A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR NAVIER-STOKES EQUATIONS APPLIED TO INCOMPRESSIBLE FLOWS *

Pavel Burda, Jaroslav Novotný, Bedřich Sousedík, Jakub Šístek

Abstract

We consider the Navier-Stokes equations for the incompressible flow in channels with forward and backward steps. The paper consists of two main parts. In the first part we investigate a posteriori error estimates for the Stokes and Navier-Stokes equations on two-dimensional polygonal domains. We apply the a posteriori estimates to solve an incompressible flow problem in a domain with corners that cause singularities in the solution. Second part of the paper stands on the result on the asymptotics of the solution in the vicinity of nonconvex internal angles. Using now a priori error estimates we suggest an alternative approach to the adaptive mesh refinement near the corners. This approach gives very precise results in a cheap way. We give numerical results and show the pros and cons of both approaches.

1. Introduction

At present various a posteriori error estimates for the incompressible flow are available, cf. e.g. references in [2]. We stress the aspect of the constant that appears in the estimate – it plays significant role in the adaptive mesh refinement, cf. also [4]. That is why we derive our own a posteriori estimate and trace carefully the role of different constants and their sources, cf. [2] where we derived an a posteriori error estimate for the Stokes problem in a 2-dimensional polygonal domain. In [3] we used similar technique to derive a posteriori error estimates also for 3-dimensional domains. Here we only comment on the accurate determination of these constants. In Section 2 we apply the estimates with the constants found numerically to the adaptive mesh refinement - we solve an incompressible flow problem in a domain with corners that cause singularities in the solution.

In Section 3 we present an alternative approach to the adaptive mesh refinement which is based on the a priori error estimate and on the knowledge of singularity near the corner. For stady Navier-Stokes equations in axially symmetric domains, we proved in [1] that for nonconvex internal angles the velocities near the corners possess an expansion $u(\rho, \vartheta) = \rho^\gamma \varphi(\vartheta)\ +\ \ldots$ (+ smoother terms), where ρ, ϑ are local spherical coordinates. E.g. for the angle $\alpha = \frac{3}{2}\pi$ we have $\gamma = 0.5444837$.

*This research has been supported partly by the grant No. 1960/G1 of the Universities Grant Agency FRVŠ, partly by the grant No. A 2120201/02 of the ASCR agency, and partly by the research project of the Ministry of Education No. J04/98/210000003.

It is well-known that using the standard finite element method on triangles with polynomials of degree $p = 1, 2, 3$ we have the a priori error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leq C\ h^{\gamma-\epsilon} \|u\|_{H^{\gamma+1-\epsilon}(\Omega)},\ \ \epsilon > 0,$$

which cannot be improved by increasing the degree of polynomials.

In this paper the local behaviour of the solution near the singular point is used to design *a priori* a mesh that is adjusted to the shape of the solution.

Subsection 3.1 is devoted to the behaviour of the singularity near the corner. The parts 3.2, 3.3 deal with the impact of the singularity on the refinement of the mesh. We show an example of the mesh with quadratic polynomials for velocity. Then we use this adjusted mesh for the numerical solution of flow in the channel with corners.

2. A posteriori estimates and adaptivity

2.1. A posteriori estimates for the steady Navier-Stokes equations in 2D

Let $\Omega \subset I\!R^2$ be a bounded domain with boundary $\partial\Omega$, $\mathbf{f} = (f_1, f_2)$ the volumetric loads, and ν is kinematic viscosity coefficient. The steady Navier-Stokes problem for the incompressible fluid consists in finding the velocity $\mathbf{v} = (v_1, v_2)$, and pressure p defined in Ω and satisfying

$$(\mathbf{v} \cdot \nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \tag{2}$$

in Ω together with boundary conditions on disjoint parts of the boundary Γ_{in}, Γ_{wall} and Γ_{out} (meaning, in turn, the inlet, the wall, and the outlet part),

$$\mathbf{v} = \mathbf{g} \text{ on } \Gamma_{in} \cup \Gamma_{wall} \tag{3}$$

$$\nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} - p\mathbf{n} = 0 \text{ on } \Gamma_{out} \qquad \text{(“do nothing” boundary condition).} \tag{4}$$

