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INSTABILITY OF MIXED FINITE ELEMENTS FOR RICHARDS' EQUATION*

Jan Březina

Abstract

Richards' equation is a widely used model of partially saturated flow in a porous medium. In order to obtain conservative velocity field several authors proposed to use mixed or mixed-hybrid schemes to solve the equation. In this paper, we shall analyze the mixed scheme on 1D domain and we show that it violates the discrete maximum principle which leads to catastrophic oscillations in the solution.

1 Introduction

A standard model for the water flow in a partially saturated porous medium is Richards' equation which can by written as the system:

$$\partial_t\theta(h) + \operatorname{div}(\boldsymbol{u}) = f \qquad \text{in } (0,T)\times\Omega, \tag{1}$$

$$\boldsymbol{u} = -k(h)\nabla(h+z) \qquad \text{in } (0,T)\times\Omega. \tag{2}$$

The unknowns are the pressure head h and the water velocity $\boldsymbol{u}$ while the other involved quantities are the density of volume water sources f, the z-coordinate, assumed to be in opposite direction to the gravity force, the water content θ and the hydraulic conductivity k, where θ and k are given nonlinear function of h. Both equations are considered on the domain $\Omega \subset \mathbf{R}^N$ and during the time interval $(0,T)$. Through this work we consider the Dirichlet boundary condition h_D on $\Gamma_D \subset \partial\Omega$, the homogeneous Neumann condition $\boldsymbol{u} = 0$ on the remaining part of the boundary, and the initial condition h_0 for the pressure head.

The characteristic functions $\theta(h)$ and $K(h)$ are empirical. We assume the most common Mualem – van Genuchten model [6], [5]:

$$\theta(h) = \theta_r + (\theta_s - \theta_r)\tilde\theta(h), \tag{3}$$

$$\tilde\theta(h) = (1 + (\alpha h)^n)^{-m}, \quad m = 1 - 1/n \tag{4}$$

$$k(h) = k_s\tilde\theta^{0.5}\left(1 - (1 - \tilde\theta^{1/m})^m\right)^2, \tag{5}$$

where θ_r, θ_s, n, α, and k_s are suitable soil parameters.

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System $(1-2)$ represents a quasilinear degenerated parabolic-elliptic equation. The existence and uniqueness of the solution as well as some regularity properties were proved by Alt, and Luckhaus [1]. When solving Richards' equation numerically, we want to obtain a discrete velocity field which satisfies a discrete version of the continuity equation (1) up to the given tolerance of the nonlinear solver. This is important for a subsequent simulation of the water transport. That is why mixed or mixed-hybrid finite elements are used by many authors, e.g. [4], [3].

Motivated by these works, we want to develop a simulator that can solve coupled Richards' equations on domains of different dimension. Since the solution of Richards' equation evolves substantially only around a small wetting front region, adaptivity is crucial to achieve reasonable performance. To meet these two requirements, we have decided to try C++ finite element library DEAL II [2]. The library allows to produce a dimension independent code with h, p, and hp versions of adaptivity and provides a rich palette of finite elements. The only but fundamental restriction of the library is that elements have to be topologically equivalent to hypercubes. However, during tests of our code we have observed serious oscillations of the solution. Aim of this paper is to present these observations and give an explanation of this behavior.

The paper is organized as follows. First, the mixed discretization is described. Then, in Section 3, we make its comparison with a primary discretization and we demonstrate the presence of instabilities. In the last section, we derive a condition under which the mixed scheme obeys a discrete maximum principle in 1D and we discuss some similar results.

2 Mixed finite elements

In order to derive mixed formulation of the system $(1-2)$, we multiply the first equation by a scalar test function φ, while in the second equation we divide by k, test by a vector valued function $\boldsymbol{\psi}$ and integrate by parts in the pressure term. Finally, we are looking for a solution $h \in L^2(\Omega)$, $\boldsymbol{u} \in H(div, \Omega)$ which satisfies

$$\int_\Omega k^{-1}(h)(\boldsymbol{u} \cdot \boldsymbol{\psi}) - \int_\Omega h \operatorname{div} \boldsymbol{\psi} = \int_\Omega z \operatorname{div} \boldsymbol{\psi} - \int_{\partial\Omega} (h_D + z)\boldsymbol{\psi} \cdot \boldsymbol{n}, \tag{6}$$

$$-\int_\Omega \partial_t \theta(h)\varphi - \int_\Omega \varphi \operatorname{div} \boldsymbol{u} = -\int_\Omega f\varphi \tag{7}$$

for all $\boldsymbol{\psi} \in H(div, \Omega)$ and $\varphi \in L^2(\Omega)$, where $H(div, \Omega)$ is a space of vector valued L^2-function with divergence in $L^2(\Omega)$.

