrt{\Big[\frac{E_1}{\zeta_1} + E_2(\frac{1}{\zeta_1}+\frac{1}{\zeta_3})\Big]^2 - 4\frac{E_1}{\zeta_1}\frac{E_2}{\zeta_3}}\Big]. \tag{5}$$

(B) **Elastic recovery regime**

At the moment $t = t_0$ the stress $\sigma(t_0) = \sigma^\star$ is removed ($\sigma(t) = 0$ for $t > t_0$) that is folloved by a jump of deformation $\varepsilon(t_0) \to \varepsilon^\star$ where $\varepsilon^\star$ represents the elastic recovery of the material. The plastic or tenacious deformation ε_p of the material is the difference of the initial ε_0 and the final ε_r deformations.

As the stress $\sigma(t) = 0$ for $t > t_0$ the left side of the equation (2) is zero and we get

$$0 = E_2\frac{d^2\varepsilon}{dt^2} + \frac{E_1}{\zeta_1}E_2\frac{d\varepsilon}{dt}. \tag{6}$$

The solution of the differential equation (6) for elastic recovery regime respecting the initial conditions $\sigma(t_0) = 0$ and deformation $\varepsilon(t_0) = \varepsilon^\star$ is

$$\varepsilon(t) = \varepsilon_3(t_0) + \varepsilon_1(t_0)e^{-\frac{E_1}{\zeta_1}(t-t_0)} = \varepsilon_p + (\varepsilon^\star - \varepsilon_p)e^{-\frac{E_1}{\zeta_1}(t-t_0)}. \tag{7}$$

3. Determination of model parameters

Analysis of measured stress-deformation and recovery curves (Fig.1) makes possible to find the input parameters E_1, E_2, ζ_1 and ζ_2 for the model. In experiments we can measure $\sigma_0, v_0, \varepsilon_0, \varepsilon_p, \varepsilon^\star, \varepsilon(t)$ and $\sigma(t)$.

The parameter E_2 is determined from the equation (1)

$$E_2 = \frac{\sigma_0}{\varepsilon_0}. \tag{8}$$

The rate E_1/ζ_1 can be determined from the elastic recovery curve

$$X = \frac{E_1}{\zeta_1} = \frac{1}{t - t_0} \ln \frac{\varepsilon^\star - \varepsilon_p}{\varepsilon(t) - \varepsilon_p}. \tag{9}$$

From the equation (1) we can find the time dependence of deformation of the viscosity element $\varepsilon_3(t)$ of the Kelvin model during the loading regime

$$\varepsilon_3(t) = \frac{1}{\zeta_3(k_2 - k_1)} \Big[\frac{\sigma_0 k_2 - v_0}{k_1}(e^{k_1 t} - 1) - \frac{\sigma_0 k_1 - v_0}{k_2}(e^{k_2 t} - 1) \Big]. \tag{10}$$

If $t \to \infty$ the elastic parts are not deformed and therefore $\varepsilon_0 = \varepsilon(\infty) = \varepsilon_3(\infty)$ and the equation (10) tends to

$$\varepsilon_3(\infty) = \frac{v_0 + \sigma_0[X + E_2(\frac{1}{\zeta_1} + \frac{1}{\zeta_2})]}{X E_2} = \varepsilon_0. \tag{11}$$

From this equation (11) we are able to find values of

$$Y = \frac{1}{\zeta_1} + \frac{1}{\zeta_3} = -\frac{v_0}{\sigma_0 E_2} = -\frac{v_0}{E_2^2 \varepsilon_0}. \tag{12}$$

Using the notation

$$Z = X + E_2 Y, \tag{13}$$

and

$$D = Z^2 - 4X\frac{E_2}{\zeta_3} > 0 \tag{14}$$

the equation (4) is changed into the form

$$\sigma(t) = e^{-\frac{Zt}{2}} \Big[\frac{\sigma_0 Y + 2v_0}{2\sqrt{D}} (e^{\frac{\sqrt{D}t}{2}} - e^{-\frac{\sqrt{D}t}{2}}) + \frac{\sigma_0}{2}(e^{\frac{\sqrt{D}t}{2}}) \Big]. \tag{15}$$

From the equation (15) it is possible to find numerically $\sqrt{D}$ and then values of all input parameters of the Maxwell-Kelvin model:
the parameter ζ_3 from the equation (14)

$$\zeta_3 = \frac{4XE_2}{Z^2 - D}, \tag{16}$$

the parameter ζ_1 from the equation (12)

$$\zeta_1 = \frac{\zeta_3}{Y\zeta_3 - 1}, \tag{17}$$

and the parameter E_1 from the equation (9)

$$E_1 = X\zeta_1. \tag{18}$$

4. Conclusion

Determination of input parameters for the M-K model from experiments enables to find a set of constants characterising highloft materials and to make computer simulations of other theoretical experiments in order to suggest their optimum design.

References

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[2] Sobotka Z.: *Reologie hmot a konstrukcí.* Academia, Praha 1981.