For the finite element discretization we use Hood-Taylor elements P2/P1 where velocities are approximated by quadratic shape functions and pressure by linear shape functions. Suppose that exact solution of the problem is denoted by (v_1, v_2, p) and the approximate finite element solution by (v_1^h, v_2^h, p^h). The exact solution differs from its approximation in the error

$$(e_{v_1}, e_{v_2}, e_p) = (v_1 - v_2^h, v_2 - v_2^h, p - p^h).$$

The norm of a vector function $\mathbf{u} = (u_1, u_2)$ in Sobolev space $\mathsf{H}^1(\Omega_l)$ is standard,

$$\|\mathbf{u}\|^2_{1,\Omega_l} = \sum_{i=1}^{2} \int_{\Omega_l} \left(u_i^2 + \sum_{k=1}^{2} \left(\frac{\partial u_i}{\partial x_k} \right)^2 \right) \mathrm{d}\Omega_l,$$

as well as the norm of $\mathbf{u} = (u_1, u_2)$ in the space $\mathsf{L}^2(\Omega_l)$,

$$\|\mathbf{u}\|_{0,\Omega_l}^2 = \sum_{i=1}^{2} \int_{\Omega_l} u_i^2 \mathrm{d}\Omega_l.$$

For the solution (v_1, v_2, p) we denote

$$\|(v_1, v_2, p)\|_V^2 = \|(v_1, v_2)\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2.$$

The estimate in [2], generalized to the Navier-Stokes equations reads:

$$\|(e_{v_1}, e_{v_2})\|_{1,\Omega}^2 + \|e_p\|_{0,\Omega}^2 \leq \mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega), \tag{5}$$

where

$$\mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega) = C \left[\sum_{l \in K} h_l^2 \int_{\Omega_l} \left(r_1^2 + r_2^2 \right) + \sum_{l \in K} \int_{\Omega_l} r_3^2 \mathrm{d}\Omega \right], \tag{6}$$

where h_l denotes the diameter of the element Ω_l and r_i mean the residuals, cf. [4]. Let us note that according to our practical experience we use only the element residuals. It is not possible to determine the constant C by analysis, the contributions to C can be seen in Theorem 1 of [2]. In [4] we presented a way how to find the constant C by a numerical experiment.

In our adaptive strategy we use the relative error given by the ratio of absolute norm of the solution error, related to unit area of the element Ω_l, $\frac{1}{|\Omega_l|}\, \mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega_l)$, to the solution norm on the whole domain Ω, related to unit area, $\frac{1}{|\Omega|}\, \|(v_1^h, v_2^h, p^h)\|_{V,\Omega}^2$, i.e.

$$\mathcal{R}^2(v_1^h, v_2^h, p^h, \Omega_l) = \frac{|\Omega|\ \mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega_l)}{|\Omega_l|\ \|(v_1^h, v_2^h, p^h)\|_{V,\Omega}^2}. \tag{7}$$

2.2. Model problem

Consider two-dimensional flow of viscous, incompressible fluid described by Navier-Stokes equations in a domain with corner singularity, cf. Fig. 1. Due to symmetry, we solve the problem only on the upper half of the channel, cf. Figs. 1, 2. On the inflow we consider parabolic velocity profile, at the outflow 'do nothing' boundary condition. On the upper wall, no-slip condition and on the lower wall, condition of symmetry. We consider data: $\nu = 0.0001$ m^2/s, $v_{in} = 1$ m/s.

