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A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAYS

Pavol Chocholatý

Abstract

It is well-known that the environments of most natural populations change with time and that such changes induce variation in the growth characteristics of population which is often modelled by delay differential equations, usually with time-varying delay. The purpose of this article is to derive a numerical solution of the delay differential system with continuously distributed delays based on a composition of p-step methods ($p = 1, 2, 3, 4, 5$) and quadrature formulas. Some numerical results are presented compared to the known ones.

1 Introduction

Delay differential equations (DDE), also called functional differential equations, time-delay systems, are widely used for describing and mathematical modeling of various processes and systems in various applied problems. Theoretical aspects of DDE theory are elaborate with almost the same completeness as corresponding parts of ordinary differential equations (ODE) theory. However, unlike ODE, even for linear DDE there are no general methods of finding solutions in explicit forms. So elaboration of numerical methods for DDE is a very important problem. Presently, various specific numerical methods are constructed for solving specific DDEs. Most of investigations are devoted to numerical methods for systems with discrete delays and Volterra integro-differential equations, see e.g. [2]. In the framework of such approach one can construct for DDE analogies of all known numerical methods of ODE case. Moreover the coefficients of the corresponding numerical methods are the same in ODE and DDE cases. The approach described in this paper was applied to numerical solution of delay differential equations with distributed delays.

2 Delay differential equations

ODE – the Cauchy problem considered is

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \geq t_0, \\ x(t_0) &= x_0, \end{aligned}$$

where f is a nonlinear function, assumed to be Lipschitz continuous in x and x_0 is a given initial value. We consider a single equation because simpler notation can be used in this case. Everything in this paper can be generalized to systems of differential equations in a straightforward manner. There is a variety of applications which are more naturally modelled as functional differential equations rather then ODEs. In such equations dependent variables are concurrently evaluated at more then one value of the independent variable.

A generic form for such equations considered here is

$$x'(t) = f(t, x(t), x(\omega_1(t)), x(\omega_2(t))), \quad t \geq t_0$$

where $\omega_1(t) < t < \omega_2(t)$. When there are $\omega_1(t)$ terms (corresponding to so-called delays), but no $\omega_2(t)$ terms (corresponding to so-called advances), then the functional differential equation is called a DDE.

DDE – the Cauchy problem considered is

$$\begin{aligned} x'(t) &= f(t, x(t+\tau_1), \dots, x(t+\tau_k)), & t \geq t_0, \\ x(t) &= \Psi(t), & t \leq t_0, \end{aligned}$$

f is a function where t is the independent variable (usually considered as time), dependent variable $x(t)$ is a phase vector, and dependent variable $x(t+\tau_i)$, $\tau_i \in \langle -r, 0 \rangle$, $i = 1, 2, \dots, k$ is the function which characterizes an influence of the pre-history of the phase vector on the dynamics of the system. A class of DDE with constant delay τ_i, $i = 1, 2, \dots, k$ is called DDEs with discrete delay. Supposed that delay $\tau_i = \tau_i(t)$ we speak about differential equations with discrete time-varying delay. One can see, it is insufficient to know only the initial value to define the phase vector $x(t)$, but it is also necessary to define an initial function (initial pre-history) $\Psi(t)$. So, DDEs are generalizations of ODEs when the velocity $x'(t)$ of a process depends also on the pre-history $x(t+\tau_i)$.

Delay can also be distributed

$$x'(t) = f(t, x(t), \int\limits_{\tau(t)}^{0} \alpha(t, s, x(t+s)) \mathrm{d}s).$$

The Volterra integro-differential equations

$$x'(t) = f(t, x(t), \int\limits_{0}^{t} \beta(t, s, x(s)) \mathrm{d}s)$$

represent a special class of DDE with distributed delays. However, in practical models distributed delays occur rather than concentrated one. Here, we study an equation which includes as special cases logistic equations with both “concentrated” and “continuous” delays.

Let us consider some of them, the delay logistic equation

$$x'(t) = r(t)x(t)\left(1 - \frac{x(g(t))}{K}\right), \quad g(t) \leq t$$

describes a delay population model and is known as Hutchinson's equation, if r and K are positive constants and function $g(t) = t + \tau$ for negative constant τ. The oscillation of solutions of this equation was studied by Gopalsamy and Zhang [4]. It is well-known that the environments of most natural populations change with time and that such changes induce variation in the growth characteristics of populations. For instance, favourable weather conditions stimulate an increase in the body size an reproduction while unfavourable environments can lead to a decline in the birth rate and increase in mortality. Temporal variations of an environment of a population are usually incorporated in model systems by the introduction of time-dependent parameters in governing equations. The reader is referred to the monograph of Gopalsamy [1] for an extensive discussion of multispecies dynamics in temporally uniform environments governed by autonomous differential equations with continuously distributed delays.

