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ON RUNGE–KUTTA, COLLOCATION AND DISCONTINUOUS GALERKIN METHODS: MUTUAL CONNECTIONS AND RESULTING CONSEQUENCES TO THE ANALYSIS

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Abstract

Discontinuous Galerkin (DG) methods are starting to be a very popular solver for stiff ODEs. To be able to prove some more subtle properties of DG methods it can be shown that the DG method is equivalent to a specific collocation method which is in turn equivalent to an even more specific implicit Runge–Kutta (RK) method. These equivalences provide us with another interesting view on the DG method and enable us to employ well known techniques developed already for any of these methods. Our aim will be proving the superconvergence property of the DG method in Radau quadrature nodes.

1. Introduction

The Discontinuous Galerkin (DG) method, either as space or time discretization, starts to play an important role in problems, where robust and highly efficient solvers are needed. Such a method enables a user to fully exploit adaptivity with higher order approximation and still it remains very robust.

The DG time discretizations are usually analyzed by similar means as the finite element method, see e.g. [9]. In such a way we obtain L^∞ estimates of order s for $s-1$ degree polynomial approximation. But numerical experiments often show better behaviour of the discrete solution in the nodes of Radau quadrature and especially in the endpoints of intervals. This phenomenon is usually called superconvergence.

Our aim will be showing some ideas how the possible analysis of superconvergence can be carried out in this case. In our approach we will focus on the Radau quadrature variant, where the integrals from the classical DG discretization are replaced by (right) Radau quadrature of suitable order, i.e. the quadrature preserves linear terms. As a first step we will show generally that this Radau quadrature variant of the DG method is equivalent to the well known Radau IIA Runge–Kutta (RK) method in Radau quadrature nodes. Then it is possible to use classical results developed for implicit RK methods to achieve superconvergence error estimates. In this part we will be mainly focused on stiff, linear problems.

2. ODE and discretizations

Let us assume the following ODE

$$y'(t) = f(t, y(t)), \quad \forall t \in (0,T), \qquad (1)$$
$$y(0) = \alpha.$$

Let us assume $t_m = m\tau$ be an equidistant partition of $(0,T)$ with time step τ. We introduce several one–step methods:

Runge–Kutta methods: Let $a_{i,j}$, b_i, c_i, $i,j = 1,\dots,s$ be suitable coefficients. Then we call the sequence y^m satisfying $y^0 = \alpha$

$$g_i^m = y^{m-1} + \tau \sum_{j=1}^{s} a_{i,j} f(t_{m-1} + \tau c_j, g_j^m), \quad \forall i = 1,\dots,s, \qquad (2)$$
$$y^m = y^{m-1} + \tau \sum_{i=1}^{s} b_i f(t_{m-1} + \tau c_i, g_i^m)$$

the RK solution of (1) approximating values $y(t_m)$.

Collocation methods: Let c_i, $i = 1,\dots,s$ be suitable coefficients. Let $y^0 = \alpha$. In every step we construct polynomial p of degree at most s such that

$$p(t_{m-1}) = y^{m-1}, \qquad (3)$$
$$p'(t_{m-1} + \tau c_i) = f(t_{m-1} + \tau c_i, p(t_{m-1} + \tau c_i)), \quad \forall i = 1,\dots,s.$$

Then we put $y^m = p(t_m)$. We call the resulting sequence the collocation solution of (1) approximating values $y(t_m)$.

Discontinuous Galerkin method: Let us denote $I_m = (t_{m-1}, t_m)$. Let us define the space

$$S^\tau = \{v \in L^2(0,T) : v|_{I_m} \in P^{s-1}\}, \qquad (4)$$

where P^{s-1} is a space of polynomials of degree $s-1$. Since the functions from S^τ are discontinuous in general in nodes of the partition, we denote the limit at nodes $v_\pm^m = v(t_m\pm)$ and the jump $\{v\}_m = v_+^m - v_-^m$. We call $u \in S^\tau$ the DG solution of (1) if $u_-^0 = \alpha$ and

$$\int_{I_m} u'(t)v(t)dt + \{u\}_{m-1} v_+^{m-1} = \int_{I_m} f(t, u(t))v(t)dt, \quad \forall v \in S^\tau, \forall m. \qquad (5)$$

For comparison with previous methods we focus mainly on endpoints of intervals: $u_-^m \approx y(t_m)$.