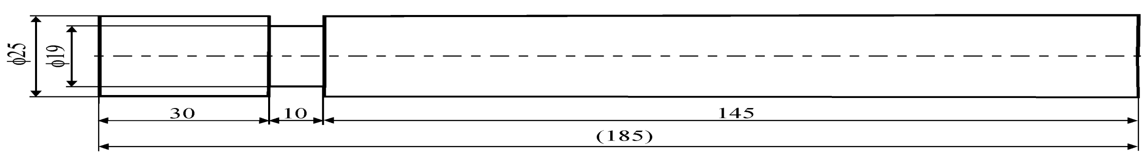

Fig. 1: *Geometry of the channel.*

2.3. Application of estimates to adaptive meshing and numerical results

We start with an initial rough mesh and calculate residuals and relative errors on elements by (7). Elements, where the relative error exceeds 3 % are refined, and new solution together with new error estimates is computed. In [5] we presented figures showing how the mesh is gradually refined. Here we show only some results. The mesh after three successive refinements is shown on Fig. 2.

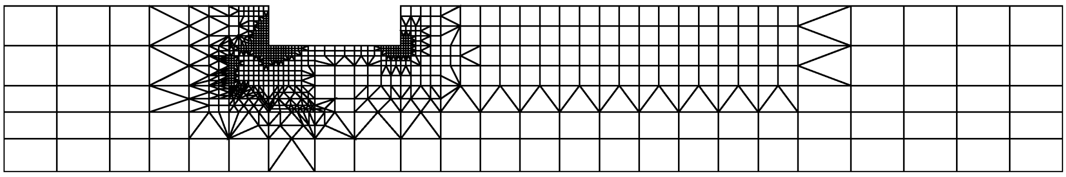

Fig. 2: *Finite element mesh after third refinement.*

The results after three refinements are on Figures 3 and 4. Fig. 3 shows pressure, Fig. 4 velocity component v_y. The singularity is well seen on the v_y component of velocity as well as on the pressure.

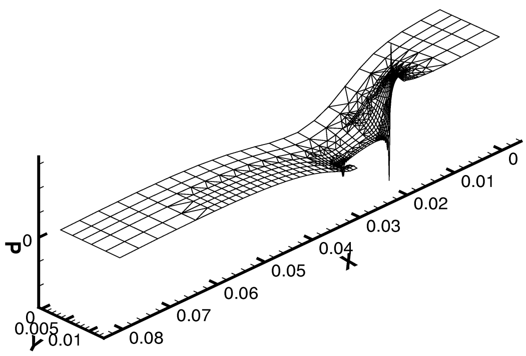

Fig. 3: *Pressure, 3-rd refinement.*

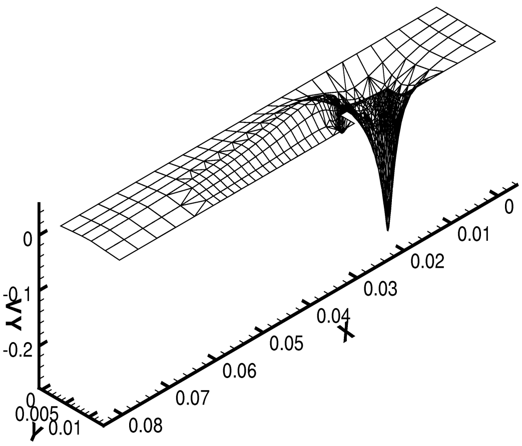

Fig. 4: *Velocity v_y, 3-rd refinement.*

3. A priori estimates and adjusted meshing

3.1. Steady Navier-Stokes equations near the corner

In this subsection we deal with *pipe flow* (axially symmetric). To study the asymptotic behaviour of the solution of the Navier-Stokes equations for incompressible fluids, we utilize the stream function - vorticity formulation, which in cylindrical geometry reads

$$\frac{\partial \omega}{\partial t} + v_1 \frac{\partial \omega}{\partial z} + v_2 \frac{\partial \omega}{\partial r} + v_2 \frac{\omega}{r} = \nu \Big(\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} \Big), \tag{8}$$

$$-r\omega = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{9}$$

$$v_1 = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v_2 = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \tag{10}$$

where r, z are cylindrical coordinates, $v_1 = v_z$, $v_2 = v_r$ are velocity components in z, r directions, respectively, ω is the vorticity, ψ is the stream function, and ν is the viscosity. We assume that all derivatives exist here at least in the generalized sense.