Next, we consider a decomposition $\mathcal{T} = \{K_i\}$ of the domain $\Omega \subset \mathbf{R}^N$ into lines $(N = 1)$, quadrilaterals $(N = 2)$ or hexahedrons $(N = 3)$. On this computational grid we use Raviart-Thomas finite elements RT_d with order d for discretization of the velocity and discontinuous polynomial finite elements P_d of order d for discretization of the pressure head. More specifically, we consider discrete solution in a form

$$\boldsymbol{u}(t,\boldsymbol{x}) = \sum_i \tilde{u}_i(t)\boldsymbol{\psi}_i(\boldsymbol{x}), \qquad h(t,\boldsymbol{x}) = \sum_i \tilde{h}_i(t)\varphi_i(\boldsymbol{x}), \tag{8}$$

where $\tilde{u}$ and $\tilde{h}$ are unknown coefficient vectors. The backward Euler is used for temporal discretization. A fully implicit scheme is necessary to avoid oscillations on the saturated part of the domain where the equation becomes elliptic. Finally, we obtain a nonlinear system of equations which we solve by simple Picard iterations. Resulting linear system for the solution $\tilde{h}^k$, $\tilde{u}^k$ in iteration k of time t_n reads

$$A(h^{k-1})\tilde{u}^k + B\tilde{h}^k = F \tag{9}$$

$$B^T\tilde{u}^k + D(h^{k-1})\tilde{h}^k = G(h^{k-1}) \tag{10}$$

with

$$\begin{aligned}
A_{i,j}(h^{k-1}) &= \sum_{K\in\mathcal{T}} \int_K k^{-1}(h^{k-1})(\boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j), \\
B_{i,j} &= -\sum_{K\in\mathcal{T}} \int_K \varphi_i \mathrm{div}\, \boldsymbol{\psi}_j\,, \\
D_{i,j}(h^{k-1}) &= \sum_{K\in\mathcal{T}} \int_K -\frac{\theta'(h^{k-1})}{dt}\varphi_i\varphi_j\,, \\
F_i &= \sum_{K\in\mathcal{T}} \int_K z\mathrm{div}\, \boldsymbol{\psi}_i - \int_{K\cap\Gamma_D} (z+h_D)\boldsymbol{\psi}_i\cdot\boldsymbol{n}, \\
G_i(h^{k-1}) &= \sum_{K\in\mathcal{T}} \int_K -\frac{\theta'(h^{k-1})h^{k-1}}{dt}\varphi_i + \frac{\theta(h^{k-1})-\theta^0}{dt}\varphi_i\,,
\end{aligned}$$

where h^{k-1} is the actual discrete pressure head field according to (8) and θ^0 is the water content field from the previous time t_{n-1}. Before solving system (9) – (10), we use the last pressure head $\tilde{h}^{k-1}$ to resolve equation (9) and compute a residuum r^{k-1} of the equation (10). Iterations are stopped, when l^2-norm of the residuum drops under the prescribed tolerance. Then the residuum is subtracted from the actual water content which forms θ^0 for the next time step. This way we achieve a perfect conservation of the total water content over the whole domain.

3 Comparison of mixed and primary discretization

The described mixed finite element approximation with the lowest element order $d = 0$ (MFE) have been compared with a mature one dimensional solver based on the primary linear finite element (FE) approximation of the pressure. The latter solver was thoroughly tested against experimental data in cooperation with Vogel et al. [7].