Here, we conclude the Lotka-Volterra-like predator-prey model, which is a system of two delay differential equations with distributed delay. This system is frequently used to describe the dynamics of biological systems in which two species interact, one a predator $x_1(t)$ and one its prey $x_2(t)$

$$x_1'(t) = \left[c - k_1 x_2(t) - \int_{-\tau}^{0} \alpha_1(x_2(t+s))\mathrm{d}s \right] x_1(t)$$

$$x_2'(t) = \left[-c + k_2 x_1(t) - \int_{-\tau}^{0} \alpha_2(x_1(t+s))\mathrm{d}s \right] x_2(t)$$

where $x_1'(t)$ and $x_2'(t)$ represent the growth of the two populations against time, c, k_i, α_i are parameters representing the interaction of the two species.

Also, one of the models for human immunodeficiency virus (HIV) in a homogeneously mixed single-gender group with distributed waiting times can be described using equations with distributed delay. Such DDEs with distributed delay arise in a number of other scientific applications. In general, it is difficult to obtain solutions of such equations for arbitrary choices of parameters. We usually resort to a numerical method for obtaining an approximate solution of the problem. And we must obtain classes of numerical methods for a specific choice of the parameters. In [3], Kim and Pimenov have proposed an exact solution to a system of DDEs with distributed delay. Then, by considering the maximum absolute errors in the solution at grid points and tabulated in tables for different choices of step size, we can conclude how further presented approaches produce accurate results in comparison with those exact ones.

A solution $(x_1(t), x_2(t))^{\mathrm{T}}$ of a nonlinear system of two DDEs with distributed delays

$$x_1'(t) = -\frac{1}{2}\int_{-\pi}^{0} x_1(t+s)\mathrm{d}s + \frac{2x_1(t) - \frac{\pi}{2}x_2(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}$$

$$x_2'(t) = -\frac{1}{2}\int_{-\pi}^{0} x_2(t+s)\mathrm{d}s + \frac{2x_2(t) + \frac{\pi}{2}x_1(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}$$

corresponding to an initial function

$$\begin{matrix} \Psi_1(s) = (1+s)\cos(1+s) \\ \Psi_2(s) = (1+s)\sin(1+s) \end{matrix}, \qquad \text{for} \quad -\pi \le s \le 0,$$

and an initial time $t_0 = 1$, has the form

$$\begin{matrix} x_1(t) = t\cos(t) \\ x_2(t) = t\sin(t) \end{matrix}, \qquad \text{for} \quad t \ge 1.$$

3 A numerical approach

The most popular numerical approaches for solving Cauchy problem of ODEs are called finite difference methods. Approximate values are obtained for the solution at a set of grid points $\{t_i : i = 0, 1, 2, \dots, N\}$ and the approximate value at each point t_{i+1} is obtained by using some of values obtained in previous steps. Single-step methods for solving ODEs require only a knowledge of the numerical solution at the point t_i in order to compute the next value at the point t_{i+1}. The best known one-step methods are Euler's methods (explicit, implicit), trapezoidal method and higher-order Runge-Kutta methods. This has obvious advantages over the p-step method that use several past values computed at the points $t_{i-p+1}\dots, t_i$. And p-step methods are used to produce predictor-corrector algorithms known as Milne's method, Milne-Simpson's method and higher-order Adams-Moulton methods.

Now, we analyze numerical methods for evaluating definite integrals $I(f) = \int_a^b f(t)\mathrm{d}t$. Most such integrals cannot be evaluated explicitly and with many others it is often faster to integrate them numerically rather than evaluating them exactly using a complicated antiderivative of $f(t)$. There are many numerical methods for evaluating $I(f)$, but most can be made to fit within the following framework. For integrand $f(t)$, find an approximating family $\{f_n : n = 1, 2, \dots\}$ and define $I_n(f) = \int_a^b f_n(t)\mathrm{d}t = I(f_n)$ and we usually require the approximations $f_n(t)$ to satisfy $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Most numerical integration formulas are based on defining $f_n(t)$ by using polynomial or piecewise polynomial interpolation. Formulas using such interpolation with evenly spaced grid points are the composite trapezoidal rule and the composite Simpson's rule, which are the first two cases $(k = 1, 2)$ of the Newton-Cotes integration formula. The complete formula for $k = 3$ is called the composite three-eights rule. Integration formulas in which one or both endpoints are missing are called open Newton-Cotes formulas, and the previous formulas are called closed formulas. Each Newton-Cotes formula ($k = 1, 2, 3$) can be used to construct a composite method with mentioned p-step methods ($p = 1, 2, 3, 4, 5$). The next question of interest is whether the obtained approximate values for the solution of our two DDEs with distributed delays converges to the exact ones.