In case of *steady Stokes flow*, substituting ω, v_1, v_2 from (9)–(10) to (8) we get

$$\begin{aligned} 0 \;=\; \nu \, \{ \frac{1}{r} (\frac{\partial^4 \psi}{\partial z^4} + 2 \frac{\partial^4 \psi}{\partial z^2 \partial r^2} + \frac{\partial^4 \psi}{\partial r^4}) - \\ - \frac{1}{r^2} (\frac{\partial^3 \psi}{\partial z^3} + \frac{\partial^3 \psi}{\partial z^2 \partial r} + \frac{\partial^3 \psi}{\partial z \partial r^2} + \frac{\partial^3 \psi}{\partial r^3}) + \frac{3}{r^3}\frac{\partial^2 \psi}{\partial z^2} - \frac{3}{r^4}\frac{\partial \psi}{\partial z} \}. \end{aligned} \tag{11}$$

One example of our solution domain is shown in Fig. 2. Investigating the equation (11) and using the technique of Kondratiev [7], we have shown in [1] that near the corners the solution ψ possesses the expansion

$$\psi(x,y) = \sum_j \sum_{s=0}^{p_j - 1} a_{js} \rho^{-i\lambda_j} \ln^s \rho \cdot \psi_{sj}(\vartheta) + w(x,y), \tag{12}$$

where ρ, ϑ are polar coordinates, w is smooth, and λ_j are the poles of multiplicity p_j of the corresponding resolvent $R(\lambda)$, i means the complex unit. More specifically, for the internal angle $\alpha = \frac{3}{2}\pi$, we proved that the leading term of the expansion for the velocity components is as follows

$$v_l(\rho, \vartheta) = \rho^{0.54448374} \varphi_l(\vartheta) + \dots, \; l = 1, 2, \tag{13}$$

Similar results were proved for the plane flow, cf. [7].

3.2. Finite element solution to steady Navier-Stokes equations

In the finite element solution of the stationary Navier-Stokes equations we intend to use the information about the asymptotics of the flow near the singular point, in order to suggest adequate local mesh refinement.

In [1] we have shown that the behaviour near the singular point, of the axisymmetric flow and of the plane flow, are the same. So in what follows, for simplicity we deal with the plane flow, in the domain Ω which has the same shape as in Fig. 2. We return to the Navier-Stokes equations (1), (2). For the finite element approximation we take Ω a polygon in $\mathbb{R}^2$ and use Hood-Taylor elements, as in Section 2.1.

3.3. Refinement of FEM mesh adjusted to singularity

Near the corner where the angle is $\frac{3}{2}\pi$, velocities have the leading term in the expansion as given in (13). There ρ is the distance from the corner, ϑ the angle. Note that $\frac{\partial v_i(\rho,\vartheta)}{\partial \rho} \rightarrow \infty$ for $\rho \rightarrow 0$.

Now, in this section we assume the Stokes flow, for simplicity. A priori estimate of the finite element error is (cf. [6])

$$\|\nabla(\mathbf{v}-\mathbf{v_h})\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \leq C\Big[\Big(\sum_T h_T^{2k} \mid \mathbf{v} \mid^2_{H^{k+1}(T)}\Big)^{1/2} + \Big(\sum_T h_T^{2k} \mid p \mid^2_{H^k(T)}\Big)^{1/2}\Big],$$

where $k = 2$. Taking into account the expansion (13), we can derive the rough estimate (cf. [1])

$$\mid \mathbf{v} \mid^2_{H^{k+1}(T)} \approx C \int_{r_T-h_T}^{r_T} \rho^{2(\gamma-k-1)}\ \rho\ d\rho \approx C\ r_T^{2(\gamma-k)}, \tag{14}$$

where h_T is the diameter of the triangle T of a triangulation $\mathcal{T}_h$, and r_T is the distance of the element T from the corner. So, in order to get the error estimate of an auxiliary order $O(h^k)$, we should guarantee on each element T,

$$h_T^{2k}\ r_T^{2(\gamma-k)} \approx h^{2k}. \tag{15}$$

This lead us in [1] to an algorithm for generating the mesh near the corner:

Algorithm. *Let r_1 be the distance of the large element from the corner. For given auxiliary stepsize h we compute recursively:*
for $i = 1, 2, \ldots, N : h_i = h \cdot (r_i)^{1-\frac{\gamma}{k}},\quad r_{i+1} = r_i - h_i$.