The setting of the one dimensional infiltration test problem was as follows: a vertical domain $(-5, 0)$ $[m]$, the constant initial pressure head $h_0 = -150$ $[m]$, the Dirichlet

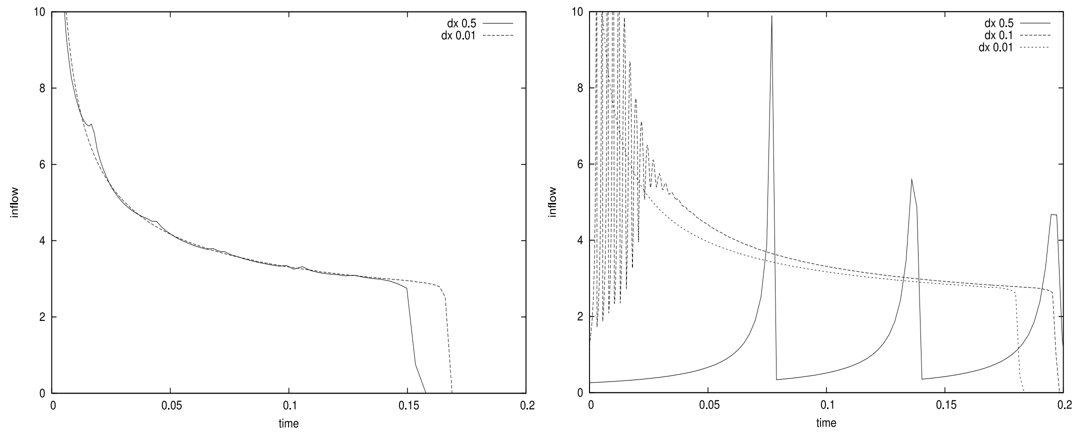

Fig. 1: *Infiltration velocity on the top of the vertical 1D domain. The stable FE scheme (left) and the unstable MFE scheme (right).*

boundary condition $h_D = 1\,[m]$ on the top and the homogeneous Neumann condition on the bottom. The parameters of the soil model were $n = 1.14$, $\alpha = 0.1\,[m^{-1}]$, $\theta_r = 0.01$, $\theta_s = 0.480$, $k_s = 2\,[mh^{-1}]$. This setting leads to a steep wetting front during the initial phase, thus we have to use short time steps. The wetting front goes from the top to the bottom so that the pressure head should be monotonous in time and space, increasing from -150 up to $1 + z$. The velocity should be always negative. The MFE code was run on meshes with steps 0.01, 0.1, and 0.5 the FE code was run only for steps 0.01 and 0.5. All simulations were started with the time step 10^{-6} and the time step is enlarged if the number of nonlinear iterations drops under 3.

Figure 1 shows the infiltration velocity on the top of the domain up to the full saturation of the domain. For the fine mesh step 0.01 the results are comparable. The infiltration computed by the MFE code takes just little bit longer compared to the FE code. On the other hand, for the coarser meshes, the MFE code produces terrible oscillations while the FE code still provides satisfactory results. The oscillations are not only in time but also in space and they get worse with shorter time steps or larger mesh steps. Values of the pressure head leave the valid interval $[-150, 1]$ and positive values of the velocity appear.

4 Discrete maximum principle

Maximum principle for elliptic PDEs states that a solution of the equation

$$\operatorname{div}(-\tilde{k}\nabla h) + \tilde{c}h = \tilde{f} \quad \text{on } \Omega, \quad h = \tilde{g} \quad \text{on } \partial\Omega, \tag{11}$$

with $\tilde{k} > 0$, $\tilde{c} \geq 0$, is non-negative provided $\tilde{f}$ and $\tilde{g}$ are non-negative. If a similar property holds for a discrete problem, we say that it obeys the discrete maximum principle (DMP).

In view of the previous section it seems that the MFE scheme violates DMP for the short time steps. To show this, we shall analyze one linear step, i.e. system (9)–(10), which can be viewed as the discretization of the linear elliptic problem (11) with $\tilde{k} = k(h)$, $\tilde{c} = \theta'(h)/dt$, and suitable positive $\tilde{f}$. We consider one dimensional domain with grid points $x_1 < x_2 < \cdots < x_n$ and the lowest order elements $d = 0$. Further, we use equivalent mixed-hybrid discretization of (11). On every element $K_i = (x_i, x_{i+1})$ the discrete solution is represented by the pressure head h_i in the center of the element, by the traces $\mathring{h}_i^{1,2}$ on the element boundary, and by the velocity $\boldsymbol{u}_i = u_i^1\psi^1 + u_i^2\psi_2$. The velocity is linear combination of discontinuous RT_0 base functions