Radau discontinuous Galerkin method: Let $r \in P^s$ be the (right) Radau polynomial, i.e. $r(0) = 1$, $r(1) = 0$, and for $s \geq 2$ let r be orthogonal to the polynomial space P^{s-2}. We can define the (right) Radau quadrature by

$$\int_0^1 F(t)dt \approx Q[F(t)] = \sum_{i=1}^{s} w_i F(x_i), \qquad (6)$$

where x_i are roots of r and w_i are chosen in such a way that the resulting quadrature is accurate for polynomials P^{2s-2}. Similarly we can define the Radau quadrature $Q_m[.]$ and Radau polynomial r_m on I_m. We can define the Radau DG solution of (1) by replacing integrals by Radau quadratures in (5)

$$Q_m[u'(t)v(t)] + \{u\}_{m-1}v_+^{m-1} = Q_m[f(t,u(t))v(t)], \quad \forall v \in S^\tau, \forall m. \tag{7}$$

3. The Radau discontinuous Galerkin method is a Runge–Kutta method

In fact we want to show this in two steps. First, when the coefficients c_i of the collocation method are chosen as Radau quadrature nodes, then there is the following relation between the collocation polynomial p and Radau DG solution u

$$p(t) = u(t) - \{u\}_{m-1}r_m(t). \tag{8}$$

From this it follows that $p(t_{m-1} + \tau c_i) = u(t_{m-1} + \tau c_i)$, since $r_m(t_{m-1} + \tau c_i) = 0$. Since $c_s = 1$ we gain the correspondence of the collocation solution and the Radau DG solution at t_m, i.e. $y^m = p(t_m) = u_-^m$.

Lemma 1. *Let $p \in P^s$ be the collocation polynomial on I_m associated to the collocation method with coefficients c_i chosen as Radau quadrature nodes, $u \in P^{s-1}$ be the Radau DG solution on I_m and $r_m \in P^s$ be the (right) Radau polynomial on I_m. Then (8) holds.*

The proof follows the ideas from [7], where a similar case (continuous Galerkin and Gauss quadrature) is considered.

Proof. Let $u \in P^{s-1}$ be the Radau DG solution on I_m. We need to verify (3).

$$p(t_{m-1}) = u|_{I_m}(t_{m-1}) - \{u\}_{m-1}r_m(t_{m-1}) = u_+^{m-1} - (u_+^{m-1} - u_-^{m-1}) = u_-^{m-1}. \tag{9}$$

We denote $\ell_{m,i}$ the Lagrange interpolation basis function

$$\ell_{m,i}(t) = \prod_{j \neq i} \frac{t - t_{m-1} - \tau c_j}{\tau(c_i - c_j)}. \tag{10}$$

We can use $\ell_{m,i}$ as test functions in (7) and we obtain

$$w_i u'(t_{m-1} + \tau c_i) + \{u\}_{m-1}\ell_{m,i}(t_{m-1}) = w_i f(t_{m-1} + \tau c_i, u(t_{m-1} + \tau c_i)). \tag{11}$$

Now it is sufficient to show that $w_i r_m'(t_{m-1} + \tau c_i) = -\ell_{m,i}(t_{m-1})$. Since the product $\ell_{m,i} r_m' \in P^{2s-2}$, Radau quadrature for such a term is exact and we obtain

$$w_i r_m'(t_{m-1} + \tau c_i) = Q_m[\ell_{m,i}(t) r_m'(t)] = \int_{I_m} \ell_{m,i}(t) r_m'(t) dt \tag{12}$$

$$= \ell_{m,i}(t_m) r_m(t_m) - \ell_{m,i}(t_{m-1}) r_m(t_{m-1}) - \int_{I_m} \ell_{m,i}'(t) r_m(t) dt = -\ell_{m,i}(t_{m-1}),$$

since $r_m(t_m) = 0$, $r_m(t_{m-1}) = 1$ and r_m is orthogonal to P^{s-2} on I_m. ☐

The second step is that every collocation method is equivalent to a suitable RK method.

Lemma 2. *Let the RK coefficients be chosen in the following way*

$$a_{i,j} = \int_0^{c_i} \ell_j(t)dt, \quad \forall i, j = 1, \ldots, s, \tag{13}$$

$$b_i = \int_0^{1} \ell_i(t)dt, \quad \forall i = 1, \ldots, s, \tag{14}$$

where ℓ_i is the Lagrange interpolation basis function

$$\ell_i(t) = \prod_{j \neq i} \frac{t - c_j}{c_i - c_j}. \tag{15}$$

Then the values g_i^m, $i = 1, \ldots, s$ and y^m produced by such a RK method are equal to the values $p(t_{m-1} + \tau c_i)$, $i = 1, \ldots, s$ and y^m produced by the collocation method with the same coefficients c_i.

Proof. The proof can be found in [4] or [10]. □

Now from Lemma 1 and Lemma 2 we can see that the values produced by the Radau DG method in Radau quadrature nodes are equal to the values produced by a suitable RK method. Such a RK method is the well known Radau IIA RK method.