Consider again two-dimensional flow as specified in Section 2.2.

The algorithm for mesh refinement described in previous section is applied to the corner where the channel or tube suddenly decreases the diameter (forward step in Figs. 1, 2).

We have $k = 2$, $\gamma = 0,5444837$, and we start with $r_1 = 0,25\ mm$, $h = 0,1732\ mm$. This corresponds to the contribution cca 3% of individual elements to the global error. This way we get ten diameters of elements, cf. Tab. 1.

i	$r_i(mm)$	$h_i(mm)$
1	0.25000	0.06316
2	0.18685	0.05110
3	0.13575	0.04050
4	0.09526	0.03129
5	0.06396	0.02342
6	0.04054	0.01681
7	0.02374	0.01138
8	0.01235	0.07077
9	0.00528	0.00381
10	0.00147	0.00147

Tab. 1: *Resulting refinement.*

3.4. Design of the mesh detail near the corner

Using the parameters from Table 1, J. Šístek [8] suggested three variants of the mesh refinement near the corner (Figures 5-7). Mesh No. 1 was a classic used

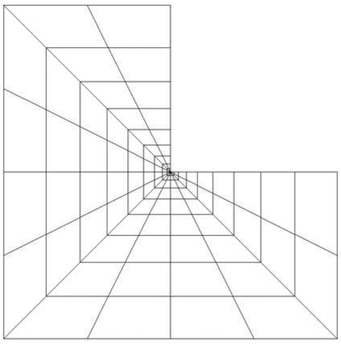

Fig. 5: *Mesh No. 1.*

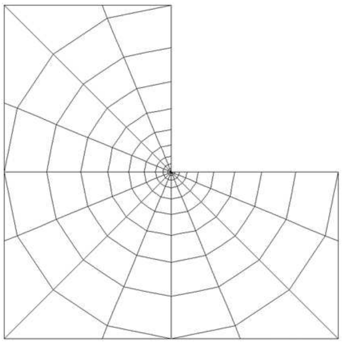

Fig. 6: *Mesh No. 2.*

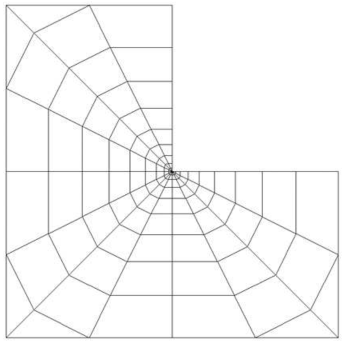

Fig. 7: *Mesh No. 3.*

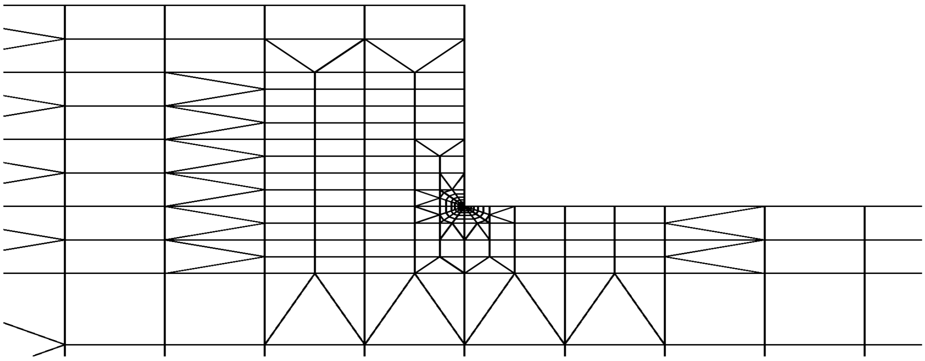

Fig. 8: *The mesh – detail.*

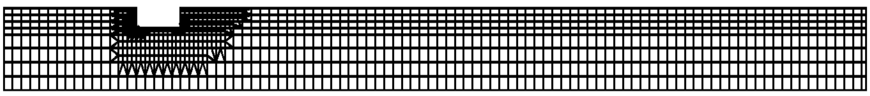

Fig. 9: *The whole computational mesh.*

before. He suggested two other variants on order to fit better to the algorithm of mesh refinement, esp. to its polar coordinates nature (Figures 6-7). In Figs. 8-9 we present the whole computational mesh and its detail, using Mesh 3.