$$\psi_i^1(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}, \quad \psi_i^2(x) = \frac{x - x_i}{x_{i+1} - x_i}$$

where coefficients $u_i^{1,2}$ are the outer normal fluxes from the element i. Proceeding similarly as in the case of mixed formulation we obtain a discrete version of (11):

$$\sum_{j=1,2} \tilde{k}_i^{-1} u_i^j \int_{K_i} \psi_i^m \psi_i^j = h_i - \mathring{h}_i^m \quad \text{for } m = 1, 2 \tag{12}$$

$$\tilde{c}_i h_i |K_i| + u_i^1 + u_i^2 = \tilde{f}_i |K_i| \tag{13}$$

$$u_i^2 = -u_{i+1}^1, \quad \mathring{h}_i^2 = \mathring{h}_{i+1}^1. \tag{14}$$

We denote $\mathring{h}_i = \mathring{h}_i^2 = \mathring{h}_{i+1}^1$. The integral in (12) evaluates to $|K_i|/3$ and $-|K_i|/6$ for $m = j$ and $m \neq j$, respectively. On the Dirichlet boundary x_n we set $\mathring{h}_n^1 = h_D$. Then, eliminating h_i and $u_i^{1,2}$ from the system, we obtain an equation for $\mathring{h}_i$:

$$a_{i-1}\mathring{h}_{i-1} + (b_{i-1} + b_i)\mathring{h}_i + a_i\mathring{h}_{i+1} = c_{i-1} + c_i \tag{15}$$

where

$$a_i = \frac{2\tilde{k}_i}{|K_i|} - \frac{\alpha_i\alpha_i}{\beta_i}, \quad b_i = \frac{4\tilde{k}_i}{|K_i|} - \frac{\alpha_i\alpha_i}{\beta_i}, \quad c_i = \frac{\alpha_i |K_i| \tilde{f}_i}{\beta_i}, \tag{16}$$

$$\alpha_i = \frac{6\tilde{k}_i}{|K_i|}, \quad \beta_i = |K_i|\tilde{c}_i + 2\alpha_i. \tag{17}$$

Equation (15) is one row of a linear system $A\mathring{h} = c$, where vector c is non-negative provided $\tilde{f}_i$ and h_D are non-negative. In order to obtain a non-negative solution $\mathring{h}$, the matrix A has to have positive inverse. This holds if A is so called M-matrix, that is a matrix with positive diagonal entries, non-positive off diagonal entries, and positive row sums. In our case this is equivalent to $a_i \leq 0$, $b_i > 0$, and $a_i + b_i > 0$. The later two inequalities are always true, while the first one holds only if

$$\frac{|K_i|^2}{6} \leq \frac{\tilde{k}_i}{\tilde{c}_i} = dt\frac{k(h_i)}{\theta'(h_i)}. \tag{18}$$

For positive $\tilde{f}$ and $\tilde{g}$, this condition implies positive nodal pressures $\mathring{h}_i$. Then the elemental pressures h_i are also positive since

$$h_i = \frac{|K_i|\tilde{f}_i + \alpha_i(\mathring{h}_i^1 + \mathring{h}_i^2)}{\beta_i}.$$

Further numerical experiments reveal that oscillations of the solution appear exactly on that elements where the condition (18) does not hold. Thus to get stable scheme one has to adapt the element size $|K_i|$ according to the condition. However, the right hand side tends to zero as $h_i \to -\infty$, at least for the soil model (3)–(5). It means we should use small mesh step on the dry region which is highly ineffective since the solution is mainly constant there. Situation is even worse for mixed elements on 2D quadrilaterals or 3D hexahedrons since they never lead to M-matrix even for $\tilde{c}_i = 0$.

In the paper due to Younes, Ackerer, and Lehmann [8] authors prove stability conditions similar to (18) for mixed-hybrid elements on triangular and tetrahedral meshes. We can conclude that the mixed scheme for the Richards' equation is stable only for large time steps and therefore is not suitable for a robust solver. However, one can try to modify the mixed scheme to make it more stable. In fact two such modifications were already proposed in [8].

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