So, the simplest way to solve our model equation

$$x'(t) = I + F(x(t)),$$

with initial function $\Psi(s)$, by applying Simpson's rule and explicit Euler's methods is outlined by the following algorithm:

```
h=pi/N;
for i=1:N+1
x{i}= Ψ(-pi+(i-1)*h);
end
for i=N+1:N+N*7
if(mod(i,2) ≈=0)
I=0;
for j=i-N:i-2
if(mod(j,2) ≈=0)
SUM=(h/3)*(x{j}+4*x{j+1}+x{j+2});
I=I+SUM;
end
end
end
if(mod(i,2) ==0)
I=0;
for j=i-N:i-2
if(mod(j,2) ==0)
SUM=(h/3)*(x{j}+4*x{j+1}+x{j+2});
I=I+SUM;
end
end
end
x{i+1}=x{i}+h*(I+F(x{i}));
end
```

4 Numerical experiments

In order to test the viability of the proposed composite methods and to demonstrate its convergence computationally we have considered several tests with some steps, to assess the convergence property and efficiency of methods developed in Section 3.

For instance, the idea is to calculate the numerical solution by Milne's predictor-corrector method with composite Simpson rule on an equidistant mesh $t_{i+1} - t_i = h$. We discretize the time-interval $t \in \langle 1, 1 + 7\pi \rangle$ on N subintervals in order to obtain the approximate values for the solution at the grid points t_i. Here we are only interested in showing the errors for the solution at some grid points. To obtain an approximation of maximum errors at the endpoint, we compare the numerical solutions on two different meshes having N and $10N$ subintervals, respectively, and having transition points $1 + c\pi/3$, $c = 1, 8, 14, 21$ at the same place (!irrational numbers) in both the meshes, with the exact solution of this problem. Numerical results are given in Table 1 for several values of h. The answers are given at only a few points, rather than at all points at which they were calculated.

This problem has been solved using our methods with different values of N, $(30, 300, 3000, 30000)$ and compared with exact solution. To illustrate the applicability and effectiveness of "the best" composite method obtained by Milne-Simpson's method of 5-th order and Simpson rule, we compare our results with exact ones in Table 2.

Tab. 1

points	$1 + \pi/3$	$1 + 8\pi/3$	$1 + 14\pi/3$	$1 + 21\pi/3$
x_1, $h = \pi/300$	-0.938061642	-9.386079514	-15.69957230	-12.67558987
x_1, $h = \pi/3000$	-0.938829650	-9.369056409	-15.64897895	-12.44655301
x_1, exact	-0.938812239	-9.367137558	-15.64332592	-12.42217059
x_2, $h = \pi/300$	1.819802987	0.476277848	0.838558832	-19.34452329
x_2, $h = \pi/3000$	1.819233853	0.445332299	0.748045279	-19.34682070
x_2, exact	1.819244182	0.442434527	0.738875399	-19.34638443

Tab. 2

grid $T(j)$	exact x_1	approx. x_1	exact x_2	approx. x_2
1	-0.93881224	-0.93882868	1.81924418	1.81923445
2	2.37949667	2.37914423	-4.61102368	-4.61175244
3	-3.82018110	-3.81898822	7.40280318	7.40510074
4	5.26086553	5.25835117	-10.19458268	-10.19927812
5	-6.70154995	-6.69723186	12.98662181	12.99428392
6	8.14223438	8.13562983	-15.77814168	-15.79011818
7	-9.58291881	-9.57354472	18.56992118	18.58678092

This table presents approximate values to the solution of our DDEs with distributed delays computed by step $h = \pi/3000$ at only a few points $T(j) = 1 + (3j-2)\pi/3$, $j = 1, 2, 3, 4, 5, 6, 7$ rather than at all points at which they were calculated.

References

[1] Gopalsamy, K.: *Stability and oscillation in delay differential equations of population dynamics.* Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

[2] Hairer, E., Norsett, S., and Wanner, G.: *Solving ordinary differential equations.* Nonstiff Problems. Springer, Berlin, 1987.

[3] Kim, A.V. and Pimenov, V.G.: Numerical methods for delay differential equations – application of i-smooth calculus. *Lecture Notes Series*, vol. 44. Seoul National University, Korea, 1999.

[4] Zhang, B. and Gopalsamy, K.: Oscillation and nonoscillation in a nonautonomous delay-logistic equation. Q. Appl. Math. **46** (1988), 267–273.