3.5. Evaluation of the approximation error

We use a posteriori error estimate (5). To evaluate the error on elements we use now the modified absolute error, defined as

$$\mathcal{A}_m^2(v_1^h, v_2^h, p^h, \Omega_l) = \frac{|\Omega|\mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega_l)}{|\overline{\Omega}_l|\|(v_1^h, v_2^h, p^h)\|_{V,\Omega}^2}, \tag{16}$$

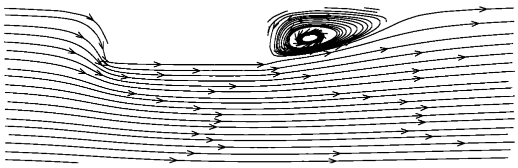

Fig. 10: *Streamlines at corners.*

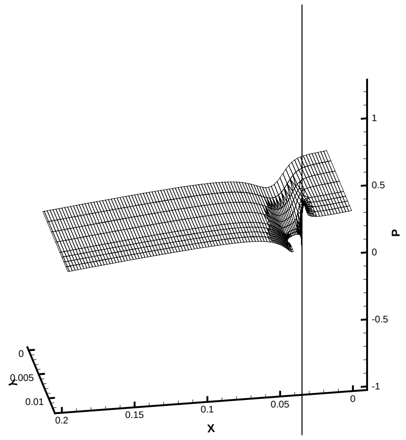

Fig. 11: *Pressure p.*

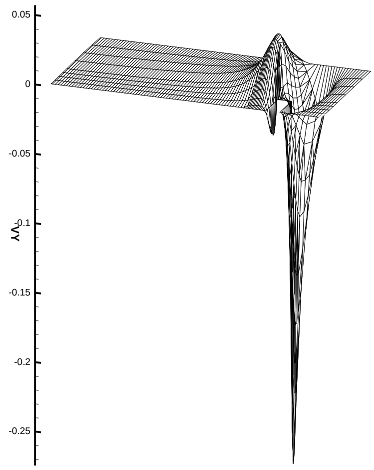

Fig. 12: *Velocity comp.* v_y.

where $|\Omega|$ is the area of the whole domain and $|\overline{\Omega}_l|$ is the mean area of elements obtained as $|\overline{\Omega}_l| = \frac{|\Omega|}{n}$. Here n means the number of all elements in the domain.

3.6. Numerical results of adjusted mesh approach

On Figures 10 - 12 we present the graphical output of entities that characterize the flow in the channel. On Figs. 11, 12 we observe how strong the singularity both for velocity and pressure is (note that here the flow is from the right to the left, to have better view). Figure 14 shows that the location of the peak of the singularity of velocity is outside the patch where the refinement was done. One can see that, again it is pressure what decides, cf. Fig. 11. On Fig. 13 we show the errors on elements of Mesh No. 2. The results on Figures 11, 12 are much more precise than those on Figures 3 and 4.