4. Analysis of the Radau IIA Runge–Kutta method

Now, we shall turn our focus on numerical analysis of linear problems

$$y'(t) = By(t) + f(t), \quad \forall t \in (0, T). \tag{16}$$

To do so, we shall focus on Dalquist's equation $y'(t) = \lambda y(t)$ with the exact solution $y(t_m) = e^{\tau\lambda} y(t_{m-1})$. For the purpose of analysis we assume $\mathrm{Re}\lambda \leq 0$, i.e. stable behaviour of the solution. Rewriting (2) in a vector–matrix formulation we obtain

$$g^m = y^{m-1}\mathbf{1} + \tau\lambda A g^m, \tag{17}$$

$$y^m = y^{m-1} + \tau\lambda b^T g^m, \tag{18}$$

where vector $\mathbf{1} = (1, \ldots, 1)^T$, matrix A and vectors b and g^m are formed by entries $a_{i,j}$, b_i and g_i^m. Eliminating inner stages g_i^m we obtain $y^m = R(\tau\lambda)y^{m-1}$, where

$$R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1} = \frac{\det(I - zA + zb^T\mathbf{1})}{\det(I - zA)}. \tag{19}$$

Following [6, Theorem 3.11] we can see that $R(z)$ is in the case of Radau IIA RK method the "subdiagonal" $(s-1, s)$–Padé approximation satisfying

$$\exp(z) - R(z) = O(z^{2s}). \tag{20}$$

Moreover, following results from [6, Chapter IV.4] we can conclude that $R(z)$ is also A–stable, i.e. $|R(z)| \leq 1$ for any $\mathrm{Re} z \leq 0$. We define the local error

$$\rho^m = y(t_m) - R(\tau\lambda)y(t_{m-1}) = (\exp(\tau\lambda) - R(\tau\lambda))y(t_{m-1}). \tag{21}$$

From (20) we can see that $|\rho^m| \leq C\tau^{2s} \max |y^{(2s)}|$. Then the error analysis follows easily from the stability of $R(z)$

$$|e^m| = |y(t_m) - y^m| = |\rho^m + R(\tau\lambda)e^{m-1}| \leq \ldots \tag{22}$$

$$\ldots \leq |R(\tau\lambda)|^m |e^0| + \sum_{i=1}^{m} |R(\tau\lambda)|^{m-i} |\rho^i| \leq |e^0| + T\frac{1}{\tau}\max_i |\rho^i|.$$

Assuming $e^0 = 0$ we gain global error estimate $e^m = O(\tau^{2s-1})$.

This result can be extended to the multidimensional case $y'(t) = By(t)$, where B is a matrix (or operator on Banach spaces in general) satisfying $\mathrm{Re}\,\langle By, y\rangle \leq 0$. The extension remains almost the same as the scalar case with the only difficulty arising from the question whether $\|R(\tau B)\| \leq 1$. The answer to this question is positive. The proof of the matrix case can be found in [6, Theorem 11.2]. The proof of the general operator case can be found in [8].

Now, we shall come back to equation (16). Unfortunately, the extension of previous results is not straightforward. According to [1] it is necessary to assume an additional assumption, otherwise the so-called order reduction phenomena occur.

Theorem 3. *Let y be the exact solution of (16) with operator B satisfying $Re\,\langle Bv, v\rangle \leq 0$. Let*

$$y^{(k)} \in Dom(B^{2s-k}), \quad \forall k = s+1, \ldots, 2s. \tag{23}$$

Then the Radau IIA RK solution y^m converges with order $2s-1$, i.e. $\|y(t_m) - y^m\| = O(\tau^{2s-1})$.

Proof. The proof can be found in [1]. □

We should mention that in the previous case $f = 0$, the additional assumption (23) was automatically satisfied for solutions with bounded derivatives, i.e. $y^{(2s)}$ bounded. For ODEs coming from PDE discretizations in space assumption (23) can be reformulated as some kind of regularity and compatibility conditions on data. In usual context of weakly formulated PDEs these conditions are considered unnatural. Assumption (23) is necessary to achieve order $2s$, but it can be relaxed to obtain reduced orders, still higher than s. For assumptions needed to obtain order $s+1$ see e.g. [3].

Up to now we have analyzed the error in the nodes t_m only. From [2] and [5] follows the local error estimate for internal stages g_i^m of implicit RK methods. In the case of Radau IIA RK method we obtain order $s+1$ there. Together with global error estimates at t_m at least of order $s+1$ we get also global error estimates at Radau quadrature nodes of order $s+1$.

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