4. Conclusions

Pros and cons of a priori approach:

- distribution of the error on elements is quite uniform (esp. for Mesh 2)
- strength of singularity (both for velocity and pressure) is very well captured
- the algorithm of adjusted mesh refinement has been confirmed

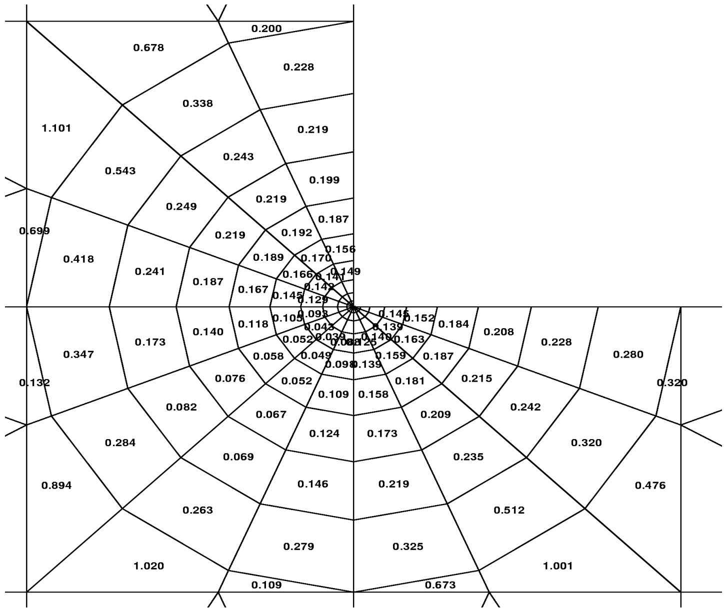

Fig. 13: *Errors on elements near the corner (forward step).*

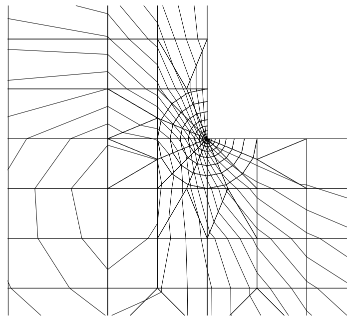

Fig. 14: *Isolines of v_y.*

- the efficiency: desired precision needs only one run (compared with adaptive approach - the same precision would need approx. 10 refinement runs)
- suitable only for singularities from "geometry",

Pros and cons of a posteriori estimate approach:

- adaptive approach is much more robust than that with adjusted mesh.
- in case of corner singularities it is more expensive

Nevertheless, efficient refinement near the corners still remains a challenge.

References

[1] P. Burda: *On the F.E.M. for the Navier-Stokes equations in domains with corner singularities.* In: Křížek M., et al. (eds), Finite Element Methods, Supeconvergence, … A Posteriori Estimates, Marcel Dekker, New York, 1998, 41–52.

[2] P. Burda: *An a posteriori error estimate for the Stokes problem in a polygonal domain using Hood-Taylor elements.* In: Neittaanmäki P., et al. (eds), ENUMATH 99, Proc. Conf. Numerical Mathematics and Advanced Applications. World Scientific, Singapore, 2000, 448–455.

[3] P. Burda: *A posteriori error estimates for the Stokes flow in 2D and 3D domains.* In: Neittaanmäki P., Křížek M. (eds), Finite Element Methods, 3D, GAKUTO Intern. Ser., Math. Sci. and Appl., **15**) Gakkotosho, Tokyo, 2001, pp. 34–44.

[4] P. Burda, J. Novotný, B. Sousedík: *Adaptive mesh refinement based on a posteriori error estimates for the Stokes flow in a 2D Problem.* In: F. Brezzi et al. (eds), ENUMATH 2001, Proc. Conf. Numerical Mathematics and Advanced Applications, Springer Verlag, 2003, 681–690.

[5] P. Burda, J. Novotný, M. Jelínek, B. Sousedík: *Adaptive mesh refinement versus refinement adjusted to singularity in 2D flow problem.* In: Programy a algoritmy numerické matematiky 11, MÚ AV ČR, Praha, 2002, 33–40.

[6] V. Girault, P.G. Raviart: *Finite element method for Navier-Stokes equations.* Springer, Berlin, 1986.

[7] V.A. Kondratiev: *Asimptotika rešenija.* Uravnienija Nav'je-Stoksa v okrestnosti uglovoj točki granicy, Prikl. Mat. i Mech., **1**, 1967, 119–123.

[8] J. Šístek: *Solution of the fluid flow in a tube with singularity using suitable mesh refinement*, (in Czech). Student Conference STČ, ČVUT Praha, 2003.